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Irregularity of Quintic Surfaces of General Type

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Introduction.

Let X be a hypersurface in $\boldsymbol{P}^3$ of degree d defined over an algebraically closed field k of characteristic 0. For $d\leq 4$, singularities on X and properties of the resolution $\tilde{X}$ of X have been studied. For example, if X is normal, then it is known that $\tilde{X}$ is birationally equivalent to one of the following surfaces:

$d=1,2$: a rational surface;

$d=3$: a rational surface or an elliptic ruled surface;

$d=4$: a $K3$ surface, a rational surface, an elliptic ruled surface or a ruled surface over a curve of genus 3.

(The case of $d=1$ or 2 is clear. For $d=3$, see Hidaka-Watanabe [3], and for $d=4$, Umezu [8]. The argument in [8] can also be applied to the case of $d\leq 3$.)

On the other hand, not many things are known about the case of higher d. The purpose of this paper is to prove the following

MAIN THEOREM. *Let X be a normal quintic surface and $\tilde{X}$ denote its resolution. If $\tilde{X}$ is of general type, then its irregularity $q(\tilde{X})$ vanishes.*

REMARK. As we see in the following example, this result is not available for $d\geq 6$.

EXAMPLE (Zariski). Let $(X_0:X_1:X_2:X_3)$ be homogeneous coordinates of $\boldsymbol{P}^3$ and put

$$X=\{X_3^6-(F(X_0,X_1,X_2)^2+G(X_0,X_1,X_2)^3)=0\}$$

where F and G are homogeneous polynomials of degree 3 and 2 respectively. Then the irregularity of a resolution $\tilde{X}$ of X is positive ([13]). The singularity of X corresponds to the singularity of the curve $C=\{F(X_0,X_1,X_2)^2+G(X_0,X_1,X_2)^3=0\}\subset\{X_3=0\}\simeq\boldsymbol{P}^2$. If F and G are general, the singularity of C is at the six points of $\{F(X_0,X_1,X_2)=0\}\cap\{G(X_0,X_1,X_2)=0\}$ and each corresponding singular point on X is defined locally by

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the equation $z^6=x^2+y^3$. The exceptional curve of the minimal resolution of this singularity is a smooth elliptic curve with self-intersection number -1. Therefore, if $\tilde{X} \xrightarrow{\pi} X$ is the minimal resolution of X, then we have $K_{\tilde{X}}^2=(\pi^*\mathcal{O}_X(2))^2-6=18$, and so $\tilde{X}$ is of general type.

This paper is based on the report [9]. Yang [11] got the same result as a corollary of his close analysis of singularities on quintic surfaces of general type. Our method is rather straightforward to the one aim, and also seems to be applicable to other surfaces of lower degree. In fact, we are to use similar method to investigate some normal quintic surfaces which are not of general type [6].

§ 1. Preliminaries.

In this section, we summarize some results about the invariants of normal two dimensional singularity. Let (Y, y) be a normal two dimensional singularity, $\pi\colon \tilde{Y}\to Y$ its minimal resolution and $A=\pi^{-1}(y)$ the exceptional set.

DEFINITION 1 (Wagreich [10]). We put

$$p_g(Y, y)=\dim_k(R^1\pi^*\mathcal{O}_{\tilde{Y}})_y\colon \quad \text{the } \textit{geometric genus} \text{ of} \quad (Y, y),$$

$$p_a(Y, y)=\sup_{\substack{D>0\\ \operatorname{supp} D\subseteq A}} p_a(D)\colon \quad \text{the } \textit{arithmetic genus} \text{ of} \quad (Y, y).$$

By the definition we have $p_g(Y, y)\geq p_a(Y, y)\geq 0$.

THEOREM (Artin [1]). *$p_g(Y, y)=0$ if and only if $p_a(Y, y)=0$.*

Since we only have to deal with singularities with positive geometric genus, we see from this theorem that it is sufficient to consider singularities with positive arithmetic genus.

DEFINITION 2. We say that the singularity (Y, y) is *numerically Gorenstein* if there exists a divisor K' on $\tilde{Y}$ which is supported on A and satisfies $A_iK'=A_iK_{\tilde{Y}}$ for any component A_i of A.

The following Lemma was pointed out by Y. Koyama.

LEMMA 1. *If (Y, y) is numerically Gorenstein, then $p_a(Y, y)\leq -K'^2/8+1$. In particular we have $p_a(Y, y)\leq 1$ if $K'^2\geq -7$.*

PROOF. Let D be a divisor with support on A. Then we have

$$p_a(D)=(D^2+DK')/2+1=(D+K'/2)^2/2-K'^2/8+1\,.$$

By the negativity of the intersection matrix of A, we get the desired inequality.

DEFINITION 3 (Wagreich [10]). If $p_a(Y, y)=1$, (Y, y) is said to be an *elliptic singularity*.

As for an elliptic singularity, there exists a unique minimal divisor E on $\tilde{Y}$ among effective divisors D which is supported on A and satisfies $p_a(D)=1$. The divisor E is called the *minimal elliptic cycle* (Laufer [4]). We define a sequence of effective divisors $\{Z_1, \cdots, Z_l\}$ on $\tilde{Y}$ which is called the *elliptic sequence* as follows (Tomari [7], Yau [12]). For Z_1 we choose the fundamental cycle. Suppose that we have defined from Z_1 to Z_k. If $Z_kE<0$, we define $\{Z_1, \cdots, Z_k\}$ as the elliptic sequence. Assume $Z_kE=0$. Then let B_{k+1} denote the connected component containing E of the sum of irreducible components A_i of A satisfying $Z_kA_i=0$. We define Z_{k+1} to be the fundamental cycle of B_{k+1}. Since $\operatorname{supp} Z_k \supsetneqq \operatorname{supp} Z_{k+1}$, we can define the elliptic sequence $\{Z_1, \cdots, Z_l\}$ as a finite sequence.

First Yau defined the elliptic sequence on the minimal good resolution. But his definition is valid for our case too, and his results which we need here are reformulated as in the Theorem below (The proof goes similarly as in [12], or even more simply.). We also refer to Tomari [7] as a direct reference, in which he extended the notion of the elliptic sequence to any resolution and showed its properties.

THEOREM (Tomari [7], Yau [12]). *Let (Y, y) be a numerically Gorenstein elliptic singularity and $\{Z_1, \cdots, Z_l\}$ its elliptic sequence. Then we have*

(1) $p_g(Y, y)\leq l$,

(2) $K'=-\sum_{i=1}^{l} Z_i$.

COROLLARY 1. *If (Y, y) is a numerically Gorenstein elliptic singularity, we have $p_g(Y, y)\leq -K'^2$.*

PROOF. From (2) of the Theorem and the definition of the elliptic sequence, we have $K'^2=\sum_{i=1}^{l} Z_i^2\leq -l$. Combining this with (1) of the Theorem, we get the result.

COROLLARY 2. *Let (Y, y) be as above. Then there exists an effective divisor E supported on A and satisfying the following conditions*:

$$p_a(E)=1\,, \qquad p_g(Y, y)E\leq -K'\,.$$

PROOF. The minimal elliptic cycle E, which is described in the definition of the elliptic sequence, satisfies these conditions. The first one is obvious. By the definition of the elliptic sequence, we see that $E\leq Z_i$ for all i. Hence the second condition follows from (1) and (2) of the Theorem.

§2. The proof of the main theorem.

In the sequel we fix the notations as follows:

$X\subset \boldsymbol{P}^3$: a normal quintic surface

$\pi: \tilde{X} \to X$: the minimal resolution
$q = q(\tilde{X}) = \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$
$p_g = p_g(\tilde{X}) = \dim H^2(\tilde{X}, \mathcal{O}_{\tilde{X}})$
$p = \dim_k R^1\pi_*\mathcal{O}_{\tilde{X}} = \sum_{x \in X} p_g(X, x)$
$H \subset X$: a general hyperplane section of X (hence a non-singular curve of genus 6 with $\omega_X \simeq \mathcal{O}_X(H)$)
$\tilde{H} = \pi^{-1}(H) \simeq H$
$\tilde{D}$: the effective divisor on $\tilde{X}$ such that $K_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}(\tilde{H} - \tilde{D})$ (the support of $\tilde{D}$ coincides with $\bigcup_{p_g(X,x)>0} \pi^{-1}(x)$)

ASSUMPTION. We assume that $\tilde{X}$ is of general type.

Let us consider the following exact sequence:

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to R^1\pi_*\mathcal{O}_{\tilde{X}} \to H^2(X, \mathcal{O}_X) \to H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to 0 .$$

Since $H^1(X, \mathcal{O}_X) = 0$, we have

$$p_g = 4 - p + q . \qquad (*)$$

On the other hand, since $\tilde{X}$ is of general type, $0 < \chi = 1 - q + p_g$ and so by $(*)$ $0 \le p \le 4$.

LEMMA 2. *If* $q > 0$, *then* $0 < q < p - 1 \le 3$.

PROOF. We have already got $p - 1 \le 3$. Suppose $q > 0$. Then we have $0 < q \le p$ from the above exact sequence. So it is enough to deduce a contradiction by assuming $q = p$ or $p - 1$. If $q = p$, then $p_g = 4$ by $(*)$ and so

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}})) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{H} - \tilde{D})) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\tilde{H})) .$$

Since $|\tilde{H}|$ has no base points, this implies $\tilde{D} = 0$ and hence $q = p = 0$, a contradiction.

If $q = p - 1$, then $p_g = 3$. Therefore there is a unique singular point $x \in X$ such that $p_g(X, x) > 0$ and moreover

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(K_{\tilde{X}})) = \{\pi^* f \mid f \in H^0(X, \mathcal{O}_X(H)), f(x) = 0\} .$$

Hence $|K_{\tilde{X}}|$ has no fixed components except for those of $\pi^{-1}(x)$. Therefore, if $\tilde{X}$ has an exceptional curve of the first kind, it must lie on $\pi^{-1}(x)$, contradicting our hypothesis that the resolution $\pi: \tilde{X} \to X$ is minimal. Hence $\tilde{X}$ is a minimal surface. But then, from $2p_g - 2 = 4 \ge \tilde{H}^2 + \tilde{D}^2 = K_{\tilde{X}}^2$, we obtain $q = 0$, again a contradiction. (For the case of $2p_g - 2 > K_{\tilde{X}}^2$, c.f. Bombieri [2]. As for the case of $2p_g - 2 = K_{\tilde{X}}^2$ in general, we refer to Miyanishi–Nakamura [5]. But in the present situation we can also prove as follows. By $4 = \tilde{H}^2 + \tilde{D}^2$ we have $\tilde{D}^2 = -1$ and hence x is an elliptic singular point by Lemma 1 ($K' = -\tilde{D}$). Applying Corollary 1, we obtain $p = p_g(X, x) = 1$ and so $q = p - 1 = 0$.)

Let $\bar{X}$ be the minimal model of $\tilde{X}$ and $\mu: \tilde{X} \to \bar{X}$ the induced blow-down. If $q > 0$, by Bombieri [2] we have $2\chi \le K_{\bar{X}}^2$, and hence, by Lemma 2 and $(*)$, one of the following cases occurs:

$$\begin{array}{cccccc} & q & p & \chi & K_{\bar{X}}^2 \\ \text{(I)} & \left\{\begin{array}{c}2\\1\end{array}\right. & \begin{array}{c}4\\4\end{array} & \begin{array}{c}1\\1\end{array} & \begin{array}{c}\geq 2\\ \geq 2\end{array} \\ \text{(II)} & 1 & 3 & 2 & \geq 4 \end{array}$$

LEMMA 3 ($q \geq 0$). *Assume that*

$$K_{\bar{X}}^2 \geq 2 \ [\textit{resp.} \ \geq 3] \quad \textit{and} \quad \tilde{D}^2 \leq -4 \ [\textit{resp.} \ \leq -3] \, .$$

Then there exists on $\tilde{X}$ a unique exceptional curve E of the first kind, and $\bar{X}$ is obtained from $\tilde{X}$ by contracting E. Moreover the following equations hold:

$$K_{\bar{X}}^2 = 2 \ [\textit{resp.} \ 3] \, , \qquad \tilde{D}^2 = -4 \ [\textit{resp.} \ -3] \, ,$$

$$\tilde{D}\mu^* K_{\bar{X}} = 2 \ [\textit{resp.} \ 1] \, , \qquad \tilde{D}E = 2 \, .$$

PROOF. Note first that $\tilde{H}$ and $\mu^* K_{\bar{X}}$ are numerically independent. In fact, if $\tilde{X} = \bar{X}$, then $\tilde{H}\tilde{D} = 0$ and $K_{\bar{X}}\tilde{D} = -\tilde{D}^2 > 0$; if $\tilde{X} \neq \bar{X}$, then for any (-1)-curve Γ on $\tilde{X}$ we have $\Gamma \mu^* K_{\bar{X}} = 0$ and $\Gamma \tilde{H} > 0$ because $\tilde{X}$ is the minimal resolution of X. Put $\tilde{H}\mu^* K_{\bar{X}} = d$ and $(\mu^* K_{\bar{X}})^2 = a$. The Hodge index theorem implies

$$\begin{vmatrix} 5 & d \\ d & a \end{vmatrix} = 5a - d^2 < 0 \, .$$

Since $a \geq 2$, we have $d \geq 4$. Moreover, let E denote the divisor on $\tilde{X}$ such that

$$\tilde{H} - \tilde{D} \simeq K_{\tilde{X}} \simeq \mu^* K_{\bar{X}} + E \, ,$$

that is, E is the sum of the total transforms on $\tilde{X}$ of the exceptional (-1)-curves, one from each stage of the successive blow-ups μ. Then we have

$$5 = \tilde{H} K_{\tilde{X}} = d + \tilde{H}E \, , \qquad \tilde{H}E \geq 0 \, .$$

If $d = 5$, then $\tilde{H}E = 0$. This means $E = 0$ and so $\tilde{X} = \bar{X}$, since otherwise E contains at least one (-1)-curve Γ, which necessarily intersects $\tilde{H}$. But this leads us to a contradiction because $\tilde{X} = \bar{X}$ is impossible under our assumption:

$$2 \ [\text{resp.} \ 3] \leq K_{\bar{X}}^2 \, , \qquad K_{\tilde{X}}^2 = 5 + \tilde{D}^2 \leq 1 \ [\text{resp.} \ 2] \, .$$

Therefore we obtain $d = 4$ and $\tilde{H}E = 1$. From $\tilde{H}E = 1$ and the definition of E, we can deduce that E is reduced and irreducible and that by contracting E we obtain $\bar{X}$. Hence we have 1 [resp. 2] $\geq 5 + \tilde{D}^2 = K_{\tilde{X}}^2 = K_{\bar{X}}^2 - 1 \geq 1$ [resp. 2], and so $K_{\bar{X}}^2 = 2$ [resp. 3], $\tilde{D}^2 = -4$ [resp. -3]. Moreover we have $\tilde{D}E = 2$ since $-1 = K_{\tilde{X}}E = \tilde{H}E - \tilde{D}E$, and $\tilde{D}\mu^* K_{\bar{X}} = 2$ [resp. 1] since 4 [resp. 3] $= K_{\tilde{X}}\tilde{D} = \tilde{D}\mu^* K_{\bar{X}} + \tilde{D}E$.

PROOF OF THE MAIN THEOREM. Case (II): If $\tilde{D}^2 \geq -2$, then by Lemma 1 and

Corollary 1 we have $p\leq 2$, a contradiction. If $\tilde{D}^2\leq -3$, we are also led to a contradiction since then we have $K_{\tilde{X}}^2=3$ by Lemma 3.

Case (I): If $\tilde{D}^2\geq -3$, we get a contradiction as above, so $\tilde{D}^2\leq -4$. Applying Lemma 3, we have that there exists on $\tilde{X}$ an exceptional curve E of the first kind whose contraction coincides with the morphism μ: $\tilde{X}\to\bar{X}$ and that $\tilde{D}^2=-4$. The arithmetic genus of each singular point of X is not greater than 1 by Lemma 1, and hence, by Corollary 2, there exist on $\tilde{X}$ effective divisors E_1, E_2, E_3, E_4 such that

(1) $p_a(E_i)=1$ $(i=1,\cdots,4)$,

(2) $\sum_{i=1}^4 E_i\leq\tilde{D}$.

Since $\sum_{i=1}^4 E_i\tilde{D}\geq\tilde{D}^2=-4$ by (2) and $0>E_i^2=E_i\tilde{D}$ by (1) and (2), we have $E_i^2=E_i\tilde{D}=-1$ $(i=1,\cdots,4)$. Hence

$$1=-E_i\tilde{D}=E_iK_{\tilde{X}}=E_i\mu^*K_{\bar{X}}+E_iE \qquad (i=1,\cdots,4)\,,$$

where we note that $E_i\mu^*K_{\bar{X}}\geq 0$ and $E_iE\geq 0$. On the other hand $\tilde{D}\mu^*K_{\bar{X}}=2$ and $\tilde{D}E=2$ by Lemma 3, and so there exists an i such that $E_iE=1$ and $E_i\mu^*K_{\bar{X}}=0$. For this i we have

$$(E_i+E)^2=0\,,\quad (\mu^*K_{\bar{X}})^2=2 \quad\text{and}\quad (E_i+E)\mu^*K_{\bar{X}}=0\,,$$

which contradicts the Hodge index theorem. Thus we complete the proof.

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