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The Sequence of Luxemburg Norms of Derivatives

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Abstract. In this paper we prove that the results obtained in [1] for L_p-norm are still valid for an arbitrary Luxemburg norm.

1. Introduction.

Let $G \subset \boldsymbol{R}$ be some domain and let $\phi(t): [0, +\infty) \to [0, +\infty]$ be an arbitrary Young function [2–3], i.e., $\phi(0)=0$, $\phi(t) \geq 0$, $\phi(t) \not\equiv 0$ and $\phi(t)$ is convex. We denote by $L_\phi(G)$ the set of all measurable functions $f(x)$ on G such that

$$\|f\|_\phi = \inf\left\{\lambda > 0 : \int_G \phi(|f(x)|/\lambda)dx \leq 1\right\} < \infty .$$

Then $L_\phi(G)$ with the Luxemburg norm $\|\cdot\|_\phi$ is a Banach space. $L_\phi(G)$ is called Orlicz space.

Recall that $\|\cdot\|_\phi = \|\cdot\|_p$ when $1 \leq p < \infty$ and $\phi(t) = t^p$; and $\|\cdot\|_\phi = \|\cdot\|_\infty$ when $\phi(t)=0$ for $0 \leq t \leq 1$ and $\phi(t)=\infty$ for $t > 1$. Orlicz spaces are often arised in the study of nonlinear problems (see, for example, [4–5]).

We obtained the following result in [1]:

THEOREM A. *Let $1 \leq p \leq \infty$ and $D^n f(x) \in L_p(\boldsymbol{R})$, $n=0, 1, \cdots$. Then there always exists the limit*

$$d_f = \lim_{n \to \infty} \|D^n f\|_p^{1/n} ,$$

and moreover

$$d_f = \sigma_f = \sup\{|\xi| : \xi \in \operatorname{supp} \tilde{f}(\xi)\} ,$$

where the last equality is the definition of σ_f and $\tilde{f}(\xi)$ is the Fourier transform of the function $f(x)$.

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In this paper we prove that Theorem A still holds when we replace Lebesgue norm $\|\cdot\|_p$ by general norm $\|\cdot\|_\phi$. In [1] we used the Kolmogorov-Stein inequality [6–8] to prove the existence of the limit d_f. Unfortunately, we do not know the generalization of the Kolmogorov-Stein inequality in the case of an arbitrary Luxemburg norm. Therefore, here we had to use a new technique for the proof of the corresponding result.

Studying the properties of functions from $L_\phi(G)$, without loss of generality we may assume that $\phi(t)$ is left continuous. Actually, in the contrary case, there exists a point $t_0>0$ such that

$$\lim_{t\to t_0-}\phi(t)<\phi(t_0)\le\infty\,,\quad \phi(t)=\infty\,,\quad t>t_0\,.$$

We put

$$\psi(t)=\begin{cases}\phi(t)\,, & t\ne t_0\\ \lim\limits_{t\to t_0-}\phi(t)\,, & t=t_0\,.\end{cases}$$

Then $\psi(t)$ is left continuous, and it is obvious that $L_\phi(G)=L_\psi(G)$ and $\|\cdot\|_\phi=\|\cdot\|_\psi$.

2. Results.

THEOREM 1. *Let $n_1<n_2<\cdots$ be some sequence of natural numbers and $f(x)\in L_\phi(\boldsymbol{R})$ such that $D^{n_k}f(x)\in L_\phi(\boldsymbol{R})$, $k=1,2,\cdots$. Then there always exists the limit*

$$d_f=\lim_{k\to\infty}\|D^{n_k}f\|_\phi^{1/n_k}\,,$$

and moreover

$$d_f=\sigma_f=\sup\{|\xi|:\xi\in\operatorname{supp}\tilde{f}(\xi)\}\,.$$

To prove this theorem we need the following results:

LEMMA 1. *Let $g(x)\in L_\phi(\boldsymbol{R})$. Then $g(x)\in L_{1,loc}(\boldsymbol{R})$.*

PROOF. Let $\varepsilon>0$ and $\gamma>0$ be some number such that

$$\phi(\gamma/(\|g\|_\phi+\varepsilon))>0\,.$$

Then it follows from $\phi(at)\ge a\phi(t)$, $a\ge1$, $t\in[0,\infty)$ that

$$\phi(\gamma/(\|g\|_\phi+\varepsilon))\int_{|g(x)|\ge\gamma}|g(x)|/\gamma dx\le\int_{-\infty}^{\infty}\phi(|g(x)|/(\|g\|_\phi+\varepsilon))dx\le1\,.$$

Hence $g(x)\in L_{1,loc}(\boldsymbol{R})$. (q.e.d.)

Let $\sigma>0$. Denote by E_σ the set of all entire functions of exponential type σ and

by $M_{\sigma,\phi}$ the space of all functions from E_σ which as functions of $x \in \boldsymbol{R}$ belong to $L_\phi(\boldsymbol{R})$.
We have the following result [9, p. 191]:

LEMMA 2. *Let* $f(z) \in E_\sigma$ *and*

$$\sup_{-\infty < s < \infty} \left\{ \int_s^{s+2\pi} |f(x)|^p dx \right\}^{1/p} \leq A < \infty$$

with some $p \geq 1$. *Then for each* $x \in \boldsymbol{R}$

$$|f(x)| \leq (2\pi)^{1/q} A (1 + \sigma^{1/p}),$$

where $p^{-1} + q^{-1} = 1$.

LEMMA 3. *Let* $f(x) \in M_{\sigma,\phi}$. *Then* $f(x)$ *is bounded on* $\boldsymbol{R}$.

PROOF. Without loss of generality we may assume that

$$\int_{-\infty}^{\infty} \phi(|f(x)|) dx \leq 1 . \tag{1}$$

Then using Jensen's inequality we get for each $s \in \boldsymbol{R}$

$$\phi\left(\frac{1}{2\pi} \int_s^{s+2\pi} |f(x)| dx\right) \leq \frac{1}{2\pi} \int_s^{s+2\pi} \phi(|f(x)|) dx$$

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(|f(x)|) dx \leq \frac{1}{2\pi} .$$

Therefore, there exists a number $A < \infty$ such that

$$\sup_{-\infty < s < \infty} \int_s^{s+2\pi} |f(x)| dx \leq A$$

because of $\lim_{t \to \infty} \phi(t) = \infty$. Therefore, it follows from Lemma 2 that $f(x)$ is bounded.
(q.e.d.)

REMARK 1. We can prove that $\lim_{|x| \to \infty} f(x) = 0$ if $f(x) \in M_{\sigma,\phi}$ and $\phi(t) > 0$, $t > 0$. Actually, without loss of generality we may suppose that (1) is satisfied. Further, assume the contrary that there exist a number $c > 0$ and a sequence $|x_n| \to \infty$ such that

$$|f(x_n)| \geq 2c , \qquad n = 1, 2, \cdots . \tag{2}$$

Taking account of

$$f(x) - f(x_n) = \int_{x_n}^{x} f'(t) dt , \qquad n = 1, 2, \cdots$$

and the Bernstein inequality [9, p. 183]

$$\|f'\|_\infty \le \sigma \|f\|_\infty ,$$

we get

$$|f(x)-f(x_n)| \le \sigma \|f\|_\infty |x-x_n| , \qquad n=1, 2, \cdots .$$

Put

$$r = c/\sigma \|f\|_\infty .$$

Then

(3) $$|f(x)| \ge c \qquad \text{for} \quad |x-x_n| \le r , \qquad n=1, 2, \cdots$$

because of (2) and

$$|f(x)-f(x_n)| \le \sigma \|f\|_\infty |x-x_n| \le c , \qquad n=1, 2, \cdots .$$

On the other hand, without loss of generality we may assume that

(4) $$x_{n+1}-x_n \ge r , \qquad n=1, 2, \cdots .$$

Combining (3) and (4), we get

$$1 \ge \int_{-\infty}^{\infty} \phi(|f(x)|)dx \ge \sum_{n=1}^{\infty} \int_{x_n}^{x_{n+1}} \phi(|f(x)|)dx \ge \sum_{n=1}^{\infty} r\phi(c) = \infty .$$

We thus arrive at a contradiction.

REMARK 2. In order that $\lim_{|x|\to\infty} f(x)=0$, the condition $\phi(t)>0$, $t>0$ is necessary because of $f(x)\equiv c \in M_{\sigma,\phi}$, $0 \le c < \infty$ in the contrary case.

We have the following Bernstein inequality for Luxemburg norm:

LEMMA 4. *Let $f(x) \in M_{\sigma,\phi}$. Then*

(5) $$\|D^n f\|_\phi \le \sigma^n \|f\|_\phi , \qquad n=1, 2, \cdots .$$

PROOF. Using Lemma 3, the following interpolation formula [9, p. 188]

$$f'(x) = \frac{\sigma}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^{k-1}}{(k-1/2)^2} f\left(x + \frac{\pi}{\sigma}(k-1/2)\right)$$

and

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k-1/2)^2} = \pi^2 ,$$

we immediately get (5). (q.e.d.)

REMARK 3. It is not difficult to show that Lemmas 1–4 and Remarks 1–2 still hold for n-dimensional case.

PROOF OF THEOREM 1. It follows from Lemma 1 and the Sobolev imbedding theorem that $f(x)\in C^\infty(\boldsymbol{R})$.

We shall begin by showing that

$$\limsup_{k\to\infty}\|D^{n_k}f\|_\phi^{1/n_k}\le\sigma_f\,. \tag{6}$$

It is enough to prove (6) for $\sigma_f<\infty$. Then using $f\in\mathscr{S}'$ (this follows from the proof of Lemma 1) and the well-known Paley-Wiener-Schwartz theorem, we obtain that f is an analytic function of exponential type σ_f. Therefore, by virtue of Lemma 4 we get (6).

Now we claim that

$$\liminf_{k\to\infty}\|D^{n_k}f\|_\phi^{1/n_k}\ge\sigma_f\,. \tag{7}$$

We divide the proof into two cases.

Case 1 ($\sigma_f<\infty$). Assume the contrary that (7) does not hold. Then there exist a number $0<\delta<\sigma_f$ and a subsequence $\{k_m\}$ (for simplicity of notation we assume that $k_m=m$, $m=1, 2, \cdots$) such that

$$\|D^{n_k}f\|_\phi^{1/n_k}\le\sigma_f-\delta\,,\qquad k=1, 2, \cdots. \tag{8}$$

Let $\varepsilon>0$ and

$$f_k(x)=k\int_0^{1/k}f(x+t)dt\,,\qquad k=1, 2, \cdots. \tag{9}$$

Then by Jensen's inequality and $f_k(x)\in C^\infty(\boldsymbol{R})$ we obtain

$$\begin{aligned}\phi\left(\frac{|D^nf_k(x)|}{\|D^nf\|_\phi+\varepsilon}\right)&\le k\int_0^{1/k}\phi\left(\frac{|D^nf(x+t)|}{\|D^nf\|_\phi+\varepsilon}\right)dt\\&\le k\int_{-\infty}^{\infty}\phi\left(\frac{|D^nf(x+t)|}{\|D^nf\|_\phi+\varepsilon}\right)dt=k\int_{-\infty}^{\infty}\phi\left(\frac{|D^nf(t)|}{\|D^nf\|_\phi+\varepsilon}\right)dt\le k\end{aligned}$$

for $k=1, 2, \cdots$ and $n=0, 1, \cdots$. Therefore, it follows from the left continuity of $\phi(t)$ that

$$\phi\left(\frac{|D^nf_k(x)|}{\|D^nf\|_\phi}\right)\le k,\qquad k=1, 2, \cdots;\quad n=0, 1, \cdots.$$

Therefore,

$$\phi\left(\frac{\|D^nf_k\|_\infty}{\|D^nf\|_\phi}\right)\le k\,,\qquad k=1, 2, \cdots;\quad n=0, 1, \cdots. \tag{10}$$

On the other hand, it is easy to check that $\sigma_f\le\liminf_{k\to\infty}\sigma_{f_k}$. Therefore, there exists a number m such that

$$\sigma_{f_m}\ge\sigma_f-\delta/4\,. \tag{11}$$

Further, it follows from Theorem A that

$$\lim_{n\to\infty} \|D^n f_m\|_\infty^{1/n} = \sigma_{f_m}.$$

Therefore, there exists a number k_0 such that

(12) $$\|D^{n_k} f_m\|_\infty^{1/n_k} \geq \sigma_{f_m} - \delta/4, \qquad k \geq k_0.$$

Combining (8), (10), (11) and (12), we get

$$m \geq \phi\left(\frac{\|D^{n_k} f_m\|_\infty}{\|D^{n_k} f\|_\phi}\right) \geq \phi\left(\left(\frac{\sigma_f - \delta/2}{\sigma_f - \delta}\right)^{n_k}\right), \qquad k \geq k_0.$$

This contradicts $\lim_{t\to\infty} \phi(t) = \infty$.

Case 2 ($\sigma_f = \infty$). Assume the contrary that (7) does not hold. Then there exist a number $C < \infty$ and a subsequence $\{k_m\}$ (for simplicity of notation we assume again that $k_m = m$, $m = 1, 2, \cdots$) such that

(13) $$\|D^{n_k} f\|_\phi^{1/n_k} \leq C, \qquad k = 1, 2, \cdots.$$

On the other hand, it is easy to check that $\lim_{m\to\infty} \sigma_{f_m} = \infty$. Therefore, there exist numbers m and k_0 such that

$$\|D^{n_k} f_m\|_\infty^{1/n_k} \geq C + 1, \qquad k \geq k_0.$$

Therefore, using (10) and (13) we get

$$m \geq \phi\left(\frac{\|D^{n_k} f_m\|_\infty}{\|D^{n_k} f\|_\phi}\right) \geq \phi\left(\left(\frac{C+1}{C}\right)^{n_k}\right), \qquad k \geq k_0.$$

This contradicts $\lim_{t\to\infty} \phi(t) = \infty$. The proof is complete.

For the periodic case, we can prove easily the following result:

THEOREM 2. *Suppose that $f(x) \in C^\infty(\mathbf{R})$ is an arbitrary 2π-periodic function and $\phi(t)$ is an arbitrary Young function. Then there exists the limit*

$$d_f = \lim_{n\to\infty} |||D^n f|||_\phi^{1/n},$$

and moreover

$$d_f = \sigma_f = \sup\{|k| : k \in \operatorname{supp} \tilde{f}(\xi)\},$$

where $|||\cdot|||_\phi$ is the $L_\phi(0, 2\pi)$-norm.

REMARK 4. Theorem 1 still holds for any (L, l^q)-amalgam cases.

REMARK 5. Theorem 1 gives us certain information about the support of the Fourier transform of a function when we know the behaviour of a subsequence of

L_ϕ-norm of its derivatives.

REMARK 6. Let $f(x) \in L_\phi(\boldsymbol{R})$ and $\sigma_f < \infty$. Then $D^n f(x) \in L_\phi(\boldsymbol{R})$, $n = 1, 2, \cdots$. Therefore, using tables of the Fourier transform, in many cases, we can find the limit d_f without any concrete calculation.

REMARK 7. We have obtained some results in this direction for n-dimensional case, but the picture is different. It will be published elsewhere.

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