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Finite Type Minimal 2-Spheres in a Complex Projective Space

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§1. Introduction.

Let M be a compact C^∞-Riemannian manifold, $C^\infty(M)$ the space of all smooth functions on M, and Δ the Laplacian on M. The Δ is a self-adjoint elliptic differential operator acting on $C^\infty(M)$, which has an infinite discrete sequence of eigenvalues:

$$\mathrm{Spec}(M)=\{0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_k<\cdots\uparrow\infty\}.$$

Let $V_k=V_k(M)$ be the eigenspace of Δ corresponding to the k-th eigenvalue λ_k. Then V_k is finite-dimensional. We define an inner product $(\ ,\ )$ on $C^\infty(M)$ by

$$(f,g)=\int_M fg\,dV,$$

where dV denotes the volume element on M. Then $\sum_{t=0}^{\infty} V_t$ is dense in $C^\infty(M)$ and the decomposition is orthogonal with respect to the inner product $(\ ,\ )$. Thus we have

$$C^\infty(M)=\sum_{t=0}^{\infty} V_t(M)\qquad (\text{in } L^2\text{-sense}).$$

Since M is compact, V_0 is the space of all constant functions which is 1-dimensional.

Let $\tilde{M}$ be a compact C^∞-Riemannian manifold, and assume that M is a submanifold of $\tilde{M}$ which is immersed by an isometric immersion φ. We have the decomposition

$$C^\infty(\tilde{M})=\sum_{s=0}^{\infty} V_s(\tilde{M})\qquad (\text{in } L^2\text{-sense})$$

with respect to the Laplacian $\Delta_{\tilde{M}}$ of $\tilde{M}$. We denote by φ^* the pull-back, i.e., φ^* is an $\boldsymbol{R}$-linear map of $C^\infty(\tilde{M})$ into $C^\infty(M)$ such that

$$(\varphi^*F)(p)=F(\varphi(p)),\qquad p\in M,\quad F\in C^\infty(\tilde{M}).$$

For each integer s, $\varphi^*V_s(\tilde{M})$ is a subspace of $C^\infty(M)$. Then we have a decomposition

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$$\varphi^* V_s(\tilde{M}) \subset \sum_{t=0}^{\infty} W_t\,, \qquad W_t = W_t(M, \tilde{M}, \varphi, s) \subset V_t(M)\,,$$

where each W_t is the minimal subspace of $V_t(M)$ such that $\sum_{t=0}^{\infty} W_t$ contains $\varphi^* V_s(\tilde{M})$.

We say that φ (or M) is *of finite-type with respect to* $V_s(\tilde{M})$, if $\#\{t \geq 1 \mid W_t \neq (0)\}$ is finite, and if it is not finite, we say that φ (or M) is *of infinite-type with respect to* $V_s(\tilde{M})$. If $\#\{t \geq 1 \mid W_t \neq (0)\}$ is equal to k, then we say that φ (or M) is *of k-type with respect to* $V_s(\tilde{M})$, and that φ (or M) is *of order* $\{t \geq 1 \mid W_t \neq (0)\}$ *with respect to* $V_s(\tilde{M})$. Furthermore, we say that φ (or M) is *of mass-symmetric with respect to* $V_s(\tilde{M})$ if $W_0 = (0)$.

In this paper, we consider the case where $\tilde{M}$ is an n-dimensional complex projective space $CP^n(4)$ of constant holomorphic sectional curvature 4, and $s = 1$. So we omit the terms "with respect to $V_1(CP^n(4))$" in conditions for immersions of M into $CP^n(4)$. These definitions are compatible with those by B. Y. Chen in [4].

A submanifold M of $CP^n(4)$ is said to be *full*, if M is not contained in any totally geodesic complex submanifold of $CP^n(4)$. In [6], A. Ros shows that a 1-type complex submanifold of $CP^n(4)$ is a totally geodesic Kähler submanifold, so that it is of order $\{1\}$. He also shows that an m-dimensional 1-type totally real minimal submanifold of $CP^n(4)$ is a totally real minimal submanifold of $CP^m(4)$ which is a totally geodesic Kähler submanifold of $CP^n(4)$. In [9, 11], S. Udagawa shows that a full Kähler submanifold $CP^n(4)$ is of 2-type if and only if it is Einstein, so that it is of order $\{1, 2\}$. He also studies compact Hermitian symmetric submanifolds of degree 3 in $CP^n(4)$. Here, for a Kähler submanifold M of $CP^n(4)$, we say that M is *of degree k* if the pure part of the $(k-2)$-nd covariant derivative of h is not zero and the pure part of the $(k-1)$-st covariant derivative of h is zero, where h is the second fundamental form. He shows that compact irreducible Hermitian symmetric submanifolds of degree 3 in $CP^n(4)$ are of order $\{1, 2, 3\}$. Moreover, we can see in [10] that there exists a compact Hermitian symmetric submanifold of degree 3 in $CP^n(4)$ which has different order, but it is reducible.

One of the most typical examples of irreducible submanifolds in $CP^n(4)$ is a 2-sphere. Let $S^2(c)$ be the 2-sphere of constant curvature $c > 0$. S. Bando and Y. Ohnita in [1] gave the family $\{\varphi_{n,k}\}$ of all full isometric minimal immersions of $S^2(c)$ into $CP^n(4)$, using irreducible unitary representations of $SU(2)$. Independently, in [2], J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward gave this family $\{\varphi_{n,k}\}$, using the method of harmonic sequence. They called this family the Veronese sequence.

The purpose of this paper is to give the type of minimal 2-spheres of constant curvature in $CP^n(4)$, and to characterize them in terms of the type.

We obtain the following main results.

THEOREM A. (1) *$\varphi_{n,k}$ is of at most n-type and mass-symmetric. For integers n, k, l with $n \geq 1$, $0 \leq k, l \leq n$, define*

$$q_l^k=\frac{1}{l!}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\prod_{j=1}^{l}(k+j-m)(n-k-j+m+1)\,.$$

Then the order of $\varphi_{n,k}$ *is* $\{l \mid 1\leq l\leq n,\ q_l^k\neq 0\}$.

(2) *A holomorphic imbedding* $\varphi_{n,0}$ *and its antipodal* $\varphi_{n,n}$ *are of n-type and of order* $\{1, 2, 3, \cdots, n\}$.

(3) *If n is even, then a totally real minimal immersion* $\varphi_{n,n/2}$ *is of n/2-type and of order* $\{2, 4, 6, \cdots, n\}$.

REMARK. Generic $\varphi_{n,k}$ is of n-type except for totally real $\varphi_{2k,k}$.

PROPOSITION B. *If a compact submanifold in* $CP^n(4)$ *is mass-symmetric, then it is fully immersed.*

THEOREM C. *Let* S^2 *be a k-type, mass-symmetric, minimal 2-sphere in* $CP^n(4)$. *Then n satisfies* $n\leq 2k$.

THEOREM D. *If a mass-symmetric, minimal 2-sphere* S^2 *in* $CP^n(4)$ *is of at most 2-type, then* S^2 *is of constant curvature, so that the immersion is congruent to either* $\varphi_{1,0}$, $\varphi_{1,1}$, $\varphi_{2,0}$, $\varphi_{2,1}$, $\varphi_{2,2}$ *or* $\varphi_{4,2}$.

Let M be a compact surface in $CP^n(4)$, and $z=x+iy$ an isothermal coordinate in M. We call the angle θ between $J\partial/\partial x$ and $\partial/\partial y$ the Kähler angle, where J is the complex structure of $CP^n(4)$. M is holomorphic (resp. anti-holomorphic) in $CP^n(4)$ if and only if θ is equal to 0 (resp. π). M is totally real in $CP^n(4)$ if and only if θ is equal to $\pi/2$.

THEOREM E. *Let* S^2 *be a mass-symmetric, minimal 2-sphere in* $CP^n(4)$. *If* S^2 *is of at most 3-type and with constant Kähler angle, then* S^2 *is of constant curvature, so that the immersion is congruent to either* $\varphi_{n,k}$ $(n=1, 2, 3,\ 0\leq k\leq n)$, $\varphi_{4,2}$ *or* $\varphi_{6,3}$.

REMARK. In [2], J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward show that, without the assumption of 3-type, Theorem E remains true if $n\leq 4$ and the immersion is neither holomorphic, antiholomorphic nor totally real.

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§2. Preliminaries.

Let M be a compact C^∞-Riemannian manifold, $C^\infty(M)$ the space of all smooth functions on M, and Δ the Laplacian on M. In a natural manner, Δ can act on $\boldsymbol{R}^N$-valued functions on M. We assume that M is a submanifold of an N-dimensional Euclidean space $\boldsymbol{R}^N$ with an isometric immersion F. Then an $\boldsymbol{R}^N$-valued function F has the decomposition

$$F=F_0+\sum_{k=1}^{\infty}F_k\,,\qquad \Delta F_k=\lambda_k F_k\,,$$

where F_0 is a constant map and λ_k is the k-th eigenvalue of Δ. Here, the center of mass of M in $\boldsymbol{R}^N$ is equal to F_0. We say that F (or M) is *of finite-type*, if $\#\{t\geq 1 \mid F_t\neq 0\}$ is finite, and if it is not finite, we say that F (or M) is *of infinite-type*. If $\#\{t\geq 1 \mid F_t\neq 0\}$ is equal to k, then we say that F (or M) is *of k-type*, and that F (or M) is *of order* $\{t\geq 1 \mid F_t\neq 0\}$. B. Y. Chen in [4] showed the following:

THEOREM 2.1. *Let $F: M\to \boldsymbol{R}^N$ be an isometric immersion of a compact Riemannian manifold M into $\boldsymbol{R}^N$. Then F is of finite-type if and only if there exists a polynomial $P(x)$ and some constant F_0 in $\boldsymbol{R}^N$ satisfying*

$$P(\Delta)(F-F_0)=0\,. \tag{2.1}$$

Moreover, F is of k-type if and only if there exists a polynomial $P(x)$ of degree k and some constant F_0 in $\boldsymbol{R}^N$ satisfying (2.1), *and any polynomial $P(x)$ of degree $<k$ and any constant F_0 in $\boldsymbol{R}^N$ do not satisfy* (2.1).

The natural Hermitian inner product in $\boldsymbol{C}^{n+1}$ is defined by

$$\langle v, w\rangle=\sum_{i=0}^{n}v_i\overline{w_i}\,,\qquad v={}^t(v_0,\cdots,v_n)\,,\quad w={}^t(w_0,\cdots,w_n)\,. \tag{2.2}$$

The unitary group $U(n+1)$ is the group of all linear transformations on $\boldsymbol{C}^{n+1}$ leaving the Hermitian inner product (2.2) invariant. An n-dimensional complex projective space $\boldsymbol{CP}^n$ is the orbit space of $\boldsymbol{C}^{n+1}-\{0\}$ under the action of the group $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$; $z\to \lambda z$ ($\lambda\in \boldsymbol{C}^*$). Let $\pi: \boldsymbol{C}^{n+1}-\{0\}\to \boldsymbol{CP}^n$ be the natural projection. Denote by $\mathscr{H}_z$ and $\mathscr{V}_z$, the horizontal and the vertical spaces of π at $z\in \boldsymbol{C}^{n+1}-\{0\}$, respectively, so that

$$T_z(\boldsymbol{C}^{n+1}-\{0\})=\mathscr{H}_z\oplus\mathscr{V}_z\,,$$

$$\mathscr{H}_z=\{v\in \boldsymbol{C}^{n+1} \mid \langle v, z\rangle=0\}\,,\qquad \mathscr{V}_z=\{\lambda z \mid \lambda\in \boldsymbol{C}\}\,.$$

Then $\pi_*: \mathscr{H}_z\to T_{\pi(z)}\boldsymbol{CP}^n$ is a linear isomorphism over $\boldsymbol{C}$. The Fubini-Study metric $\tilde{g}$ of constant holomorphic sectional curvature $\tilde{c}$ in $\boldsymbol{CP}^n$ is given by

$$\tilde{g}(\pi_*(v), \pi_*(w))=\frac{4}{\tilde{c}}\operatorname{Re}\frac{\langle v, w\rangle}{|z|^2}\,,\qquad z\in \boldsymbol{C}^{n+1}-\{0\}\,,\quad v, w\in\mathscr{H}_z\,,$$

where $|z|^2=\langle z, z\rangle$. $U(n+1)$ acts on $\boldsymbol{CP}^n$ as follows:

$$U\pi(z)=\pi(Uz)\,,\qquad U\in U(n+1)\,,\quad z\in \boldsymbol{C}^{n+1}-\{0\}\,,$$

so that this action leaves the metric $\tilde{g}$ invariant. We denote by $\boldsymbol{CP}^n(\tilde{c})$ an n-dimensional complex projective space equipped with the metric $\tilde{g}$.

Let $HM(n+1, C)$ be the set of all Hermitian $(n+1, n+1)$-matrices over C, which can be identified with R^N, $N=(n+1)^2$. For $X, Y \in HM(n+1, C)$, the natural inner product is given by

$$(2.3) \qquad (X, Y)=\frac{2}{\tilde{c}}\operatorname{Re}(\operatorname{tr} XY).$$

$U(n+1)$ acts on $HM(n+1, C)$ by $X \to UXU^*$, $U \in U(n+1)$, $X \in HM(n+1, C)$, where $U^*={}^t\bar{U}$, so that this action leaves the inner product (2.3) invariant. Define two linear subspaces of $HM(n+1, C)$ as follows:

$$HM_0=HM_0(n+1, C)=\{X \in HM(n+1, C) \mid \operatorname{tr} X=0\},$$

$$HM_R=HM_R(n+1, C)=\{aI \mid a \in R\},$$

where I is the $(n+1, n+1)$-identity matrix. Both of them are invariant under the action of $U(n+1)$, and irreducible. We get the orthogonal decomposition of $HM(n+1, C)$ as follows:

$$HM(n+1, C)=HM_0 \oplus HM_R.$$

It is well-known that HM_0 (resp. HM_R) is identified with the first eigenspace $V_1(CP^n(\tilde{c}))$ (resp. the set of all constant functions, i.e., $V_0(CP^n(\tilde{c}))$). The first standard imbedding Ψ of $CP^n(\tilde{c})$ is defined by

$$(2.4) \qquad \Psi(\pi(z))=\frac{1}{|z|^2}zz^* \in HM(n+1, C), \qquad z \in C^{n+1}-\{0\}.$$

Ψ is $U(n+1)$-equivariant and the image of $CP^n(\tilde{c})$ under Ψ is given as follows:

$$\Psi(CP^n(\tilde{c}))=\{A \in HM(n+1, C) \mid A^2=A, \operatorname{tr} A=1\},$$

so that it is contained fully in a hyperplane

$$HM_1=HM_1(n+1, C)=\{A \in HM(n+1, C) \mid \operatorname{tr} A=1\}$$
$$=\left\{A+\frac{1}{n+1}I \,\middle|\, A \in HM_0\right\}$$

of $HM(n+1, C)$. Denote by $S^{N-2}(\tilde{c}(n+1)/(2n))$ the hypersphere in $HM_1(n+1, C)$ centered at $(1/(n+1))I$ with radius $\sqrt{2n/(\tilde{c}(n+1))}$. Thus we obtain that Ψ is a minimal immersion of $CP^n(\tilde{c})$ into $S^{N-2}(\tilde{c}(n+1)/(2n))$, and that the center of mass of $CP^n(\tilde{c})$ is $(1/(n+1))I$. In fact, Ψ satisfies the equation $\Delta\Psi=\tilde{c}(n+1)(\Psi-(1/(n+1))I)$, so that Ψ is of order 1. Moreover, all coefficients of $\Psi-(1/(n+1))I$ span the first eigenspace $V_1(CP^n(\tilde{c}))$. For details, see [4].

From now on, we assume that M is a submanifold of $CP^n(\tilde{c})$ with an isometric immersion φ. Then $F=\Psi \circ \varphi$ is an isometric immersion of M into $HM(n+1, C)$, and the set of all coefficients of $F-(1/(n+1))I$ spans the pull-back $\varphi^* V_1(CP^n(\tilde{c}))$. Therefore, the conditions "of finite-type", "of infinite-type", "of k-type" and "mass-symmetric" for φ defined in § 1 are compatible with those for F, and so is "order", so that we obtain the following proposition:

PROPOSITION 2.2. *Let $\varphi: M \to CP^n(\tilde{c})$ be an isometric immersion of a compact Riemannian manifold M into $CP^n(\tilde{c})$. Then φ is mass-symmetric and of finite-type if and only if there exists a polynomial $P(x)$ satisfying*

$$P(\Delta)\left(F-\frac{1}{n+1}I\right)=0, \tag{2.5}$$

where $F=\Psi\circ\varphi$. Moreover, φ is mass-symmetric and of k-type if and only if there exists a polynomial $P(x)$ of degree k satisfying (2.5), *and any polynomial $P(x)$ of degree $<k$ do not satisfy* (2.5).

REMARK. φ is mass-symmetric if and only if the center of mass of M in $HM(n+1, C)$ is equal to that of $CP^n(\tilde{c})$.

Now we prove Proposition B. Let M be a compact Riemannian submanifold of $CP^n(\tilde{c})$, which is fully contained in a totally geodesic complex submanifold $CP^m(\tilde{c})$ of $CP^n(\tilde{c})$. We can assume that

$$\Psi(CP^m(\tilde{c}))=\left\{\begin{pmatrix} A' & 0 \\ 0 & 0\end{pmatrix} \,\middle|\, A'\in HM(m+1, C),\ A'^2=A',\ \mathrm{tr}A'=1\right\}.$$

Let $\varphi: M\to CP^n(\tilde{c})$ be an isometric immersion, and for $x\in M$, set $\Psi\circ\varphi(x)=A(x)=\begin{pmatrix} A'(x) & 0 \\ 0 & 0\end{pmatrix}$. Then the center of mass of M is given by

$$\frac{1}{\mathrm{vol}(M)}\int_{x\in M} A(x)dv_M=\frac{1}{\mathrm{vol}(M)}\begin{pmatrix} \int_{x\in M} A'(x)dv_M & 0 \\ 0 & 0\end{pmatrix}.$$

If M is mass-symmetric in $CP^n(\tilde{c})$, then this is equal to $(1/(n+1))I$. Therefore, we get $m=n$ so that M is full in $CP^n(\tilde{c})$.

§ 3. Minimal 2-spheres with constant curvature in $CP^n(\tilde{c})$.

The purpose of this section is to prove Theorem A. First, we review S. Bando and Y. Ohnita's results for minimal 2-spheres of constant curvature.

$SU(2)$ is defined by

$$SU(2)=\left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix} \,\middle|\, a, b\in C,\ |a|^2+|b|^2=1\right\}.$$

The Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ is given by

$$\mathfrak{su}(2)=\left\{\begin{pmatrix}\sqrt{-1}x & y\\ -\bar{y} & -\sqrt{-1}x\end{pmatrix}\,\middle|\, x, y', y''\in \boldsymbol{R},\ y=y'+\sqrt{-1}y''\right\}.$$

Define a basis $\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$ of $\mathfrak{su}(2)$ by

$$\varepsilon_0=\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix},\qquad \varepsilon_1=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\qquad \varepsilon_2=\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}.$$

Then these satisfy

$$[\varepsilon_0, \varepsilon_1]=2\varepsilon_2\,,\qquad [\varepsilon_1, \varepsilon_2]=2\varepsilon_0\,,\qquad [\varepsilon_2, \varepsilon_0]=2\varepsilon_1\,.$$

Let V_n be an $(n+1)$-dimensional complex vector space of all complex homogeneous polynomials of degree n with respect to z_0, z_1. We define a Hermitian inner product $\langle\,,\rangle$ of V_n in such a way that

$$\{u_k^{(n)}=z_0^k z_1^{n-k}/\sqrt{k!(n-k)!}\ |\ 0\le k\le n\}$$

is a unitary basis for V_n. We define a real inner product by $(\,,)=\mathrm{Re}\langle\,,\rangle$. A unitary representation ρ_n of $SU(2)$ on V_n is defined by

$$\rho_n(g)f(z_0, z_1)=f((z_0, z_1)g)=f(az_0-\bar{b}z_1, bz_0+\bar{a}z_1)$$

for $g\in SU(2)$ and $f\in V_n$. We also denote by ρ_n the action of $\mathfrak{su}(2)$ on V_n, so that

(3.1)
$$\rho_n(X)(u_k^{(n)})=(k-(n-k))\sqrt{-1}xu_k^{(n)} -\sqrt{k(n-k+1)}\bar{y}u_{k-1}^{(n)}+\sqrt{(k+1)(n-k)}yu_{k+1}^{(n)}\,,$$

for $0\le k\le n$ and $X\in\mathfrak{su}(2)$. It is well-known that $\{(\rho_n, V_n)\,|\, n=0, 1, 2, \cdots\}$ is the set of all inequivalent irreducible unitary representations of $SU(2)$.

Put $T=\{\exp(t\varepsilon_0)\in\mathfrak{su}(2)\,|\, t\in\boldsymbol{R}\}$ and we have $S^2=\boldsymbol{CP}^1=SU(2)/T$. We identify the tangent space of S^2 at $o=\{T\}\in S^2=SU(2)/T$ with a subspace $\mathfrak{m}=\mathrm{span}\{\varepsilon_1, \varepsilon_2\}$ of $\mathfrak{su}(2)$. We fix a complex structure on S^2 so that $\varepsilon_1-\sqrt{-1}\varepsilon_2$ is a vector of type (1, 0). Let g_c be an $SU(2)$-invariant Riemannian metric on S^2 defined by

$$g_c(X, Y)=-\frac{2}{c}\,\mathrm{tr}\,XY$$

for X and $Y\in\mathfrak{m}$ and c is a positive constant. It is the restriction of $SU(2)$-invariant inner product on $\mathfrak{su}(2)$. Clearly, $\{(\sqrt{c}/2)\varepsilon_1, (\sqrt{c}/2)\varepsilon_2\}$ forms an orthonormal basis of $\mathfrak{m}\cong T_oS^2$ and (S^2, g_c) has the constant curvature c, so that we denote this by $S^2(c)$. The spectrum of the Laplacian Δ of $S^2(c)$ is given by $\mathrm{Spec}(S^2(c))=\{\lambda_l=cl(l+1)\,|\, l\ge 0\}$.

Put $S^{2n+1}=\{v\in V_n\,|\,\langle v, v\rangle=4/\tilde{c}\}$ where $\tilde{c}$ is a positive constant. Let $\pi: S^{2n+1}\to \boldsymbol{CP}^n(\tilde{c})$ be the Hopf fibration, so that the action of $\rho_n(SU(2))$ on S^{2n+1} induces the action on $\boldsymbol{CP}^n(\tilde{c})$ through π. Thus, for any non-negative integers n and k with $0\le k\le n$, denote by $\varphi_{n,k}$ the $SU(2)$-equivariant mapping of a Riemann sphere $S^2(c)$ into $CP^n(\tilde{c})$ defined by

$$\varphi_{n,k} : S^2(c) = SU(2)/T \ni gT \mapsto \pi\left(\rho_n(g)\frac{2}{\sqrt{\tilde{c}}}u_k^{(n)}\right) \in CP^n(\tilde{c}). \tag{3.2}$$

Bando and Ohnita in [1] show the following:

THEOREM 3.1. (1) *$\varphi_{n,k}$ is a full isometric immersion.*

(2) *c is equal to $\tilde{c}/(2k(n-k)+n)$.*

(3) *$\varphi_{n,k}$ is a minimal immersion.*

(4) (a) *If $k=0$ (resp. $k=n$), then $\varphi_{n,k}$ is holomorphic (resp. anti-holomorphic).*

(b) *If n is even and $k=n/2$, then $\varphi_{2k,k}$ is totally real and $\varphi_{2k,k}(S^2(c))$ is contained in a totally geodesic totally real submanifold $RP^{2k}(\tilde{c}/4)$ of $CP^{2k}(\tilde{c})$.*

(c) *Otherwise, $\varphi_{n,k}$ is neither holomorphic, anti-holomorphic nor totally real.*

(5) *$\varphi_{n,k}(S^2(c)) = \varphi_{n,n-k}(S^2(c))$.*

Moreover, they show the following rigidity theorem.

THEOREM 3.2. *Let $\varphi : S^2(c) \to CP^n(\tilde{c})$ be a full isometric minimal immersion. Then there exists an integer k with $0 \le k \le n$ such that $c = \tilde{c}/(2k(n-k)+n)$ and φ is congruent to $\varphi_{n,k}$ up to a holomorphic isometry of $CP^n(\tilde{c})$.*

We identify V_n with C^{n+1} such that $\{u_0^{(n)}, u_1^{(n)}, \cdots, u_n^{(n)}\}$ is the canonical basis of C^{n+1}, so that we can regard $\rho_n(g)$, $g \in SU(2)$, as an element of $U(n+1)$.

Put $\tilde{V} = HM(n+1, C)$. Let $\tilde{\rho} : SU(2) \to GL(\tilde{V})$ be a real representation defined by $\tilde{\rho}(g)X = \rho_n(g)X\rho_n(g)^*$ for $g \in SU(2)$ and $X \in \tilde{V}$. Let $(\tilde{\rho}, \tilde{V}^C)$ be the complexification of $(\tilde{\rho}, \tilde{V})$. It is easy to see that $(\tilde{\rho}, \tilde{V}^C) = (\tilde{\rho}, \mathfrak{gl}(n+1, C))$ is $SU(2)$-equivalent to $(\rho_n \otimes \rho_n, V_n \otimes V_n)$, since the dual representation (ρ_n^*, V_n^*) of (ρ_n, V_n) is $SU(2)$-equivalent to (ρ_n, V_n). By Clebsch-Gordan's theorem, we have the following decomposition $\tilde{V}^C = \tilde{V}_0 \oplus \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_n$, where $(\tilde{\rho}, \tilde{V}_l)$ is $SU(2)$-equivalent to (ρ_{2l}, V_{2l}) for each l with $0 \le l \le n$. Set $W_l = \tilde{V} \cap \tilde{V}_l$. Then each $(\tilde{\rho}, W_l)$ is an irreducible real representation, and $\tilde{V}$ is decomposed into $\tilde{V} = W_0 \oplus W_1 \oplus \cdots \oplus W_n$. Let $C_{\tilde{\rho}}$ be the Casimir operator of $\tilde{\rho}$, which is a real operator on $\tilde{V}^C$ defined by $C_{\tilde{\rho}} = \sum_{i=0}^{2} \tilde{\rho}((\sqrt{c}/2)\varepsilon_i)^2$. Then each W_l is characterized by the eigenspace of $C_{\tilde{\rho}}$ in $\tilde{V}$ with the eigenvalue $-cl(l+1)$.

Let $\tilde{V}_T$ be the set of all $\tilde{\rho}(T)$-invariant elements of $\tilde{V}$, i.e., $\tilde{V}_T = \{v \in \tilde{V} \mid \tilde{\rho}(t)v = v$ for any $t \in T\}$. For integers i and j with $0 \le i, j \le n$, let E_{ij} be the matrix in $\tilde{V}^C$ whose $(i+1, j+1)$-coefficient is 1 and others are zero, so that E_{ij} is equal to $u_i^{(n)}(u_j^{(n)})^*$ and $\tilde{V}$ is spanned by $\{E_{ii}, (1/2)(E_{ij}+E_{ji}), (\sqrt{-1}/2)(E_{ij}-E_{ji}) \mid 0 \le i < j \le n\}$ over R. By the definition, for $t = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in T$, we get $\rho_n(t)u_k^{(n)} = e^{i(2k-n)\theta}u_k^{(n)}$. Therefore, we obtain $\tilde{V}_T$ is spanned by $\{E_{ii} \mid 0 \le i \le n\}$ over R, i.e., $\tilde{V}_T$ is the set of all diagonal matrices in $\tilde{V}$. Since (ρ_{2l}, V_{2l}) is a spherical representation, $\tilde{V}_T \cap W_l$ is 1-dimensional, so that there exists an element Q_l such that $\tilde{V}_T \cap W_l = \boldsymbol{R}\{Q_l\}$. Since $C_{\tilde{\rho}}$ and $\tilde{\rho}(SU(2))$ are commutable, $\tilde{V}_T$ is invariant under $C_{\tilde{\rho}}$. Therefore, each Q_l is characterized by an eigenvector of $C_{\tilde{\rho}}$ in $\tilde{V}_T$ with the eigenvalue $-cl(l+1)$.

For $v \in \tilde{V}_T$, f_v denotes a $\tilde{V}$-valued function on S^2 defined by $f_v(go) = \tilde{\rho}(g)v$ for $g \in SU(2)$. Then the action of Δ for f_v is give by $\Delta f_v = f_{-C_{\tilde{\rho}}v}$. Thus, v has the decomposition $v = \sum_{l=0}^n v_l$, $v_l \in W_l \cap \tilde{V}_T$ if and only if f_v is the sum of the λ_l-eigenfunctions f_{v_l}. Now, we define the order of f_v (or $v \in \tilde{V}_T$) by $\mathrm{Ord}(f_v) = \mathrm{Ord}(v) = \{l \mid 1 \leq l \leq n,\ v_l \neq 0\}$.

For integers n and k with $0 \leq k \leq n$, we set $F_{n,k} = \Psi \circ \varphi_{n,k}$. By the definition of $\varphi_{n,k}$, we get $F_{n,k} = f_{E_{kk}}$. Since $E_{kk} \in \tilde{V}_T$, we have $\mathrm{Ord}(F_{n,k}) = \{l \mid 1 \leq l \leq n,\ (E_{kk}, Q_l) \neq 0\}$, so that $\varphi_{n,k}$ is at most n-type. Put $Q_l = \sum_{k=0}^n q_l^k E_{kk}$, $q_l^k \in \boldsymbol{R}$. Then the order of $\varphi_{n,k}$ is given by $\{l \mid 1 \leq l \leq n,\ q_l^k \neq 0\}$.

We can easily see that the identity matrix I in $\tilde{V}$ is a 0-eigenvector of $C_{\tilde{\rho}}$, and so we put $Q_0 = I$. Since the W_0-part of E_{kk} is equal to $(1/(n+1))I$, the constant term of $F_{n,k} - (1/(n+1))I = f_{E_{kk} - (1/(n+1))I}$ vanishes. Therefore, $\varphi_{n,k}$ is always mass-symmetric.

To prove Theorem A (1), we shall give q_l^k explicitly. First, we restrict $C_{\tilde{\rho}}$ to $\tilde{V}_T$.

LEMMA 3.3. *For $A = \sum_{l=0}^n a_l E_{ll}$ and $B = \sum_{l=0}^n b_l E_{ll} \in \tilde{V}_T$, $B = C_{\tilde{\rho}} A$ if and only if*

$$b_l = -c\{(2l(n-l)+n)a_l - l(n-l+1)a_{l-1} - (l+1)(n-l)a_{l+1}\}$$

for $0 \leq l \leq n$.

PROOF. By (3.1) we get

$$\rho_n(\varepsilon_1)u_l^{(n)} = -\sqrt{l(n-l+1)}\,u_{l-1}^{(n)} + \sqrt{(l+1)(n-l)}\,u_{l+1}^{(n)},$$

$$\rho_n(\varepsilon_2)u_l^{(n)} = \sqrt{l(n-l+1)}\sqrt{-1}\,u_{l-1}^{(n)} + \sqrt{(l+1)(n-l)}\sqrt{-1}\,u_{l+1}^{(n)},$$

so that

$$\begin{aligned}\rho_n(\varepsilon_1)^2 u_l^{(n)} = &-(2l(n-l)+n)u_l^{(n)} + \sqrt{l(l-1)(n-l+1)(n-l+2)}\,u_{l-2}^{(n)}\\ &+\sqrt{(l+1)(l+2)(n-l)(n-l-1)}\,u_{l+2}^{(n)},\\ \rho_n(\varepsilon_2)^2 u_l^{(n)} = &-(2l(n-l)+n)u_l^{(n)} - \sqrt{l(l-1)(n-l+1)(n-l+2)}\,u_{l-2}^{(n)}\\ &-\sqrt{(l+1)(l+2)(n-l)(n-l-1)}\,u_{l+2}^{(n)}.\end{aligned}$$

Thus simple computation gives

$$\sum_{i=1}^{2} \rho_n(\varepsilon_i)^2 u_l^{(n)} = -2(2l(n-l)+n)u_l^{(n)},$$

$$u_l^{(n)*} \sum_{i=1}^{2} \rho_n(\varepsilon_i)^2 = \left(\sum_{i=1}^{2} \rho_n(\varepsilon_i)^2 u_l^{(n)}\right)^* = -2(2l(n-l)+n)u_l^{(n)*}.$$

Since $E_{ll} = u_l^{(n)} u_l^{(n)*}$, we get

$$C_{\tilde{\rho}}A=\frac{c}{4}\sum_{i=1}^{2}\tilde{\rho}(\varepsilon_i)^2\sum_{l=0}^{n}a_l u_l^{(n)}u_l^{(n)*}$$

$$=\frac{c}{4}\sum_{l=0}^{n}a_l\left\{\left(\sum_{i=1}^{2}\rho_n(\varepsilon_i)^2u_l^{(n)}\right)u_l^{(n)*}+2\sum_{i=1}^{2}(\rho_n(\varepsilon_i)u_l^{(n)})(\rho_n(\varepsilon_i)u_l^{(n)})^*+u_l^{(n)}\left(\sum_{i=1}^{2}\rho_n(\varepsilon_i)^2u_l^{(n)}\right)^*\right\}$$

$$=c\sum_{l=0}^{n}a_l\{-(2l(n-l)+n)u_l^{(n)}u_l^{(n)*}+l(n-l+1)u_{l-1}^{(n)}u_{l-1}^{(n)*}+(l+1)(n-l)u_{l+1}^{(n)}u_{l+1}^{(n)*}\}\,.$$

This implies Lemma 3.3 immediately. □

We identify $\tilde{V}_T$ with $\boldsymbol{R}^{n+1}$ such that $\{E_{00}, E_{11}, \cdots, E_{nn}\}$ is the canonical basis of $\boldsymbol{R}^{n+1}$. Define an $(n+1, n+1)$-matrix $R=(r_{ij})_{0\le i,j\le n}$ by

$$r_{ij}=\begin{cases}-i(n-i+1)\,, & \text{if } j=i-1\,,\\ 2i(n-i)+n\,, & \text{if } j=i\,,\\ -(i+1)(n-i)\,, & \text{if } j=i+1\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$

and put $q_l={}^t(q_l^0, q_l^1, \cdots, q_l^n)$. Then, from Lemma 3.3, q_l and R are corresponding to Q_l and $-(1/c)C_{\tilde{\rho}}$, respectively. Therefore, each q_l is characterized by an eigenvector of R with the eigenvalue $l(l+1)$. Notice that $q_0={}^t(1, 1, \cdots, 1)$ is a 0-eigenvector of R.

In order to prove Theorem A (1), it is sufficient to show the following lemma.

LEMMA 3.4. *Let $q_l={}^t(q_l^0, q_l^1, \cdots, q_l^n)$, $1\le l\le n$, be a vector in $\boldsymbol{R}^{n+1}$ defined by*

$$q_l^k=\frac{1}{l!}\sum_{m=0}^{l}(-1)^m\binom{l}{m}\prod_{j=1}^{l}(k+j-m)(n-k-j+m+1)\,. \tag{3.3}$$

Then for each l with $0\le l\le n$, q_l is an eigenvector of R with an eigenvalue $l(l+1)$.

To prove this lemma, we need some lemmas. Put $r_j=j(n-j+1)$, so that

$$R=\begin{pmatrix} r_0+r_1 & -r_1 & & & 0\\ -r_1 & r_1+r_2 & -r_2 & & \\ & -r_2 & \ddots & \ddots & \\ & & \ddots & r_{n-1}+r_n & -r_n\\ 0 & & & -r_n & r_n+r_{n+1}\end{pmatrix}.$$

It is easy to see that

$$r_{k+l}+r_{k-l}-2r_k=-2l^2\,, \qquad \text{for all } k, l\,. \tag{3.4}$$

In particular, we have

(3.5) $$r_{k+1}+r_{k-1}-2r_k=-2\,, \qquad \text{for all } k\,.$$

LEMMA 3.5. (1) *For any integers k, l and p with* $0\leq p\leq l$, *we have*

(3.6) $$r_{k+l-p}=-(l-p)(l-p+1)+(l-p+1)r_k-(l-p)r_{k-1}\,.$$

For any k and p with $p\geq 1$, *we have*

(3.7) $$pr_k-(p+1)r_{k-1}=-p(p+1)-r_{k-p-1}\,.$$

(2) *For any integers k, l and p with* $0\leq p\leq l$, *we have*

(3.8) $$r_{k-l+p}=-(l-p)(l-p+1)+(l-p+1)r_k-(l-p)r_{k+1}\,.$$

For any k and p with $p\geq 1$, *we have*

(3.9) $$pr_k-(p+1)r_{k+1}=-p(p+1)-r_{k+p+1}\,.$$

PROOF. We shall prove (1). We get

$$\begin{aligned} r_{k+l-p}=&(r_{k+l-p}-2r_{k+l-p-1}+r_{k+l-p-2})\\ &+2(r_{k+l-p-1}-2r_{k+l-p-2}+r_{k+l-p-3})\\ &+3(r_{k+l-p-2}-2r_{k+l-p-3}+r_{k+l-p-4})+\cdots\\ &+(l-p-1)(r_{k+2}-2r_{k+1}+r_k)\\ &+(l-p)(r_{k+1}-2r_k+r_{k-1})\\ &+(l-p+1)r_k-(l-p)r_{k-1}\,,\end{aligned}$$

which, together with (3.5), implies

$$\begin{aligned} r_{k+l-p}&=-2(1+2+\cdots+(l-p))+(l-p+1)r_k-(l-p)r_{k-1}\\ &=-(l-p)(l-p+1)+(l-p+1)r_k-(l-p)r_{k-1}\,.\end{aligned}$$

Next, we show (3.7). Similarly, we get

$$\begin{aligned} pr_k-(p+1)r_{k-1}=&p(r_k-2r_{k-1}+r_{k-2})\\ &+(p-1)(r_{k-1}-2r_{k-2}+r_{k-3})\\ &+(p-2)(r_{k-2}-2r_{k-3}+r_{k-4})+\cdots\\ &+2(r_{k-p+2}-2r_{k-p+1}+r_{k-p})\\ &+(r_{k-p+1}-2r_{k-p}+r_{k-p-1})\\ &-r_{k-p-1}\,,\end{aligned}$$

which, together with (3.5), implies

$$\begin{aligned} pr_k-(p+1)r_{k-1}&=-2(p+(p-1)+\cdots+2+1)-r_{k-p-1}\\ &=-p(p+1)-r_{k-p-1}\,.\end{aligned}$$

(2) is proved similarly. □

LEMMA 3.6. (1) *For each* $p=0, 1, \cdots, [(l-2)/2]$, *we have*

$$-l(l+1)\sum_{m\in I_p}(-1)^m\binom{l-1}{m}\prod_{\substack{1\le j\le l\\ j\ne m+1}} r_{k+j-1-m} \tag{3.10}$$

$$=\sum_{m\in J_p}(-1)^m\binom{l+1}{m}\prod_{1\le j\le l} r_{k+j-m}$$

$$+(-1)^{p+2}\binom{l}{p+1} r_{k+l-p-1}\cdots r_{k+1}r_{k-1}\cdots r_{k-p-1}$$

$$+(-1)^{l-p-1}\binom{l}{l-p-1} r_{k+p+1}\cdots r_{k+1}r_{k-1}\cdots r_{k-l+p+1},$$

where $I_p=\{0, 1, \cdots, p, l-p-1, \cdots, l-1\}$ *and* $J_p=\{0, 1, \cdots, p+1, l-p, \cdots, l+1\}$.

(2) *We have*

$$-l(l+1)\sum_{m=0}^{l-1}(-1)^m\binom{l-1}{m}\prod_{\substack{1\le j\le l\\ j\ne m+1}} r_{k+j-1-m} \tag{3.11}$$

$$=\sum_{m=0}^{l+1}(-1)^m\binom{l+1}{m}\prod_{1\le j\le l} r_{k+j-m}.$$

PROOF. (1) We shall prove (3.10) by induction on p. Assume $p=0$. By (3.6), we get

$$r_{k+l}=-l(l+1)+(l+1)r_k-lr_{k-1},$$

which implies

$$\begin{aligned} -l(l+1)r_{k+l-1}\cdots r_{k+1}&=r_{k+l}\cdots r_{k+1}\\ &\quad -(l+1)r_{k+l-1}\cdots r_k+lr_{k+l-1}\cdots r_{k+1}r_{k-1}. \end{aligned} \tag{3.12}$$

Similarly, from (3.8), we get

$$\begin{aligned} -l(l+1)r_{k-1}\cdots r_{k-l+1}&=r_{k-1}\cdots r_{k-l}\\ &\quad -(l+1)r_k\cdots r_{k-l+1}+lr_{k+1}r_{k-1}\cdots r_{k-l+1}. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we obtain (3.10).

We assume $p>0$. From (3.6) and (3.7), we have

$$(3.14)\qquad \binom{l}{p}r_{k+l-p}-\binom{l+1}{p+1}r_k$$
$$=\binom{l}{p+1}pr_k-\binom{l}{p+1}(p+1)r_{k-1}-\binom{l}{p}(l-p)(l-p+1)$$
$$=-l(l+1)\binom{l-1}{p}-\binom{l}{p+1}r_{k-p-1}\,.$$

Similarly, from (3.8) and (3.9), we have

$$(3.15)\qquad \binom{l}{l-p}r_{k-l+p}-\binom{l+1}{l-p}r_k$$
$$=-l(l+1)\binom{l-1}{l-p-1}-\binom{l}{l-p-1}r_{k+p+1}\,.$$

By the assumption of induction, we obtain

$$-l(l+1)\sum_{m\in I_p}(-1)^m\binom{l-1}{m}\prod_{\substack{1\le j\le l\\ j\ne m+1}}r_{k+j-1-m}-\sum_{m\in J_p}(-1)^m\binom{l+1}{m}\prod_{1\le j\le l}r_{k+j-m}$$
$$-(-1)^{p+2}\binom{l}{p+1}r_{k+l-p-1}\cdots r_{k+1}r_{k-1}\cdots r_{k-p-1}$$
$$-(-1)^{l-p-1}\binom{l}{l-p-1}r_{k+p+1}\cdots r_{k+1}r_{k-1}\cdots r_{k-l+p+1}\,,$$
$$=(-1)^{p+1}r_{k+l-p-1}\cdots r_{k+1}r_{k-1}\cdots r_{k-p}$$
$$\times\left\{\binom{l}{p}r_{k+l-p}-\binom{l+1}{p+1}r_k+\binom{l}{p+1}r_{k-p-1}+l(l+1)\binom{l-1}{p}\right\}$$
$$+(-1)^{l-p}r_{k+p}\cdots r_{k+1}r_{k-1}\cdots r_{k-l+p+1}$$
$$\times\left\{\binom{l}{l-p}r_{k-l+p}-\binom{l+1}{l-p}r_k+\binom{l}{l-p-1}r_{k+p+1}+l(l+1)\binom{l-1}{l-p-1}\right\}.$$

Combining (3.14) and (3.15), we obtain (3.10).

(2) Put $p=[(l-2)/2]$. If l is even, we get $p=l/2-1$. Then we obtain (3.11) from (3.10) immediately. Therefore, we assume that l is odd. In this case, we get $p=(l-3)/2$ (or $l=2p+3$), so that $I_p\cup\{p+1\}=\{0, 1, \cdots, l-1\}$ and $J_p\cup\{p+2\}=\{0, 1, \cdots, l+1\}$. From (3.10), we have

$$-l(l+1)\sum_{m=0}^{l-1}(-1)^m\binom{l-1}{m}\prod_{\substack{1\le j\le l\\ j\ne m+1}} r_{k+j-1-m}-\sum_{m=0}^{l+1}(-1)^m\binom{l+1}{m}\prod_{1\le j\le l} r_{k+j-m}$$

$$=-l(l+1)(-1)^{p+1}\binom{l-1}{p+1}\prod_{\substack{1\le j\le l\\ j\ne p+2}} r_{k+j-p-2}$$

$$-(-1)^{p+2}\binom{l+1}{p+2}\prod_{1\le j\le l} r_{k+j-p-2}$$

$$+(-1)^{p+2}\binom{l}{p+1}r_{k+l-p-1}\cdots r_{k+1}r_{k-1}\cdots r_{k-p-1}$$

$$+(-1)^{l-p-1}\binom{l}{l-p-1}r_{k+p+1}\cdots r_{k+1}r_{k-1}\cdots r_{k-l+p+1}$$

$$=(-1)^{p+2}\binom{2p+3}{p+1}\prod_{\substack{1\le j\le l\\ j\ne p+2}} r_{k+j-p-2}(2(p+2)^2-2r_k+r_{k+p+2}+r_{k-p-2}),$$

which, combined with (3.4), implies (3.11). □

PROOF OF LEMMA 3.4. For any n, k and l with $0\le k, l\le n$, we get by simple computation,

$$l!q_l^k=\sum_{m=0}^{l}(-1)^m\binom{l}{m}\prod_{j=1}^{l} r_{k+j-m}$$

$$=\sum_{m=0}^{l-1}(-1)^m\binom{l-1}{m}\prod_{j=1}^{l} r_{k+j-m}+\sum_{m=1}^{l}(-1)^m\binom{l-1}{m-1}\prod_{j=1}^{l} r_{k+j-m}$$

$$=\left(\sum_{m=0}^{l-1}(-1)^m\binom{l-1}{m}\prod_{\substack{1\le j\le l\\ j\ne m+1}} r_{(k+1)+j-1-m}\right)r_{k+1}$$

$$-\left(\sum_{m=0}^{l-1}(-1)^m\binom{l-1}{m}\prod_{\substack{1\le j\le l\\ j\ne m+1}} r_{k+j-1-m}\right)r_k.$$

On the other hand, direct computation gives

$$l!(q_l^k-q_l^{k-1})=\sum_{m=0}^{l+1}(-1)^m\binom{l+1}{m}\prod_{1\le j\le l} r_{k+j-m},$$

which, combined with (3.11), implies

$$-l(l+1)l!q_l^k=l!(q_l^{k+1}-q_l^k)r_{k+1}-l!(q_l^k-q_l^{k-1})r_k$$

$$=-l!(-r_kq_l^{k-1}+(r_k+r_{k+1})q_l^k-r_{k+1}q_l^{k+1}).$$

Therefore, we obtain $Rq_l = l(l+1)q_l$. □

To prove Theorem A (2) and (3), we need more detailed properties for q_l.

LEMMA 3.7. (1) $q_l^0 = n!/(n-l)!$ *for all* n *and* l *with* $0 \le l \le n$.
(2) $q_l^{n-k} = (-1)^l q_l^k$ *for all* n, l *and* k *with* $0 \le k, l \le n$.
(3) *If* n *is even and* l *is odd with* $0 \le l \le n$, *then* $q_l^{n/2} = 0$.
(4) *If* n *and* l *are even with* $0 \le l \le n$, *then* $q_l^{n/2} \ne 0$.

PROOF. (1) follows immediately from (3.3). Also from (3.3), we have

$$l!q_l^{n-k} = \sum_{m=0}^{l} (-1)^m \binom{l}{m} \prod_{j=1}^{l} r_{n-k+j-m}.$$

Put $j' = l - j + 1$ and $m' = l - m$, and we obtain

$$\begin{aligned} l!q_l^{n-k} &= \sum_{m'=0}^{l} (-1)^{l-m'} \binom{l}{l-m'} \prod_{j'=1}^{l} r_{k+j'-m'} \\ &= (-1)^l l! q_l^k . \end{aligned}$$

So (2) holds. (2) implies (3) immediately.

Assume that $q_l^{n/2} = 0$, for some even n and l with $0 \le l \le n$. Put $k = n/2 - j$, $j = 0, 1, \cdots, n/2$. Then (2) implies that $q_l^{n/2+j} = q_l^{n/2-j}$. From Lemma 3.4, we get

$$\begin{aligned} -\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)(q_l^{n/2-1} + q_l^{n/2+1}) &= -r_{n/2} q_l^{n/2-1} + (r_{n/2} + r_{n/2+1}) q_l^{n/2} - r_{n/2+1} q_l^{n/2+1} \\ &= l(l+1) q_l^{n/2} = 0 . \end{aligned}$$

These imply

$$q_l^{n/2-1} = q_l^{n/2} = q_l^{n/2+1} = 0 . \tag{3.16}$$

Now, from Lemma 3.4, for any k, q_l^k satisfies

$$-r_k q_l^{k-1} + (r_k + r_{k+1}) q_l^k - r_{k+1} q_l^{k+1} = l(l+1) q_l^k ,$$

which, combined with (3.16), implies $q_l^k = 0$ for all k with $0 \le k \le n$, i.e., $q_l = 0$. This contradicts (1). Therefore, (4) holds. □

From Lemma 3.7 (1) and (2), we have $q_l^0 \ne 0$ and $q_l^n \ne 0$ so that the order of $\varphi_{n,0}$ and $\varphi_{n,n}$ are $\{1, 2, \cdots, n\}$. Similarly, from Lemma 3.7 (3) and (4), if n is even, then the order of $\varphi_{n,n/2}$ is $\{2, 4, \cdots, n\}$. So Theorem A (2) and (3) are proved completely.

By Theorem A (1), if integers n and k with $0 \le k \le n$ are explicitly given, then we can obtain the order of $\varphi_{n,k}$. The following proposition is used in the later section.

Proposition 3.8. *For $n \leq 6$, the order of $\varphi_{n,k}$ is given as follows:*

$\varphi_{n,k}$	order	type
$\varphi_{1,0}$ and $\varphi_{1,1}$	$\{1\}$	1-type
$\varphi_{2,0}$ and $\varphi_{2,2}$	$\{1, 2\}$	2-type
$\varphi_{2,1}$	$\{2\}$	1-type
$\varphi_{3,k}$ $(0 \leq k \leq 3)$	$\{1, 2, 3\}$	3-type
$\varphi_{4,k}$ $(0 \leq k \leq 4, k \neq 2)$	$\{1, 2, 3, 4\}$	4-type
$\varphi_{4,2}$	$\{2, 4\}$	2-type
$\varphi_{5,k}$ $(0 \leq k \leq 5)$	$\{1, 2, 3, 4, 5\}$	5-type
$\varphi_{6,k}$ $(0 \leq k \leq 6, k \neq 1, 3, 5)$	$\{1, 2, 3, 4, 5, 6\}$	6-type
$\varphi_{6,1}$ and $\varphi_{6,5}$	$\{1, 3, 4, 5, 6\}$	5-type
$\varphi_{6,3}$	$\{2, 4, 6\}$	3-type

Remark. From Lemmas 3.4 and 3.7, we have $q_l^1 = (-1)^l q_l^{n-1} = ((n-1)!/(n-l)!)(n-l(l+1))$. Therefore, we see that $\varphi_{n,1}$ and $\varphi_{n,n-1}$ with $n = l(l+1)$ are of order $\{1, 2, \cdots, l-1, l+1, \cdots, n\}$ and of $(n-1)$-type, and that other $\varphi_{n,1}$ and $\varphi_{n,n-1}$ are of order $\{1, 2, \cdots, n\}$ and of n-type.

§4. Minimal surfaces in CP^n and harmonic sequence.

In this section, we consider minimal immersions of S^2 into CP^n in the context of harmonic maps.

Let M be a smooth manifold and V be a complex vector subbundle of the trivial bundle $\underline{C}^{n+1} = M \times C^{n+1}$ over M. Then V has a connection ∇, induced from the trivial connection on $\underline{C}^{n+1}$, given by $\nabla s = \pi_V ds$, where s is a section of V and $\pi_V : \underline{C}^{n+1} \to V$ denotes orthogonal projection onto V.

Let L be the universal line bundle over CP^n defined by $L = \{(p, v) \in CP^n \times C^{n+1} \mid v \in p\}$ then both L and its orthogonal complement $L^\perp$ have induced connections and Hermitian metrics. Let $T^{(1,0)}CP^n$ (resp. $T^{(0,1)}CP^n$) denote the (1, 0)-part (resp. (0, 1)-part) of the complexification TCP^{nC} of TCP^n. Thus we have a Hermitian metric and a connection of $\mathrm{Hom}(L, L^\perp)$ and there is a canonical isomorphism $h: T^{(1,0)}CP^n \to \mathrm{Hom}(L, L^\perp)$ given by $h(X)s = \pi_{L^\perp} ds(X)$, where $X \in T^{(1,0)}CP^n$ and s is a local section of L. Under this isomorphism, the complex structure, the metric and the connection on $\mathrm{Hom}(L, L^\perp)$ correspond respectively to the complex structure, the Fubini Study metric and the connection on CP^n with constant holomorphic sectional curvature 4.

For a smooth manifold M, there is a bijective correspondence between (smooth) complex line subbundles of $\underline{C}^{n+1}$ and smooth maps $\varphi : M \to CP^n$, given by $\varphi \leftrightarrow \varphi^* L$. Let $d^{(1,0)}\varphi : TM^C \to T^{(1,0)}CP^n$ be the (1, 0)-part of the derivative of φ. Then $h \circ d^{(1,0)}\varphi$ is a bundle map covering φ and the corresponding section δ of $\mathrm{Hom}(TM^C \otimes \varphi^* L, \varphi^* L^\perp)$

is given by $\delta(X\otimes s)=\pi_{L^\perp}ds(X)$, where a section s of φ^*L is considered as C^{n+1}-valued function defined on M. If M is a Riemann surface, the holomorphic part

$$\partial : T^{(1,0)}M\otimes\varphi^*L\to\varphi^*L^\perp$$

of δ is given in terms of a local complex coordinate z on M by

$$\partial(\partial/\partial z\otimes s)=(h\circ d^{(1,0)}\varphi(\partial/\partial z))(s)=\pi_{L^\perp}ds(\partial/\partial z)\,,$$

and the antiholomorphic part

$$\bar\partial : T^{(0,1)}M\otimes\varphi^*L\to\varphi^*L^\perp$$

of δ is given by

$$\bar\partial(\partial/\partial\bar z\otimes s)=(h\circ d^{(1,0)}\varphi(\partial/\partial\bar z))(s)=\pi_{L^\perp}ds(\partial/\partial\bar z)\,.$$

For any complex vector bundle V over the Riemann surface M, by Koszul-Malgrange theorem, each connection on V determines a holomorphic structure on V. Thus we have holomorphic structures on φ^*L and $\varphi^*L^\perp$, and Wolfson shows that φ is harmonic if and only if ∂ (resp. $\bar\partial$) is a holomorphic (resp. an antiholomorphic) bundle map. Using these ideas, for a harmonic map φ, Wolfson in [12] goes on to construct inductively an associated sequence

$$\cdots,\ L_{-2},\ L_{-1},\ L_0,\ L_1,\ L_2,\ \cdots$$

of complex line subbundles of $\underline{C}^{n+1}$ and bundle maps

$$\partial_p : T^{(1,0)}M\otimes L_p\to L_{p+1}\quad\text{and}\quad\bar\partial_p : T^{(0,1)}M\otimes L_p\to L_{p-1}\,.$$

Here $L_p=\varphi_p^*L$ for a suitable harmonic map $\varphi_p : M\to CP^n$ and ∂_p (resp. $\bar\partial_p$) is essentially the map ∂ (resp. $\bar\partial$) defined above for the map φ_p. Then ∂_p (resp. $\bar\partial_p$) is a holomorphic (resp. antiholomorphic) bundle map. If $\partial_p\equiv 0$ but $\partial_{p-1}\not\equiv 0$ (resp. $\bar\partial_p\equiv 0$ but $\bar\partial_{p+1}\not\equiv 0$) then the sequence terminates with L_p at the right (resp. left) hand end, and the corresponding harmonic map φ_p is antiholomorphic (resp. holomorphic). The set of points of M over which ∂_p (resp. $\bar\partial_p$) is singular is a set of isolated points and, except these points, L_{p+1} (resp. L_{p-1}) is the image of ∂_p (resp. $\bar\partial_p$). (Also, see [2, 3].)

We call the sequence $\{\varphi_p\}$ the *harmonic sequence* determined by φ with $\varphi=\varphi_p$ for some p, and the sequence $\{L_p\}$ the *associated bundle sequence*. φ_p is conformal if and only if L_{p+1} is orthogonal to L_{p-1}.

If the harmonic sequence $\{\varphi_p\}$ terminates at one end, then it terminates at both ends and all the elements of the associated bundle sequence $\{L_p\}$ are mutually orthogonal, i.e., L_p is orthogonal to L_q for $p\neq q$. If the harmonic sequence of φ satisfies this condition, φ is called *isotropic*, so that each φ_p is conformal. Moreover, in this case, φ is full in CP^n if and only if the sequence $\{\varphi_p\}$ has length exactly $n+1$, which is equivalent to the fact that $\underline{C}^{n+1}$ is an orthogonal sum of some $n+1$ consecutive bundles of the bundle sequence.

Now we need a local description of the harmonic sequence of an isotropic harmonic map φ. Let z be a local complex coordinate on M. Then, for each p, we can choose a meromorphic local section f_p of L_p such that

$$f_{p+1}=\partial_p(\partial/\partial z \otimes f_p).$$

Define functions γ_p by

$$\gamma_p=\begin{cases} \dfrac{|f_{p+1}|^2}{|f_p|^2}, & \text{if } f_p\not\equiv 0, \\ 0 & \text{if } f_p\equiv 0, \end{cases}$$

then we have

$$\frac{\partial}{\partial z}f_p=f_{p+1}+\frac{\partial}{\partial z}\log|f_p|^2 f_p, \tag{4.1}$$

$$\frac{\partial}{\partial \bar{z}}f_p=-\gamma_{p-1}f_{p-1}. \tag{4.2}$$

Since $(\partial^2/\partial z\partial\bar{z})f_p=(\partial^2/\partial\bar{z}\partial z)f_p$, we have

$$\frac{\partial^2}{\partial z\partial\bar{z}}\log|f_p|^2=\gamma_p-\gamma_{p-1} \tag{4.3}$$

and the unintegrated Plücker formulae

$$\frac{\partial^2}{\partial z\partial\bar{z}}\log\gamma_p=\gamma_{p+1}-2\gamma_p+\gamma_{p-1}. \tag{4.4}$$

If φ is conformal, then φ is minimal if and only if φ is harmonic. Therefore, in order to prove Theorems C, D and E, we use the method of the harmonic sequence. Notice that in [2], J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward show Theorems 3.1 and 3.2 using this method.

By Riemann-Roch theorem, every harmonic map of a 2-sphere S^2 into CP^n is isotropic. Therefore, we will prove Theorems C, D and E for a compact isotropic minimal surface in CP^n.

From now on, we assume that $\varphi: M\to CP^n$ be an isotropic conformal minimal immersion of a compact Riemann surface M into CP^n, and that $\{\varphi_p\}$ is the corresponding sequence determined by φ with $\varphi=\varphi_0$. Then each φ_p is also an isotropic conformal minimal immersion of M (perhaps with isolated singularities). Let g_p and θ_p denote the induced metric of M by φ_p and the Kähler angle of φ_p, respectively. Let Δ_p and K_p denote the Laplacian and the Gaussian curvature of (M, g_p), respectively. Then we have

$$g_p=\sigma_p dz d\bar{z}, \qquad \sigma_p=\gamma_p+\gamma_{p-1}, \tag{4.5}$$

(4.6) $$\tan^2\frac{\theta_p}{2}=\frac{\gamma_{p-1}}{\gamma_p},$$

(4.7) $$\Delta_p=-\frac{4}{\sigma_p}\frac{\partial^2}{\partial z\partial\bar{z}},$$

(4.8) $$K_p=-\frac{2}{\sigma_p}\frac{\partial^2}{\partial z\partial\bar{z}}\log\sigma_p\,.$$

Set $F_p=\Psi\circ\varphi_p$. By the definition of Ψ, we have

(4.9) $$F_p=\frac{1}{|f_p|^2}f_pf_p^*\,.$$

From (4.1), (4.2) and (4.7), we will inductively show that

(4.10) $$\Delta_0^lF_0=\sum_{|p|,|q|\le l}\alpha^{pq}f_pf_q^*$$

for any nonnegative integer l, where α^{pq} is a $\boldsymbol{C}$-valued function on M. Note that a matrix $f_pf_q^*$ acts on $\boldsymbol{C}^{n+1}$ as $(f_pf_q^*)f_r=\langle f_r, f_q\rangle f_p$.

Theorem C follows from the following theorem.

THEOREM C′. *Let M be a compact, k-type, mass-symmetric, isotropic, minimal surface in $\boldsymbol{C}P^n(4)$. Then n satisfies $n\le 2k$.*

PROOF. By Proposition 2.2, there exist real constants a_l, $1\le l\le k$, such that the matrix-valued function

$$P=\Delta_0^kF_0+a_1\Delta_0^{k-1}F_0+\cdots+a_{k-1}\Delta_0F_0+a_k(F_0-(1/(n+1))I)$$

is identically zero. Since φ is exactly k-type, we have $a_k\neq 0$. From (4.10), we get

(4.11) $$P=\sum_{|p|,|q|\le k}\alpha^{pq}f_pf_q^*-\frac{a_k}{n+1}I$$

where α^{pq} is a $\boldsymbol{C}$-valued function on M. Since φ is isotropic, f_p is orthogonal to f_q for $p\neq q$, so that (4.11) implies that if $|p|\ge k+1$, then $f_p=-((n+1)/a_k)Pf_p=0$. By Proposition B, φ is isotropic and full. Therefore, there exist nonnegative integers l and l' such that $n=l+l'$, $f_p\not\equiv 0$ for $-l'\le p\le l$ and other f_p's are identically zero. Thus we get l, $l'\le k$ so that $n\le 2k$. □

(4.1), (4.2) and (4.9) imply that

(4.12) $$\frac{\partial}{\partial z}F_p=\frac{1}{|f_p|^2}f_{p+1}f_p^*-\frac{\gamma_{p-1}}{|f_p|^2}f_pf_{p-1}^*\,,$$

$$\frac{\partial}{\partial\bar{z}}F_p=\frac{1}{|f_p|^2}f_pf^*_{p+1}-\frac{\gamma_{p-1}}{|f_p|^2}f_{p-1}f^*_p\,, \tag{4.13}$$

$$\frac{\partial^2}{\partial z\partial\bar{z}}F_p=-(\gamma_p+\gamma_{p-1})F_p+\gamma_pF_{p+1}+\gamma_{p-1}F_{p-1}\,, \tag{4.14}$$

which, combined with (4.7), yields

$$\Delta_0F_p=(t_p+t_{p-1})F_p-t_pF_{p+1}-t_{p-1}F_{p-1} \tag{4.15}$$

where $t_p=4\gamma_p/(\gamma_0+\gamma_{-1})$. After simple computation, these imply that

$$\begin{aligned}\Delta_0^2F_p&=\Delta_0(t_p+t_{p-1})F_p\\&\quad-\frac{4}{\sigma_0}(t_p+t_{p-1})_z(F_p)_{\bar z}-\frac{4}{\sigma_0}(t_p+t_{p-1})_{\bar z}(F_p)_z+(t_p+t_{p-1})\Delta_0F_p\\&\quad-\Delta_0t_pF_{p+1}+\frac{4}{\sigma_0}(t_p)_z(F_{p+1})_{\bar z}+\frac{4}{\sigma_0}(t_p)_{\bar z}(F_{p+1})_z-t_p\Delta_0F_{p+1}\\&\quad-\Delta_0t_{p-1}F_{p-1}+\frac{4}{\sigma_0}(t_{p-1})_z(F_{p-1})_{\bar z}+\frac{4}{\sigma_0}(t_{p-1})_{\bar z}(F_{p-1})_z-t_{p-1}\Delta_0F_{p-1}\,.\end{aligned} \tag{4.16}$$

PROPOSITION 4.1. *If a compact, mass-symmetric, isotropic, minimal surface M in $CP^n(4)$ is of at most 2-type, then M has constant curvature and constant Kähler angle.*

PROOF. By Proposition 2.2, there exist real constants b and c such that the matrix-valued function

$$P=\Delta_0^2F_0+b\Delta_0F_0+c(F_0-(1/(n+1))/I)$$

is identically zero. Since φ is isotropic, f_p is orthogonal to f_q for $p\neq q$. Since $t_0+t_{-1}=4$, from (4.12), (4.13), (4.15) and (4.16), we have

$$Pf_0=\left(16+t_0^2+t_{-1}^2+4b+c\frac{n}{n+1}\right)f_0-\frac{4}{\sigma_0}(t_0)_{\bar z}f_1+\frac{4}{\sigma_0}(t_{-1})_z\gamma_{-1}f_{-1}\,.$$

Since φ is not a constant map, we see that $f_0\not\equiv0$, and either f_1 or f_{-1} is not identically zero. From $P\equiv0$, we see that either $(t_0)_{\bar z}$ or $(t_{-1})_z$ is vanishing. Since each t_p is a real-valued function, we see that either t_0 or t_{-1} is constant, so that there exist real constants α and β such that

$$\alpha\gamma_0+\beta\gamma_{-1}\equiv0 \tag{4.17}$$

with $(\alpha,\beta)\neq(0,0)$ and both t_0 and t_{-1} are constant. (4.6) and (4.17) imply that M has constant Kähler angle.

Since t_0 and t_{-1} are constant, we have

$$\Delta_0^2 F_0 = 4\Delta_0 F_0 - t_0 \Delta_0 F_1 - t_{-1}\Delta_0 F_{-1}\,,$$

so that

$$(4.18)\qquad Pf_1 = \left(-4t_0 - (t_1 + t_0)t_0 - bt_0 - c\frac{1}{n+1}\right) f_1\,,$$

$$(4.19)\qquad Pf_{-1} = \left(-4t_{-1} - (t_{-1} + t_{-2})t_{-1} - bt_{-1} - c\frac{1}{n+1}\right) f_{-1}\,.$$

Assume that $f_1 \not\equiv 0$. Then from (4.17), we have $\gamma_0 \neq 0$ and $\gamma_{-1} = \nu\gamma_0$ with some constant $\nu > 0$. Since $P \equiv 0$, (4.18) implies that t_1 is constant so that there exists a constant μ such that $\gamma_1 = \mu\gamma_0$. Then from (4.8) and (4.4), we get

$$\begin{aligned} K_0 &= -\frac{2}{(1+\nu)\gamma_0}\frac{\partial^2}{\partial z \partial \bar{z}}\log(1+\nu)\gamma_0 \\ &= -\frac{2}{(1+\nu)\gamma_0}(\gamma_1 - 2\gamma_0 + \gamma_{-1}) \\ &= -\frac{2(\nu+\mu-2)}{1+\nu}\,. \end{aligned}$$

Therefore, M has constant curvature.

Similarly, from (4.19), even if $f_{-1} \not\equiv 0$, M has constant curvature. □

PROPOSITION 4.2. *Compact, totally real, minimal flat surfaces in $CP^n(4)$ are never isotropic.*

PROOF. Let $\varphi : M \to CP^n$ be a totally real minimal immersion of a flat compact Riemann surface M in CP^n, and $\{\varphi_p\}$ the corresponding harmonic sequence determined by φ with $\varphi = \varphi_0$.

Since $\theta_0 = \pi/2$, (4.6) implies $\gamma_0 = \gamma_{-1}$. Applying $\partial^2/\partial z \partial \bar{z}$, and using (4.4), we get $\gamma_1 = \gamma_{-2}$. Since $K_0 = 0$, (4.8) implies $\gamma_1 = \gamma_0$. Therefore, we have $\gamma_1 = \gamma_0 = \gamma_{-1} = \gamma_{-2}$ ($\not\equiv 0$) so that (4.6) and (4.8) imply that both φ_1 and φ_{-1} are also totally real minimal immersions of M in CP^n, and the induced metrics are flat. Inductively, we obtain that each φ_p is totally real. Therefore, the sequence $\{\varphi_p\}$ never terminates so that φ is not isotropic. □

In [5], Y. Ohnita showed the following:

THEOREM 4.3. *Let M be a minimal surface with constant curvature K immersed fully in CP^n. Assume that the Kähler angle of M is constant. Then the following hold:*

(1) *If $K > 0$, then there exists some k with $0 \leq k \leq n$ such that M is an open submanifold of $\varphi_{n,k}(S^2)$.*

(2) *If* $K=0$ (*i.e.*, M *is flat*), *then* M *is totally real.*

(3) $K<0$ *is impossible.*

Let $\varphi : M \to CP^n$ be a mass-symmetric, 2-type, isotropic, minimal immersion of a compact surface M in $CP^n(4)$. From Proposition B, φ is full. Then, from Propositions 4.1, 4.2 and Theorem 4.3, we obtain that M has positive constant curvature, and that $\varphi : M \to CP^n$ is congruent to $\varphi_{n,k} : S^2 \to CP^n$ for some k with $0 \le k \le n$. On the other hand, from Theorem C′, we get $n \le 4$. Therefore, from Proposition 3.8, we obtain the following:

THEOREM D′. *If a compact, mass-symmetric, isotropic, minimal surface* M *in* $CP^n(4)$ *is of at most* 2*-type, then* M *is of positive constant curvature, so that the immersion is congruent to either* $\varphi_{1,0}$, $\varphi_{1,1}$, $\varphi_{2,0}$, $\varphi_{2,1}$, $\varphi_{2,2}$ *or* $\varphi_{4,2}$.

Theorem D follows immediately from this theorem.

PROPOSITION 4.4. *Let* M *be a compact, mass-symmetric, isotropic, minimal surface in* $CP^n(4)$. *If* M *is of at most* 3*-type and with constant Kähler angle, then* M *is of constant curvature.*

PROOF. From (4.6), both t_0 and t_{-1} are constant so that we have

$$\Delta_0 F_0 = 4F_0 - t_0 F_1 - t_{-1} F_{-1}\,,$$

$$\Delta_0^2 F_0 = 4\Delta_0 F_0 - t_0 \Delta_0 F_1 - t_{-1} \Delta_0 F_{-1}\,,$$

$$\Delta_0^3 F_0 = 4\Delta_0^2 F_0 - t_0 \Delta_0^2 F_1 - t_{-1} \Delta_0^2 F_{-1}\,.$$

By Proposition 2.2, there exist real constants a, b and c such that the matrix-valued function

$$P = \Delta_0^3 F_0 + a\Delta_0^2 F_0 + b\Delta_0 F_0 + c\left(F_0 - \frac{1}{n+1} I\right)$$

is identically zero. Since the Kähler angle is constant, from (4.6), there exist real constants α and β such that $\alpha\gamma_0 + \beta\gamma_{-1} \equiv 0$ with $(\alpha, \beta) \ne (0, 0)$.

Assume that φ is not antiholomorphic. Then we have $f_1 \not\equiv 0$ and $\gamma_{-1} = \nu\gamma_0$ for some $\nu > 0$ so that from (4.8) and (4.4),

$$K_0 = -\frac{2}{(1+\nu)\gamma_0}(\gamma_1 + (\nu-2)\gamma_0)\,.$$

If $f_2 \equiv 0$, then $\gamma_1 \equiv 0$ so that M has constant curvature $K_0 = -2(\nu-2)/(1+\nu)$. So we assume $f_2 \not\equiv 0$. Simple computation implies

$$Pf_2 = 2t_0 t_1 (t_1)_z f_1 + \left(t_0(t_1(t_2 + 2t_1 + 4) + \Delta t_1) + a t_0 t_1 - \frac{c}{n+1}\right) f_2 - \frac{4}{\sigma_0} t_0 (t_1)_{\bar z} f_3\,,$$

and since $f_1 \not\equiv 0$ and $f_2 \not\equiv 0$, we have $t_0 \neq 0$ and $t_1 \neq 0$. Then $P \equiv 0$ implies $(t_1)_z \equiv 0$ so that t_1 is constant and there exists a constant μ such that $\gamma_1 = \mu\gamma_0$. Therefore, we obtain $K_0 = -2(\nu + \mu - 2)/(1 + \nu)$ so that M has constant curvature.

Similarly, we see that if φ is not holomorphic, then M has constant curvature. Therefore, Proposition 4.4 is proved completely. □

By an argument similar to Theorem D′, Proposition 4.4 implies the following theorem, from which Theorem E follows immediately.

THEOREM E′. *Let M be a compact, mass-symmetric, isotropic, minimal surface in* $CP^n(4)$. *If M is of at most* 3-*type and with constant Kähler angle, then M is of positive constant curvature, so that the immersion is congruent to either* $\varphi_{n,k}$ $(n = 1, 2, 3, 0 \leq k \leq n)$, $\varphi_{4,2}$ *or* $\varphi_{6,3}$.

REMARK. There exists a compact, mass-symmetric, finite-type minimal surface in CP^n which is not isotropic. From example, a totally real flat minimal torus $T^2 = \pi(S^1(3) \times S^1(3) \times S^1(3))$ in $CP^2(4)$ is mass-symmetric, 1-type and its harmonic sequence is a cyclic infinite sequence, where $\pi : C^3 - \{0\} \to CP^2$ is the projection.

References

[1] S. BANDO and Y. OHNITA, Minimal 2-spheres with constant curvature in $P_n(C)$, J. Math. Soc. Japan **39** (1987), 477–487.

[2] J. BOLTON, G. R. JENSEN, M. RIGOLI and L. M. WOODWARD, On conformal minimal immersions of S^2 in CP^n, Math. Ann. **279** (1988), 599–620.

[3] J. BOLTON and L. M. WOODWARD, Congruence theorems for harmonic maps from a Riemann surface into CP^n and S^n, J. London Math. Soc. (2) **45** (1992), 363–376.

[4] B. Y. CHEN, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific (1984).

[5] Y. OHNITA, Minimal surfaces with constant curvature and Kähler angle in complex space forms, Tsukuba J. Math. **13** (1989), 191–207.

[6] A. ROS, Spectral geometry of CR minimal submanifolds in the complex projective space, Kodai Math. J. **6** (1983), 88–99.

[7] A. ROS, On spectral geometry of Kaehler submanifolds, J. Math. Soc. Japan **36** (1984), 433–448.

[8] T. TAKAHASHI, Isometric immersion of Riemannian homogeneous manifolds, Tsukuba J. Math. **12** (1988), 231–233.

[9] S. UDAGAWA, Spectral geometry of Kaehler submanifolds of a complex projective space, J. Math. Soc. Japan **38** (1986), 453–472.

[10] S. UDAGAWA, Spectral geometry of compact Hermitian symmetric submanifolds, Math. Z. **192** (1986), 57–72.

[11] S. UDAGAWA, Einstein parallel Kaehler submanifolds in a complex projective space, Tokyo J. Math. **9** (1986), 335–340.

[12] J. G. WOLFSON, Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifolds, J. Differential Geom. **27** (1988), 161–178.

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