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A Note on L^2 Harmonic Forms on a Complete Manifold

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Introduction.

In this note, we shall show a nonexistence result for harmonic forms with values in a vector bundle equipped with a Riemannian structure over a complete manifold. Moreover in relation with the result, we shall construct examples of harmonic mappings of finite total energy. This note is motivated by a recent paper of Elworthy and Rosenberg [5].

Vanishing theorems and growth properties for such forms (including harmonic mappings for example) have been investigated extensively by many authors from various points of views. Donnelly and Xavier [4] studied, for instance, the spectrum of the Laplacian acting on the square integrable forms on a negatively curved manifold and showed a sharp lower bound for the spectrum under a certain pinching condition on curvature. Their result gives in particular vanishing of L^2 harmonic forms. An integral identity on differential forms, (1.1) in Section 1, plays a crucial role in their paper. We remark that this formula was also obtained by Karcher and Wood [8] to study the growth properties for harmonic forms. In this note, we shall derive a consequence of the formula, which is stated in the following:

THEOREM 1. *Let M be a complete Riemannian manifold of dimension m, and let E be a real vector bundle endowed with a Riemannian structure. Suppose M possesses a pole o (a point at which the exponential mapping induces a diffeomorphism). Then there are no nontrivial square integrable, E-valued harmonic q-forms ($q=p$ or $m-p$) for a positive integer p less than $m/2$ if the radial curvature K_r of M satisfies either of the following conditions*:

$$-\left(\frac{m-p-1}{p}\right)^2 \leq K_r \leq -1 \,, \tag{1}$$

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$$(2) \qquad -\frac{a}{1+r^2} \leq K_r \leq \frac{a'}{1+r^2},$$

where $a \geq 0$ *and* $a' \in [0, 1/4]$ *are constants chosen in such a way that*

$$2+(m-p-1)\{1+(1-4a')^{1/2}\}-p\{1+(1+4a)^{1/2}\} \geq 0 ;$$

$$(m-p)\{1+(1-4a')^{1/2}\}-(p-1)\{1+(1+4a)^{1/2}\}-2 \geq 0 .$$

Using Witten's deformation of Laplacian on forms, Elworthy and Rosenberg [5] showed, among other things, vanishing of L^2 harmonic p-forms under the assumption that

$$-\left(\frac{m-p-1}{p}\right)^2 + \varepsilon \leq K_r \leq -1$$

for some positive ε. In fact, it follows from this pinching condition that the lower bound of the spectrum of the Laplacian acting on square integrable p-forms is positive. To be precise, we shall prove the following:

THEOREM 2. *Let M and E be as in Theorem 1. Let σ_p be the lower bound of the spectrum of the Laplacian acting on square integrable, E-valued p-forms of M. Then*

$$\sigma_q \geq \frac{1}{4}(m-1-p-ap)^2 > 0 ,$$

if $p < (m-1)/2$ *and the radial curvature* K_r *of* M *satisfies*

$$-a^2 \leq K_r \leq -1 \qquad \text{with} \quad 1 \leq a < \frac{m-1-p}{p},$$

where $q=p$ *or* $q=m-p$.

We remark that the pinching condition in the case (1) of Theorem 1 is sharp for $p=1$. Indeed, a theorem due to Anderson [1] states that for any $m \geq 2$, $0<p<m$ and $a > |m-2p|$ with $a \geq 1$, there exist complete simply connected manifolds M of dimension m with

$$-a^2 \leq \text{the sectional curvature of } M \leq -1$$

such that the set of the L^2 harmonic p-forms on M forms an infinite dimensional space. As for the case (2) of Theorem 1, it is not clear whether the pinching condition there could be relaxed. In this regard, we shall construct examples of harmonic mappings of finite total energy in the last section of this note.

The author would like to thank Professor Steven Rosenberg for sending a preprint of the paper [5]. A main result of this note, Theorem 1, was obtained after his lectures at Kanazawa University in June, 1992. Recently the author has been informed that the

integral formula proves useful in the study on the spectrum of nonnegatively curved manifolds. The author would like to express his thanks to Professor José F. Escobar for sending a reprint of the paper [6].

1. An integral identity.

Let M be a smooth m-dimensional Riemannian manifold and E a real vector bundle equipped with metric $\langle\ ,\ \rangle$ and metric connection ∇. These have natural extensions to E-valued forms and other tensors. Let $C^\infty(\bigwedge^p T^*M\otimes E)$ be the space of smooth p-forms on M with values in the vector bundle E. Relative to the connection ∇, we define the exterior differential and codifferential operators $d\colon C^\infty(\bigwedge^p T^*M\otimes E)\to C^\infty(\bigwedge^{p+1} T^*M\otimes E)$ and $\delta\colon C^\infty(\bigwedge^p T^*M\otimes E)\to C^\infty(\bigwedge^{p-1} T^*M\otimes E)$ respectively by

$$d\omega(X_0, \cdots, X_p)=\sum_{j=0}^{p}(-1)^j(\nabla_{X_j}\omega)(X_0, \cdots, \hat{X}_j, \cdots, X_p)\,,$$

$$\delta\omega(X_2, \cdots, X_p)=-\sum_{i=1}^{m}(\nabla_{e_i}\omega)(e_i, X_2, \cdots, X_p)\,,$$

where $\{e_1, \cdots, e_m\}$ is an orthonormal basis of the tangent space at a point of M. The Laplacian on E-valued differential forms on M is given by

$$\Delta=d\delta+\delta d\colon C^\infty(\bigwedge^p T^*M\otimes E)\to C^\infty(\bigwedge^p T^*M\otimes E)\,.$$

Those $\omega\in C^\infty(\bigwedge^p T^*M\otimes E)$ such that $\Delta\omega=0$ are called *E-valued harmonic p-forms*. Let us denote by $C_0^\infty(\bigwedge^p T^*M\otimes E)$ (resp. $L^2(\bigwedge^p T^*M\otimes E)$) the space of E-valued smooth p-forms with compact support (resp. E-valued square integrable p-forms). Then Δ is a positive symmetric operator on $C_0^\infty(\bigwedge^p T^*M\otimes E)$. When M is complete, it is known that Δ extends uniquely to a self-adjoint operator, denoted by the same letter Δ, on $L^2(\bigwedge^p T^*M\otimes E)$, and further that for $\omega\in L^2(\bigwedge^p T^*M\otimes E)$, ω is harmonic if and only if ω is closed and coclosed, $d\omega=\delta\omega=0$. See [10] for these facts.

Given an E-valued smooth p-form ω, we define a symmetric covariant tensor of degree 2 by

$$S_\omega(X, Y)=\tfrac{1}{2}|\omega|^2 g_M(X, Y)-\langle\iota_X\omega, \iota_Y\omega\rangle\,.$$

The divergence of the tensor S_ω is given by

$$\operatorname{div} S_\omega(X)=\langle\iota_X(d\omega), \omega\rangle+\langle\iota_X\omega, \delta\omega\rangle\,.$$

Moreover for a vector field V, the divergence of the one-form $\iota_V S_\omega$ is given by

$$\operatorname{div}(\iota_V S_\omega)=(\operatorname{div} S_\omega)(V)+\tfrac{1}{2}|\omega|^2\operatorname{div} V-\langle\omega^{\nabla V}, \omega\rangle\,.$$

Here we use the following notation:

$$\omega^{\nabla V}(X_1, \cdots, X_p) = \omega(\nabla_{X_1} V, X_2, \cdots, X_p) + \cdots + \omega(X_1, \cdots, \nabla_{X_p} V).$$

Hence given a compact domain D of M with smooth boundary ∂D, we derive

$$\int_{\partial D} S_\omega(V, v) = \int_D \left((\operatorname{div} S_\omega)(V) + \frac{1}{2} |\omega|^2 \operatorname{div} V - \langle \omega^{\nabla V}, \omega \rangle \right), \tag{1.1}$$

where v stands for the outer normal to ∂D (cf. [4, (2.6)], [8, Proposition 2.5]). In particular for $\omega \in C_0^\infty(\bigwedge^p T^* M \otimes E)$, this reads as follows:

$$\int_M \langle \iota_V d\omega, \omega \rangle + \langle \iota_V \omega, \delta\omega \rangle + \frac{1}{2} |\omega|^2 \operatorname{div} V - \langle \omega^{\nabla V}, \omega \rangle = 0. \tag{1.2}$$

Now we shall assume that M is complete and noncompact, ω is square integrable, and V satisfies

$$|V| \leq C_1(r+1)$$

for some constant C_1, where r denotes the distance function to a fixed basepoint of M. We take a smooth function ρ on M which approximates the distance function r in such a way that

$$|\nabla \rho| \leq C_2, \qquad |\rho - r| \leq C_2$$

for some constant C_2. Then we can choose a divergent sequence t_k of regular values of ρ such that

$$\lim_{t_k \to \infty} \int_{\partial D(t_k)} |V||\omega|^2 = 0,$$

where we set $D(t) = \{x \in M : \rho(x) \leq t\}$. Indeed, by the coarea formula and the above assumption of V, we have

$$\int_{-\infty}^{\infty} (t^2+1)^{-1/2} \int_{\partial D(t)} |V||\omega|^2 dt = \int_M \frac{|V||\nabla\rho||\omega|^2}{(|\rho|^2+1)^{1/2}} \leq C_3 \int_M |\omega|^2 < +\infty$$

for some constant C_3. Since

$$\left| \int_{\partial D(t_k)} S_\omega(V, v_k) \right| \leq 2 \int_{\partial D(t_k)} |V||\omega|^2,$$

it follows from (1.1) that

$$\lim_{t_k \to \infty} \int_{D(t_k)} \left((\operatorname{div} S_\omega)(V) + \frac{1}{2} |\omega|^2 \operatorname{div} V - \langle \omega^{\nabla V}, \omega \rangle \right) = 0. \tag{1.3}$$

Now we shall assume further that V is given by the gradient ∇f of a smooth function f on M. In this case, taking an orthonormal basis $\{e_1, \cdots, e_m\}$ so that the hessian $\nabla^2 f$ of f is diagonal with respect to this basis, we have

$$\langle \omega^{\nabla V}, \omega \rangle = \sum_{i_1 < \cdots < i_p} (\lambda_{i_1} + \cdots + \lambda_{i_p}) | \omega(e_{i_1}, \cdots, e_{i_p}) |^2 ,$$

where λ_i are the eigenvalues of the hessian of f. Putting them in order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$, we have

$$\langle \omega^{\nabla V}, \omega \rangle \leq (\lambda_1 + \cdots + \lambda_p) | \omega |^2 .$$

Hence we get

$$\tfrac{1}{2} | \omega |^2 \operatorname{div} V - \langle \omega^{\nabla V}, \omega \rangle \geq \delta_{f,p} | \omega |^2 ,$$

where we set

$$\delta_{f,p} = \tfrac{1}{2} \{ \lambda_{p+1} + \cdots + \lambda_m - \lambda_1 - \cdots - \lambda_p \} .$$

Based on the observations which we have made, we can prove the following:

THEOREM 3. *Let M be a complete Riemannian manifold of dimension m and E a real vector bundle over M endowed with a Riemannian structure. Let p be a positive integer less than $m/2$.*

(I) *Suppose there exists a smooth function f on M such that* (i) *$\delta_{f,p}$ is nonnegative everywhere on M and positive somewhere*; (ii) $|\nabla f| \leq C_1(r+1)$ *for a positive constant C_1, where r stands for the distance to a fixed point of M. Then there are no nontrivial square integrable, E-valued harmonic q-forms, where $q = p$ or $q = m - p$.*

(II) *Suppose there exists a smooth function f on M such that* (iii) *$\delta_{f,p} \geq C_2$ for a positive constant C_2*; (iv) $|\nabla f| \leq C_3$ *for a positive constant C_3. Then the lower bound σ_q of the spectrum of the Laplacian Δ on $L^2(\bigwedge^q T^* M \otimes E)$ $(q = p$ or $m - p)$ is bounded from below by $(C_2/C_3)^2$.*

PROOF. Without loss of generality, we may assume $q = p$. (Otherwise we consider $*\omega$.) We shall first prove the assertion (I). Let ω be a square integrable, E-valued harmonic p-form. Since M is complete, ω is closed and coclosed. In particular, $\operatorname{div} S_\omega$ vanishes identically. Hence by the assumption (ii) and (1.3), we have

$$\limsup_k \int_{D_k} \delta_{f,p} | \omega | \leq 0$$

for some sequence of domains D_k which exhausts M. This implies together with the assumption (i) that ω must vanish on an open set and hence everywhere on M, since ω is harmonic. This completes the proof of (I).

So far as the proof of the second assertion (II) is concerned, we apply (1.2) and obtain

$$\int_M (| d\omega | + | \delta\omega |) | \omega | | \nabla f |^2 \geq \int_M \delta_{f,p} | \omega |^2$$

for compactly supported ω. Therefore we have by the assumptions (iii), (iv) and the Cauchy-Schwartz inequality

$$\int_M \langle \omega, \Delta\omega \rangle = \int_M |d\omega|^2 + |\delta\omega|^2 \geq \left(\frac{C_2}{C_3}\right)^2 \int_M |\omega|^2$$

for compactly supported ω. This holds for ω which belongs to the domain $D(\Delta)$ of Δ in $L^2(\bigwedge^p T^*M \otimes E)$. This proves the assertion (II).

The differential $d\phi$ of a smooth mapping ϕ of M into another Riemannian manifold N is considered as a $\phi^* TN$-valued 1-form over M. In this case, the tensor $S_{d\phi}$ is called the *stress-energy tensor* of ϕ and the divergence is given by

$$\operatorname{div} S_{d\phi} = -\langle \tau_\phi, d\phi \rangle ,$$

where τ_ϕ denotes the tension field of the mapping ϕ (cf. [2]). Therefore if ϕ is harmonic, namely, $\tau_\phi \equiv 0$, then $\operatorname{div} S_{d\phi} \equiv 0$. Thus we have the following.

COROLLARY 4. *Let M be a complete Riemannian manifold of dimension m which admits a smooth function f as in Theorem* 3 (I) *with $p=1$. Then any harmonic mapping of M into another Riemannian manifold has infinite total energy unless the mapping is constant.*

REMARK. The identity (1.1) still holds for the case $p=0$, namely, for a smooth section ω of E, if we understand $\omega^{\nabla V}=0$. Therefore Theorem 3 is true for this case. We note that $\delta_{f,0} = (1/2)\Delta f$.

2. Proofs of Theorems 1 and 2.

Let $M=(M, g)$ be a complete Riemannian manifold of dimension m with a pole o. We take first two continuous functions k_1 and k_2 on $[0, \infty)$ in such a way that

$$k_2 \circ r \leq \text{the radial curvature } K_r \leq k_1 \circ r ,$$

where r stands for the distance function to the pole o and the radial curvature is by definition the sectional curvature of planes tangent to the radial vector ∇r. Let J_i $(i=1, 2)$ be the solutions of the classical Jacobi equations:

$$J_i'' + k_i J_i = 0 ,$$

subject to the initial conditions:

$$J_i(0) = 0 , \qquad J_i'(0) = 1 .$$

We assume that J_1 is positive on $(0, \infty)$. Then it follows from the hessian comparison theorem (cf. [7]) that

$$\frac{J'_1(r)}{J_1(r)}(g-dr\otimes dr)\le\nabla^2 r\le\frac{J'_2(r)}{J_2(r)}(g-dr\otimes dr)$$

on $M\setminus\{o\}$. Hence if we take $f=\frac{1}{2}r^2$, we can derive a lower bound for $\delta_{f,p}$ as follows:

$$\delta_{f,p}\ge\frac{1}{2}\min\left\{1+(m-1-p)r\frac{J'_1\circ r}{J_1\circ r}-pr\frac{J'_2\circ r}{J_2\circ r},\ -1+(m-p)r\frac{J'_1\circ r}{J_1\circ r}-(p-1)r\frac{J'_2\circ r}{J_2\circ r}\right\}.$$

Now in the case:

$$k_1(t)\equiv-1,\qquad k_2(t)=-\left(\frac{m-1-p}{p}\right)^2,$$

it is easy to see that $\delta_{f,p}>1$. As for the case:

$$k_1(r)=\frac{a'}{1+r^2}\quad\left(0\le a'\le\frac{1}{4}\right),\qquad k_2(r)=-\frac{a}{1+r^2}\quad(a\ge0),$$

we can deduce that

$$\frac{J'_1(r)}{J_1(r)}>\frac{A'}{r}\quad\text{if } 0<a'\le\frac{1}{4};\qquad\frac{J'_1(r)}{J_1(r)}=\frac{1}{r}\quad\text{if } a'=0,$$

where we set $A'=\frac{1}{2}\{1+(1-4a')^{1/2}\}$, and also that

$$\frac{J'_2(r)}{J_2(r)}<\frac{A}{r}\quad\text{if } a>0;\qquad\frac{J'_2(r)}{J_2(r)}=\frac{1}{r}\quad\text{if } a=0,$$

where we set $A=\frac{1}{2}\{1+(1+4a)^{1/2}\}$. Therefore we see that

$$\delta_{f,p}>\tfrac{1}{2}\min\{1+(m-1-p)A'-pA,\ -1+(m-p)A'-(p-1)A\}.$$

Hence if a and a' satisfy the condition that the right-hand side of this inequality is non-positive, then $\delta_{f,p}$ is positive on M. Thus Theorem 1 follows from Theorem 3 (I). This completes the proof of Theorem 1.

Suppose now that $p<(m-1)/2$ and

$$-a^2\le K_r\le-1\qquad\text{with}\quad 1\le a<\frac{m-1-p}{p}.$$

Then choosing $f=r$ in the above proof, we have

$$\delta_{r,p}\ge\frac{1}{2}\left\{(m-1-p)\frac{\cosh r}{\sinh r}-ap\frac{\cosh ar}{\sinh ar}\right\}$$

on $M\setminus\{o\}$. Since the right-hand side of this inequality is bounded from below by $(m-1-p-ap)/2$, we obtain

$$\delta_{r,p} > \tfrac{1}{2}(m-1-p-ap) > 0 \,.$$

Although the distance function r is not smooth at the pole o, it is easy to see that the identity (1.2) still holds with $V=\nabla r$, because $|\nabla r|=1$ on $M\setminus\{o\}$. Hence we have

$$\int_M \langle \omega, \Delta\omega \rangle = \int_M |d\omega|^2 + |\delta\omega|^2 \geq \frac{1}{4}(m-1-p-ap)^2 \int_M |\omega|^2$$

for $\omega \in D(\Delta) \subset L^2(\bigwedge^p T^*M \otimes E)$. This proves Theorem 2.

REMARK. The estimate in Theorem 2 is sharper than that in Theorem 3.2 of Donnelly and Xavier [4], because they use $(\frac{1}{2})\{\sum_1^m \lambda_i - 2p\lambda_1\}$ instead of $\delta_{r,p}$, where $\lambda_i : \lambda_m = 0 < \lambda_{m-1} \leq \cdots \leq \lambda_1$ are as before the eigenvalues of the hessian of r.

3. Examples.

In relation to Corollary 4, we first show examples of harmonic maps with finite total energy.

We shall denote by g_ξ a rotationally symmetric Riemannian metric on R^m which can be expressed as

$$g_\xi = dt^2 + \xi(t)^2 d\theta^2$$

in polar coordinates (t, θ), where ξ is a smooth function on $[0, \infty)$ such that

$$\xi(0)=0\,, \quad \xi'(0)=1 \quad \text{and} \quad \xi>0 \ \text{on} \ (0, \infty)\,. \tag{3.1}$$

We write R_ξ^m for the Riemannian manifold (R^m, g_ξ). Let $\{Z_i\}_{i=1,2,3}$ be a left invariant orthonormal frame field on the unit 3-sphere S^3 such that $[Z_1, Z_2]=2Z_3$, $[Z_2, Z_3]=2Z_1$, and $[Z_3, Z_1]=2Z_2$. We denote by Ω_i the dual forms of Z_i and consider a (unitary symmetric) Riemannian metric $g_{\eta,\lambda}$ on R^4 of the form:

$$g_{\eta,\lambda} = dt^2 + \eta(t)^2(\Omega_1^2 + \Omega_2^2) + \lambda(t)^2\Omega_3^2\,,$$

where η and λ also satisfy (3.1). Let $\psi : S^3 \to S^2$ be the Hopf fibering and assume that $Ker(d\psi) = Z_3$. Given a smooth function $\alpha : (0, T) \to (0, \infty)$, define a map $\Psi : R_{\eta,\lambda}^4(T) \setminus \{0\} \to R_\xi^3$ by

$$\Psi(t, \theta) = (\alpha(t), \psi(\theta))\,,$$

where $0 < T \leq +\infty$, $R_{\eta,\lambda}^4 = (R^4, g_{\eta,\lambda})$ and $R_{\eta,\lambda}^4(T)$ stands for the metric ball around the origin of radius T. Then the condition that Ψ be harmonic is described in the following equation on α:

$$\alpha''(t) + p(t)\alpha'(t) - q(t, \alpha) = 0 \qquad \text{on} \quad (0, T)\,, \tag{3.2}$$

where we set

$$p(t)=2\frac{\eta'}{\eta}+\frac{\lambda'}{\lambda}, \qquad q(t,s)=8\frac{\xi'(s)\xi(s)}{\eta(t)^2}.$$

Since the energy density of Ψ is given by

$$e(\Psi)=\alpha'(t)^2+4\frac{\xi(\alpha(t))^2}{\eta(t)^2},$$

Ψ: $R^4_{\eta,\lambda}(T)\setminus\{0\}\to R^3_\xi$ has finite total energy if and only if

$$\int_0^T \alpha'(t)^2\eta(t)^2\lambda(t)\,dt<+\infty, \qquad \int_0^T \xi(\alpha(t))^2\lambda(t)\,dt<+\infty.$$

In what follows, we assume that $0\leq\xi'<C$ for some constant C. Then given two positive numbers t_0 and s_0, there exists a monotone increasing solution α: $(0,\infty)\to(0,\infty)$ of equation (3.2) with $T=\infty$ such that $\alpha(t_0)=s_0$ and $\lim_{t\to 0}\alpha(t)=0$ (cf. [9]). Hence the fundamental regularity theory of harmonic mappings asserts that Ψ defines a harmonic map of $R^4_{\eta,\lambda}$ into R^3_ξ.

Now in the following two cases, the harmonic map Ψ: $R^4_{\eta,\lambda}\to R^3_\xi$ has finite total energy.

Case 1: $\eta(t)=c_1t^a$ and $\lambda(t)=c_2t^b$ for large t, where a, b and c_* are constants chosen in such a way that

$$c_*>0, \quad 2a+b-1>0, \quad b+1<0.$$

Case 2: $\eta(t)=c_1e^{at}$ and $\lambda(t)=c_2e^{bt}$ for large t, where a, b and c_* are constants chosen in such a way that

$$c_*>0, \quad 2a+b>0, \quad b<0.$$

REMARKS. (i) In the both cases, the harmonic maps have bounded images. We recall here a result by Cheng and Yau [3] saying that a complete Riemannian manifold admits no nonconstant subharmonic functions bounded from above, if the volume of the metric ball of radius r around a basepoint is bounded from above by Cr^2 for some constant C.

(ii) The radial curvature K_r of the plane spanned by the radial vector $\partial/\partial r$ and a vector $X=x_1Z_1/\eta+x_2Z_2/\eta+x_3Z_3/\lambda$ is given by

$$K_r=-\frac{\eta''}{\eta}(x_1^2+x_2^2)-\frac{\lambda''}{\lambda}x_3^2.$$

Hence in Case 1, the radial curvature K_r satisfies

$$-\frac{\max\{a(a-1),b(b-1)\}}{r^2}\leq K_r\leq-\frac{\min\{a(a-1),b(b-1)\}}{r^2}$$

outside a compact set. Moreover, the sectional curvature of $R^4_{\eta,\lambda}$ decays to zero at a quadratic rate $O(r^{-2})$. As for Case (2), the radial curvature K_r satisfies

$$-\max\{a^2, b^2\} \le K_r \le -\min\{a^2, b^2\}$$

outside a compact set.

Finally we shall consider a warped product $M = \boldsymbol{R} \times_{w^2} N$ of $\boldsymbol{R}$ and a compact Riemannian manifold N of dimension $m-1$ with warping function w, $0 < w \in C^\infty(\boldsymbol{R})$. Then we have a harmonic function h on M defined by

$$h(t, x) = a + b\int_0^t \frac{ds}{w^{m-1}(s)}$$

for $(t, x) \in M$, where a and b are constants. The Dirichlet integral of h is given by

$$\int_M |\nabla h|^2 = \mathrm{Vol}(N)\int_{-\infty}^{\infty}\left|\frac{du}{dt}\right|^2 w^{m-1}(t)dt = \mathrm{Vol}(N)\int_{-\infty}^{\infty}\frac{dt}{w^{m-1}(t)}\,.$$

Hence M admits nontrivial square integrable harmonic one-forms if

$$\int_{-\infty}^{\infty}\frac{dt}{w^{m-1}(t)}dt < +\infty\,.$$

We observe that the eigenvalues of the hessian of a function $f(t, x) = t$ are given by $\{0, w'(t)/w(t), \cdots, w'(t)/w(t)\}$, and hence those of the hessian of $f(t, x) = |t|$ on $M \setminus \{0\} \times N$ are 0, $\mathrm{sign}(t)w'(t)/w(t), \cdots, \mathrm{sign}(t)w'(t)/w(t)$.

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