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2-Type Surfaces in S_1^3 and H_1^3

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1. Introduction.

In a series of recent papers ([4], [5], [14]) the technique of finite type immersions (see [9] for details) has been sistematically used to characterize certain interesting families of Riemannian submanifolds. The authors have used these arguments to try to classify surfaces satisfying certain characteristic differential equations in the Lorentzian space forms (see [2], [3] and [12]). It is well known that the shape operator of a pseudo-Riemannian surface does not need to be diagonalizable; because of this fact there are substantial differences between the definite and indefinite cases. Actually, it is possible to find a wide family of examples of surfaces in indefinite space forms having no Riemannian counterparts; the *B*-scrolls ([10] and [13]) and the complex circles ([16]) are some of these examples.

The finite type immersion tool allows to discover certain hidden facts in non flat Lorentzian ambient spaces $\bar{M}_1^3(c)$, with $c=\pm 1$. For instance, a totally umbilical surface does not need to be of 1-type; however both conditions are equivalent if and only if the surface is non flat. Actually, the following two quite interesting facts can be obtained from the pseudo-Riemannian version of Takahashi's theorem ([6] and [17]): (i) a surface in $\bar{M}_1^3$ is of 1-type if and only if it is either minimal or non flat totally umbilical in $\bar{M}_1^3$; and (ii) there exist flat totally umbilical surfaces in $\bar{M}_1^3$ which are *biharmonic*, i.e. its mean curvature vector field is harmonic, and therefore they are of infinite type.

On the other hand, *B*-scrolls as well as complex circles come out as surfaces in $\bar{M}_1^3$; it seems then reasonable to try to characterize them according to its finite type character. It should be noticed that *B*-scrolls already appeared in studying surfaces satisfying the condition $\Delta H=\lambda H$ in Lorentzian space forms.

In a more general situation, we look for 2-type isometric immersions into $\bar{M}_1^3$. The equation $\Delta H=\lambda H$ allows to reach only up to surfaces of 2-type with a zero eigenvalue

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(the so called *null 2-type surfaces*); therefore a natural extension of that equation should be considered. On the other hand, going back to the Riemannian case, it is known that the only 2-type surfaces in S^3 are the non minimal products of two plane circles ([5], [7] and [14]). Then a first question naturally arises as follows:

What is the family of 2-type surfaces in S_1^3 or H_1^3?

We know that there exist no surfaces of null 2-type in S^3 and H^3 ([11]). Comparing the Riemannian and the Lorentzian cases it could be interesting to state this other question:

Are there null 2-type surfaces in S_1^3 and H_1^3? If the answer is affirmative, *Would it be possible to compare the size of this family with that of 2-type?*

It is worthwhile pointing out that the key to obtain the characterization of 2-type surfaces is to show that they are isoparametric. This property can be deduced from the following more general result (see Lemma 2.1): A surface in S_1^3 or H_1^3 satisfying the equation $\Delta H = \lambda H + \mu(x - x_0)$ has constant mean curvature. To get a complete classification of 2-type surfaces, the isoparametric ones are studied in section 4 (see Proposition 4.1). As a consequence of above results we solve all stated questions (see Theorems 4.2 and 4.3 and Corollary 4.4).

The first question, for space-like surfaces, has been solved in [8] and, in some sense, the last two questions have been also considered in [15] for space-like surfaces in L^3 and S_1^3.

2. Preliminaries.

Let us denote by $\bar{M}_1^3$ the standard model of a 3-dimensional Lorentz space with constant curvature $c = \pm 1$, i.e., the De Sitter space $S_1^3 \subset R_1^4$ and the anti De Sitter space $H_1^3 \subset R_2^4$, respectively. Let R_t^4 be the corresponding pseudo-Euclidean space where $\bar{M}_1^3$ is lying. Let $x: M_s^2 \to \bar{M}_1^3 \subset R_t^4$ be an isometric immersion of a surface M_s^2 into $\bar{M}_1^3$ and let N be a unit vector field normal to M_s^2 in $\bar{M}_1^3$. Then we have

$$H = \alpha N - cx, \tag{2.1}$$

where H is the mean curvature vector field of M_s^2 in R_t^4 and α is the mean curvature of M_s^2 in $\bar{M}_1^3$.

By supposing that M_s^2 is a 2-type surface, it is well known that

$$\Delta H = \lambda H + \mu(x - x_0), \tag{2.2}$$

where x_0 is a constant vector and λ and μ are two real constants such that the polynomial $t^2 - \lambda t + 2\mu$ has exactly two distinct real roots. Now, from (2.1) and the formula for ΔH given in [6] we find that (2.2) holds if and only if the following set of equations is valid

$$(\Delta H)^T = -\mu x_0^T = 2S(\nabla\alpha) + 2\varepsilon\alpha\nabla\alpha\ , \tag{2.3}$$

$$\lambda\alpha - \varepsilon\mu\langle x_0, N\rangle = \Delta\alpha + \varepsilon\alpha \operatorname{tr}(S^2) + 2c\alpha\ , \tag{2.4}$$

$$\lambda - c\mu + \mu\langle x_0, x\rangle = 2(c + \varepsilon\alpha^2)\ , \tag{2.5}$$

where $(\ \ )^T$ is written down for tangential components, S stands for the shape operator of M_s^2 in $\bar{M}_1^3$, $\nabla\alpha$ is the gradient of α and $\varepsilon = \langle N, N\rangle$.

Then, for any vector field X tangent to M_s^2, we have from (2.3) that $\langle \Delta H, X\rangle = -\mu X(\langle x_0, x\rangle)$. By using (2.5) we get $\mu X(\langle x_0, x\rangle) = 2\varepsilon X(\alpha^2)$, which along with (2.3) leads to $(\Delta H)^T = -4\varepsilon\alpha\nabla\alpha$ and

$$S(\nabla\alpha) = -3\varepsilon\alpha\nabla\alpha\ . \tag{2.6}$$

Now we are ready to prove the following useful result.

LEMMA 2.1. *Let $x: M_s^2 \to \bar{M}_1^3 \subset \mathbf{R}_t^4$ be an isometric immersion satisfying the equation $\Delta H = \lambda H + \mu(x - x_0)$. Then M_s^2 has constant mean curvature in $\bar{M}_1^3$.*

PROOF. Our goal is to prove that the set $\mathcal{U} = \{p \in M_s^2 : \nabla\alpha^2(p) \neq 0\}$ is empty. Otherwise $\mathcal{U}$ is an open subset of M_s^2 where, by (2.6), $\nabla\alpha$ is an eigenvector of S with associate non-zero eigenvalue $-3\varepsilon\alpha$. Therefore, the shape operator is diagonalizable on $\mathcal{U}$ and we can choose a local orthonormal frame $\{E_1, E_2, E_3, E_4\}$, such that $E_3 = N$, $E_4 = x$ and $\{E_1, E_2\}$ are eigenvectors of S, E_1 being parallel to $\nabla\alpha$, $SE_1 = -3\varepsilon\alpha E_1$ and $SE_2 = 5\varepsilon\alpha E_2$. Let $\{\omega^i\}$ and $\{\omega_i^j\}$ be the dual frame and the connection forms of the chosen frame, respectively. Then we see that

$$\omega_3^1 = 3\varepsilon\alpha\omega^1\ , \tag{2.7}$$

$$\omega_3^2 = -5\varepsilon\alpha\omega^2\ , \tag{2.8}$$

$$\mathrm{d}\alpha = \varepsilon_1 E_1(\alpha)\omega^1\ , \tag{2.9}$$

where $\varepsilon_i = \langle E_i, E_i\rangle$.

Taking exterior differentiation in (2.7) and using the structure equations, we have $\mathrm{d}\omega^1 = 0$ and thus there locally exists a function u such that $\omega^1 = \mathrm{d}u$. From (2.9) we get $\mathrm{d}\alpha \wedge \mathrm{d}u = 0$ and therefore α depends on u, $\alpha = \alpha(u)$, and $E_1(\alpha) = \varepsilon_1\alpha'$.

Differentiating in (2.8) and using again the structure equations we deduce that

$$8\alpha\omega_2^1 = 5\varepsilon_1\alpha'\omega^2\ . \tag{2.10}$$

A straightforward computation from (2.10) yields the following differential equation

$$\alpha\alpha'' - \frac{13}{8}(\alpha')^2 - \frac{8}{5}c\varepsilon_1\alpha^2 + 24\varepsilon\varepsilon_1\alpha^4 = 0\ , \tag{2.11}$$

whose solution in the new variable $\beta = (\alpha')^2$ is given by

$$\beta = C_1 \alpha^{13/4} - 64\varepsilon\varepsilon_1 \alpha^4 - \frac{64}{65} c\varepsilon_1 \alpha^2 , \tag{2.12}$$

C_1 being a real constant.

On the other hand, from the expression of $\Delta\alpha$ in $\{E_1, E_2\}$, the fact that E_1 is parallel to $\nabla\alpha$ and (2.10) we have

$$\alpha\Delta\alpha = -\varepsilon_1 \alpha\alpha'' + \frac{5}{8}\varepsilon_1 (\alpha')^2 . \tag{2.13}$$

Now from (2.3) and (2.6) we deduce that

$$\mu X(\langle x_0, N\rangle) = \langle \Delta H, SX\rangle = 4X(\alpha^3) ,$$

for any tangent vector field X, which along with (2.4) and (2.13) leads to

$$\alpha\alpha'' - \frac{5}{8}(\alpha')^2 + (\lambda - 2c)\varepsilon_1 \alpha^2 - 38\varepsilon\varepsilon_1 \alpha^4 + C_2 \alpha , \tag{2.14}$$

C_2 being a real constant. Then from (2.11) and (2.14) we have

$$\beta = 62\varepsilon\varepsilon_1 \alpha^4 + \left(\frac{2}{5}c - \lambda\right)\varepsilon_1 \alpha^2 - C_2 \alpha ,$$

and by (2.12) and this equation we deduce that α is locally constant on $\mathscr{U}$, which contradicts its own definition.

3. Some examples.

In this section we will describe some examples of surfaces into $\bar{M}^3_1$ which will be useful later in order to give the classification results. We will show examples not only of 2-type surfaces but also surfaces satisfying the condition (2.2) and being of infinite type.

EXAMPLE 3.1. An easy computation shows that the following pseudo-Riemannian products, with an appropriate choice of $r > 0$ to avoid minimality,

1) $\boldsymbol{H}^1(-r) \times \boldsymbol{S}^1(\sqrt{1+r^2})$ and $\boldsymbol{S}^1_1(r) \times \boldsymbol{S}^1(\sqrt{1-r^2})$ into $\boldsymbol{S}^3_1$, and
2) $\boldsymbol{H}^1_1(-r) \times \boldsymbol{S}^1(\sqrt{r^2-1})$, $\boldsymbol{S}^1_1(r) \times \boldsymbol{H}^1(-\sqrt{1+r^2})$ and $\boldsymbol{H}^1(-r) \times \boldsymbol{H}^1(-\sqrt{1-r^2})$ into $\boldsymbol{H}^3_1$,

are all 2-type surfaces into $\bar{M}^3_1$. We will refer them as the *non-minimal standard products*. Notice that all of them have diagonalizable shape operators.

EXAMPLE 3.2. Let $\gamma(s)$ be a null curve in $\bar{M}^3_1 \subset \boldsymbol{R}^4_t$ with an associated Cartan frame $\{A, B, C\}$, i.e., $\{A, B, C\}$ is a pseudo-orthonormal frame of vector fields along $\gamma(s)$

$$\langle A, A\rangle = \langle B, B\rangle = 0\,, \qquad \langle A, B\rangle = -1\,,$$
$$\langle A, C\rangle = \langle B, C\rangle = 0\,, \qquad \langle C, C\rangle = 1\,,$$

such that

$$\dot{\gamma}(s) = A(s)\,,$$
$$\dot{C}(s) = -aA(s) - k(s)B(s)\,,$$

where a is a nonzero constant and $k(s) \neq 0$ for all s. Then the map x: $(s, u) \rightarrow \gamma(s) + uB(s)$ parametrizes a Lorentzian surface into $\bar{M}^3_1$ which is called a *B-scroll* (see [10]).

It is not difficult to see that a unit normal vector field is given by

$$N(s, u) = -auB(s) + C(s)\,,$$

and the shape operator can be put in the usual frame $\{\partial x/\partial s, \partial x/\partial u\}$ as

$$S = \begin{pmatrix} a & 0 \\ k(s) & a \end{pmatrix}.$$

Thus the B-scroll has non-diagonalizable shape operator with minimal polynomial $P_S(t) = (t-a)^2$. It has constant mean and Gaussian curvatures $\alpha = a$ and $K = c + a^2$, respectively, and satisfies $\Delta H = 2KH$. Therefore, one sees that a non-flat B-scroll is a null 2-type surface into $\bar{M}^3_1$, whereas a flat B-scroll, i.e., $c = -1$ and $a^2 = 1$, is a biharmonic surface into H^3_1 and of infinite type.

EXAMPLE 3.3. Let a and b be two real numbers such that $a^2 - b^2 = -1$ and $ab \neq 0$. Then the map $x : R^2_1 \rightarrow H^3_1 \subset R^4_2$, $x = (x^1, x^2, x^3, x^4)$, given by

$$x^1(u_1, u_2) = b\cosh u_2 \cos u_1 - a\sinh u_2 \sin u_1\,,$$
$$x^2(u_1, u_2) = a\sinh u_2 \cos u_1 + b\cosh u_2 \sin u_1\,,$$
$$x^3(u_1, u_2) = a\cosh u_2 \cos u_1 + b\sinh u_2 \sin u_1\,,$$
$$x^4(u_1, u_2) = a\cosh u_2 \sin u_1 - b\sinh u_2 \cos u_1\,,$$

where (u_1, u_2) is the usual coordinate system in R^2_1, parametrizes a non-minimal flat surface into H^3_1 whose shape operator is given, in the usual frame $\{\partial x/\partial u_1, \partial x/\partial u_2\}$, by

$$S = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

with $\alpha = 2ab/(a^2 + b^2)$ and $\beta = -1/(a^2 + b^2)$. Magid, [16], called this surface a *complex circle* of radius $a + bi$.

It is not difficult to show that a complex circle satisfies the condition (2.2) with $x_0 = 0$, $\lambda = -4/(a^2 + b^2)^2$ and $\mu = 2/(a^2 + b^2)^2$. However, it is not a finite type surface because the discriminant of $t^2 - \lambda t + 2\mu$ vanishes.

REMARK 3.4. From the pseudo-Riemannian version of Takahashi's theorem one knows that M_s^2 is a 1-type surface if and only if it is minimal in $\bar{M}_1^3$ or an open piece of a non-flat totally umbilical surface in $\bar{M}_1^3$, i.e., M_s^2 is nothing but $S_s^2(r)$ or $H_s^2(r)$. It is worth pointing out that there exists a flat totally umbilical surface in both S_1^3 and H_1^3, which is explicitly given by $x : R_s^2 \to \bar{M}_1^3 \subset R_{s+1}^4$, $x = f - x_0$, x_0 being a fixed vector and $f : R_s^2 \to R_{s+1}^4$ the function defined by $f(u_1, u_2) = (q(u_1, u_2), u_1, u_2, q(u_1, u_2))$, where $q(u) = a\langle u, u\rangle + \langle b, u\rangle + c$, $a \neq 0$. This surface is of infinite type with $\Delta x = (-4a, 0, 0, -4a)$.

4. Main results.

Let M_s^2 be a 2-type surface into $\bar{M}_1^3$. As a consequence of Lemma 2.1 we know that M_s^2 has non-zero constant mean curvature and then from (2.4) and (2.5) we deduce that $\mathrm{tr}(S^2)$ is constant, i.e., M_s^2 is a non-minimal isoparametric surface into $\bar{M}_1^3$.

Now let us discuss according to the character of its shape operator S. First, if S is diagonalizable we have from Remark 3.4 that M_s^2 is not totally umbilical and by [1] it is an open piece of one of the non-minimal standard products. Secondly, if S is not diagonalizable with a double real eigenvalue, then M_1^2 can be locally parametrized as a B-scroll over a null curve, as it is shown in the following result.

PROPOSITION 4.1. *Let M_1^2 be a Lorentzian surface in $\bar{M}_1^3 \subset R_t^4$ and let $(t-a)^2$, a being a non-zero constant, be the minimal polynomial of its shape operator. Then, in a neighborhood of any point, M_1^2 is a B-scroll over a null curve.*

PROOF. Pick a point p in M_1^2 and choose a pseudo-orthonormal frame $\{A, B\}$ of tangent vector fields in a neighborhood of p such that

$$SA = aA + kB\,,$$
$$SB = aB\,,$$

where $k \neq 0$. Let N be a unit vector field normal to M_1^2 into $\bar{M}_1^3$. Considering M_1^2 as an embedded surface into $\bar{M}_1^3$, we can take an integral curve $\gamma(s)$ of A starting from p. For short, let us write $A(s) = A(\gamma(s))$, $B(s) = B(\gamma(s))$, $C(s) = N(\gamma(s))$ and $k(s) = k(\gamma(s))$. Then

$$\dot{C}(s) = \frac{\tilde{D}C}{ds}(s) = -aA(s) - k(s)B(s)\,.$$

For each s, let $x_s(t)$ denote an integral curve of B starting from $\gamma(s)$. Then taking covariant derivate we get

$$\frac{\tilde{D}B}{dt}(x_s(t)) = \tilde{\nabla}_{\dot{x}_s(t)} B(x_s(t)) = \tilde{\nabla}_B B(x_s(t)) = \nabla_B B(x_s(t))\,.$$

By using now Codazzi's equation we have $\nabla_B B$ is in span $\{B\}$, and then the above equation yields

$$\frac{\tilde{D}B}{dt}(x_s(t))=f(x_s(t))B(x_s(t)),$$

for a certain diferentiable function f. It is not difficult to see that the solution of that differential equation is given by

$$B(x_s(t))=g_s(t)B(s),$$

for a certain positive function $g_s(t)$ with $g_s(0)=1$. Then we get

$$x_s(t)=\gamma(s)+\int_0^t g_s(v)dv\,B(s),$$

and M_1^2 is, in a neighborhood of p, a B-scroll as in Example 3.2.

Finally, suppose S has complex eigenvalues and choose a local orthonormal frame $\{E_1, E_2\}$ such that

$$SE_1=aE_1+bE_2,$$
$$SE_2=-bE_1+aE_2$$

where a and b are two non-zero constants. Now from Codazzi's equations we deduce $\omega_2^1=0$ and therefore M_1^2 is a flat Lorentzian surface in $\boldsymbol{H}_1^3$ with parallel second fundamental form in $\boldsymbol{R}_2^4$. Thus from [16] M_1^2 is congruent to a complex circle, which is not of 2-type as we have already seen.

Summing up we have proved the following main results, which solve both questions stated in §1.

THEOREM 4.2. *A surface M_s^2 into $\boldsymbol{S}_1^3$ is of 2-type if and only if it is an open piece of one of the following surfaces*:

1) $\boldsymbol{H}^1(-r)\times\boldsymbol{S}^1(\sqrt{1+r^2})$,
2) $\boldsymbol{S}_1^1(r)\times\boldsymbol{S}^1(\sqrt{1-r^2})$, *and*
3) *a B-scroll over a null curve.*

THEOREM 4.3. *A surface M_s^2 into $\boldsymbol{H}_1^3$ is of 2-type if and only if it is an open piece of one of the following surfaces*:

1) $\boldsymbol{H}_1^1(-r)\times\boldsymbol{S}^1(\sqrt{r^2-1})$,
2) $\boldsymbol{S}_1^1(r)\times\boldsymbol{H}^1(-\sqrt{1+r^2})$,
3) $\boldsymbol{H}^1(-r)\times\boldsymbol{H}^1(-\sqrt{1-r^2})$, *and*
4) *a non-flat B-scroll over a null curve.*

Putting together the above theorems, we see that the families of 2-type surfaces into $\boldsymbol{S}_1^3$ and $\boldsymbol{H}_1^3$ are essentially the same. Actually, it consists of non-minimal standard products and non-flat B-scrolls over null curves. Once again B-scrolls make the difference with regard to the Riemannian case and, as we point out in the following corollary, they solve the second question.

COROLLARY 4.4. *The only null 2-type surfaces in S_1^3 and H_1^3 are non-flat B-scrolls over null curves.*

References

[1] N. ABE, N. KOIKE and S. YAMAGUCHI, Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form, Yokohama Math. J. **35** (1987), 123–136.

[2] L. J. ALÍAS, A. FERRÁNDEZ and P. LUCAS, Classifying pseudo-Riemannian hypersurfaces by means of certain characteristic differential equations, *The Problem of Plateau: A tribute to Jesse Douglas and Tibor Radó* (ed. by Th. M. Rassias), 53–75, World Scientific (1992).

[3] L. J. ALÍAS, A. FERRÁNDEZ and P. LUCAS, Surfaces in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta x = Ax + B$, Pacific J. Math. **156** (1992), 201–208.

[4] C. BAIKOUSSIS and D. E. BLAIR, 2-Type integral surfaces in S^5, Tokyo J. Math. **14** (1991), 345–356.

[5] M. BARROS and O. J. GARAY, 2-type surfaces in S^3, Geom. Dedicata **24** (1987), 329–336.

[6] B. Y. CHEN, Finite-type pseudo-Riemannian submanifolds, Tamkang J. Math. **17** (1986), 137–151.

[7] B. Y. CHEN, Local rigidity theorems of 2-type hypersurfaces in a hypersphere, Nagoya Math. J. **122** (1991).

[8] B. Y. CHEN, Submanifolds of finite type in hyperbolic spaces, preprint (1991).

[9] B. Y. CHEN, *Total Mean Curvature and Submanifolds of Finite Type*, Series in Pure Math. **1**, World Scientific (1984).

[10] M. DAJCZER and K. NOMIZU, On flat surfaces in S_1^3 *and* H_1^3, *Manifolds and Lie Groups*, 71–108, Univ. Notre Dame, Birkhäuser (1981).

[11] A. FERRÁNDEZ and P. LUCAS, Null finite type hypersurfaces in space forms, Kodai Math. J. **14** (1991), 406–419.

[12] A. FERRÁNDEZ and P. LUCAS, On surfaces in the 3-dimensional Lorentz-Minkowski space, Pacific J. Math. **152** (1992), 93–100.

[13] L. GRAVES, Codimension one isometric immersions between Lorentz spaces, Trans. Amer. Math. Soc. **252** (1979), 367–392.

[14] T. HASANIS and T. VLACHOS, A local classification of 2-type surfaces in S^3, Proc. Amer. Math. Soc. **112** (1991), 533–538.

[15] C. S. HOUH, Null 2-type surfaces in R_1^3 and in S_1^3, *Algebra, Analysis and Geometry* (ed. by M. C. Kang and K. W. Lih), 19–37, World Scientific (1988).

[16] M. A. MAGID, Isometric immersions of Lorentz space with parallel second fundamental forms, Tsukuba J. Math. **8** (1984), 31–54.

[17] S. MARKVORSEN, A characteristic eigenfunction for minimal hypersurfaces in space forms, Math. Z. **202** (1989), 375–382.

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