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On the Existence and Conformal Equivalence of Extremal Maps of Various Types

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1. Introduction.

Let (M, g) and (N, h) be compact smooth Riemannian manifolds of dimension m and n, respectively and $f: M \to N$ be a smooth map. As a natural generalization of a harmonic map, an exponentially harmonic map and a p-harmonic map $(1 < p < \infty)$ are defined by extremals of the functionals $\int_M e^{|df|^2/2} v_g$ and $\int_M |df|^p v_g$, respectively. And a $(1, p)$-harmonic map is defined by an extremal of the less degenerate functional $\int_M (1 + |df|^2)^{p/2} v_g$ as in [D-F]. We call them extremal maps of various types.

We will study the following problem; for a given smooth map $\varphi: M \to N$, does there exist an extremal map f such that f is homotopic to φ?

Eells and Ferreira [E-F] gave some positive answer to the above problem allowing conformal change of the Riemannian metric g in the case of harmonic maps. Hong [H] also solved in the case of exponentially harmonic, by the method of the conformal equivalence of harmonic maps and exponentially harmonic maps. On the other hand, there are some results due to H. Takeuchi [T] about the conformal equivalence of p-harmonic maps and p'-harmonic for some p and p'.

In this paper, we give a positive answer to the above problem allowing conformal change of g in the case of $(1, p)$-harmonic for any p $(1 < p < \infty)$, and obtain conformal equivalence among extremal maps of various types each others.

We also study their stability under the conformal change of g in the last section.

Now, we will illustrate our results in this paper as the following diagram.

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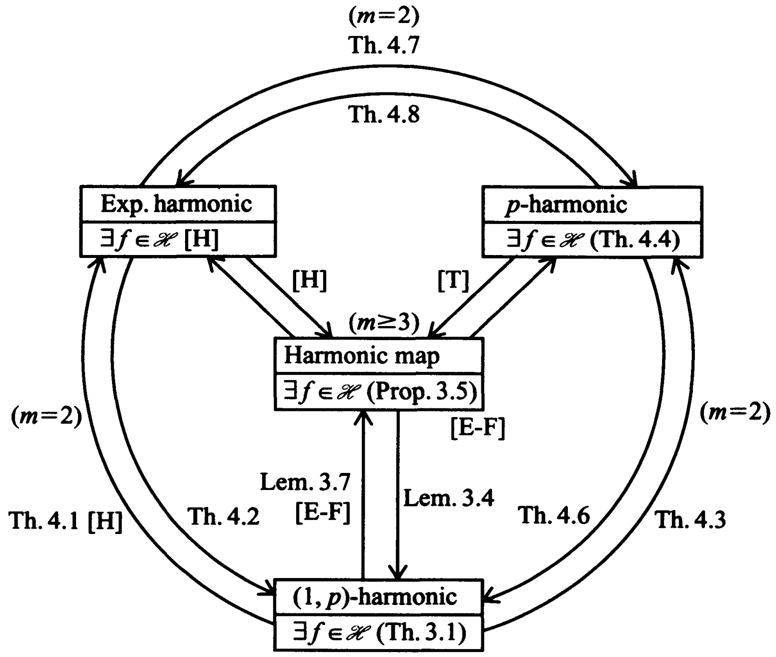

In this diagram, the arrows indicate conformal equivalence between two of extremal maps. And the under part of each box indicates existence theorem of each extremal map.

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2. Preliminaries.

Let (M, g) and (N, h) be compact smooth Riemannian manifolds of dimension m and n, respectively. For a smooth map $f:(M, g)\to(N, h)$, we define the *energy*, *exponential energy*, *p-energy* and *(1, p)-energy* functionals of f by

$$E(f)=\int_M |df|^2 v_g\,,\qquad E_e(f)=\int_M e^{|df|^2/2} v_g\,,$$

$$E_p(f)=\int_M |df|^p v_g\,,\quad \text{and}\quad E_p^1(f)=\int_M (1+|df|^2)^{p/2} v_g\,,$$

respectively, where $|df|$ denotes the Hilbert-Schmidt norm of the differential $df\in\Gamma(T^*M\otimes f^{-1}TN)$ with respect to g and h, and v_g is the volume element of (M, g) and p is a real number with $1<p<\infty$.

DEFINITION 2.1. A smooth map f is said to be *harmonic*, *exponential harmonic*, *p-harmonic* and *(1, p)-harmonic* if it is an extremal of the functionals $E(f)$, $E_e(f)$, $E_p(f)$

and $E_p^1(f)$, respectively.

Direct computation of the first variation of the functionals yields the Euler-Lagrange equation;

$$div(Adf)=0\,.$$

In local coordinates (x^i) and (f^α) on M and N, this is equivalent to

$$-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(A\sqrt{g}\,g^{ij}\frac{\partial f^\gamma}{\partial x^j}\right)=Ag^{ij\,N}\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\qquad(1\le\gamma\le n)\,,\tag{2.1}$$

where ${}^N\Gamma^\gamma_{\alpha\beta}$ are Christoffel's symbols of the connection on N, and A stands for

$$1\,,\quad e^{|df|^2/2}\,,\quad |df|^{p-2}\quad\text{and}\quad(1+|df|^2)^{(p-2)/2}$$

accordingly for f to be harmonic, exponentially harmonic, p-harmonic and $(1,p)$-harmonic, respectively. Thus if a smooth map $f\colon(M,g)\to(N,h)$ satisfies (2.1), we call it an extremal map.

Put $\tilde{g}=\psi g$ for a smooth positive function $\psi\colon M\to\boldsymbol{R}$ and a smooth Riemannian metric g on M. Substituting $g=\psi^{-1}\tilde{g}$ into (2.1), we have

$$-\frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}\left(A\psi^{(2-m)/2}\sqrt{\tilde{g}}\,\tilde{g}^{ij}\frac{\partial f^\gamma}{\partial x^j}\right)=A\psi^{(2-m)/2}\tilde{g}^{ij\,N}\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\qquad(1\le\gamma\le n)\,.\tag{2.2}$$

3. Main theorem.

Let (M,g) and (N,h) be compact smooth Riemannian manifolds of dimension m and n, respectively, and $\mathscr{H}$ be a homotopy class of a smooth given map $\varphi\colon(M,g)\to(N,h)$.

Our first main theorem is as follows:

THEOREM 3.1. *Suppose $m=\dim M\ge 3$. Then there exists a smooth metric $\tilde{g}$ conformally equivalent to g and a map $f\in\mathscr{H}$ such that $f\colon(M,\tilde{g})\to(N,h)$ is $(1,p)$-harmonic for any p $(1<p<\infty)$.*

In order to prove Theorem 3.1, we prepare the following proposition and some lemmata.

LEMMA 3.2. *Suppose that a smooth map $f\colon(M,g)\to(N,h)$ is harmonic. Then for any $\varepsilon>0$, there exists a smooth metric $\tilde{g}$ conformally equivalent to g such that $f\colon(M,\tilde{g})\to(N,h)$ is harmonic and $|df|^2_{\tilde{g}}\le\varepsilon$.*

For the proof, refer to [H, Lemma 2, p. 489].

LEMMA 3.3. *Define a function y in the variable ψ by $y(\psi)=\psi^\alpha-\psi$. Then the function $y(\psi)$ has a positive smooth inverse function $\Phi(y)$ on the following interval depending on* *the real number* α.

i) *When* $\alpha>1$, $y(\psi)$ *has a positive smooth inverse function on* $y\geq 0$.

ii) *When* $0<\alpha<1$, $y(\psi)$ *has the one on* $0\leq y\leq(1/\alpha)^{1/(\alpha-1)}(1/\alpha-1)$.

iii) *When* $\alpha=0$, $y(\psi)$ *has the one on* $0\leq y<1$.

iv) *When* $\alpha<0$, $y(\psi)$ *has the one on* $y\geq 0$.

PROOF. Calculating the derivative of $y(\psi)$ and drawing the graph on each case, we can easily prove this lemma.

LEMMA 3.4. *Suppose* $m=\dim M\geq 3$ *and* $f:(M,g)\to(N,h)$ *is harmonic. Then there exists a smooth metric* $\tilde{g}$ *conformally equivalent to* g *such that* $f:(M,\tilde{g})\to(N,h)$ *is* $(1,p)$*-harmonic for any* p $(1<p<\infty)$.

PROOF. Let $f:(M,g)\to(N,h)$ be a harmonic map. Putting $A=1$ at (2.2), we have

$$-\frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}\left(\psi^{(2-m)/2}\sqrt{\tilde{g}}\,\tilde{g}^{ij}\frac{\partial f^\gamma}{\partial x^j}\right)=\psi^{(2-m)/2}\tilde{g}^{ij}{}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\,.$$

To get the $(1,p)$-harmonic map $f:(M,\tilde{g})\to(N,h)$, we want to get a positive smooth function ψ such that $\psi^{(2-m)/2}=(1+|df|^2_{\tilde{g}})^{(p-2)/2}$. Since

$$|df|^2_{\tilde{g}}=\tilde{g}^{ij}h_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}=\psi^{-1}|df|^2\,,$$

we need

$$\psi^{(p-m)/(p-2)}-\psi=|df|^2\,.$$

Putting $\alpha:=(p-m)/(p-2)$, where $p\neq 2$, the above equality coincides with

$$\psi^\alpha-\psi=|df|^2\,. \tag{3.1}$$

Setting $\psi:=\Phi(|df|^2)$, where $\Phi(y)$ is given in Lemma 3.3, it is easily verified that ψ is a positive smooth function and satisfies (3.1) in any cases in Lemma 3.3.

To check the conditions of Lemma 3.3, note that since $m=\dim\geq 3$, the cases $\alpha>1$, $0<\alpha<1$, $\alpha=0$ and $\alpha<0$ in Lemma 3.3 correspond to $p<2<m$, $p>m>2$, $p=m\geq 3$ and $2<p<m$, respectively. We also note that in the cases $0<\alpha<1$ and $\alpha=0$, we can suppose $|df|^2\leq\varepsilon<(1/\alpha)^{1/(\alpha-1)}(1/\alpha-1)$ and $0\leq|df|^2\leq\varepsilon<1$, respectively, by Lemma 3.2.

Substituting ψ obtained above and $A=1$ into (2.2), we have the Euler-Lagrange equation for $(1,p)$-harmonic maps with respect to $\tilde{g}$ and h. Since $(1,2)$-harmonic map is nothing but harmonic map, this lemma is valid also in the case $p=2$. q.e.d.

PROPOSITION 3.5. *Suppose* $m=\dim\geq 3$. *Then there exists a smooth metric* $\tilde{g}$ *conformally equivalent to* g *and a map* $f\in\mathscr{H}$ *such that* $f:(M,\tilde{g})\to(N,h)$ *is harmonic.*

PROOF. We prove Proposition 3.5 in an analogous way as in [E-F].

LEMMA 3.6. *Suppose* $p/2>m=\dim M$. *Then there exists a smooth map* $f\in\mathscr{H}$ *such* *that* $f:(M,g)\to(N,h)$ *is* $(1,p)$*-harmonic.*

For the proof, refer to [E-F, Step 1, p. 160].

LEMMA 3.7. *Suppose* $m=\dim\geq 3$ *and* $f:(M,g)\to(N,h)$ *is* $(1,p)$*-harmonic for* $p/2>m=\dim M$*. Then there exists a smooth metric* $\tilde{g}$ *conformally equivalent to* g *such that* $f:(M,\tilde{g})\to(N,h)$ *is harmonic.*

PROOF. Putting $A=(1+|df|^2)^{(p-2)/2}$ and $\psi:=(1+|df|^2)^{(p-2)/(m-2)}$ at (2.2), we can prove that ψ is a positive smooth function and satisfies

$$(1+|df|^2)^{(p-2)/2}\cdot\psi^{(2-m)/2}=1\ .$$

So (2.2) implies $f:(M,\tilde{g})\to(N,h)$ is harmonic. q.e.d.

Combining Lemma 3.6 with Lemma 3.7, we can prove Proposition 3.5. q.e.d.

PROOF OF THEOREM 3.1. In the case $p/2>m$, Lemma 3.6 implies Theorem 3.1 without conformal change of metric g. In the other case, applying Lemma 3.4 to the map obtained in Proposition 3.5, we get Theorem 3.1. q.e.d.

Here are a few observations about the conformal equivalence of harmonic map and other extremal maps in the case $m=\dim\geq 3$. H. Takeuchi [T] showed the following;

THEOREM 3.8. *If* $f:(M,g)\to(N,h)$ *is a* p'*-harmonic map* $(1<p'<\infty)$*, then there is a smooth metric* $\tilde{g}$ *conformally equivalent to* g *on* M_+ *such that* $f:(M_+,\tilde{g})\to(N,h)$ *is* p*-harmonic* $(p\neq m)$*, where* $M_+=\{x\in M;|df(x)|\neq 0\}$.

In his theorem, putting $p=2$ or $p'=2$, we can easily get conformal equivalence of harmonic map and p-harmonic map. Together with Hong's theorems [H, Theorem 2, p. 489 and Theorem 3, p. 490] and Lemma 3.4 and Lemma 3.7 to these, we have accomplished to show results in the diagram of the introduction concerned to conformal equivalence among extremal maps of various types in the case $m=\dim M\geq 3$.

4. In the case $m=\dim M=2$.

If $m=\dim M=2$, note that the energy functional is conformal invariant of the metric on M. So it turns out to be impossible to get the conformal equivalence of harmonic map and the other extremal maps. But there is a theorem about exponentially harmonic map due to Hong as follows:

THEOREM 4.1. *Suppose that* $m=\dim M=2$*. Then there exists a continuous metric* $\tilde{g}$ *conformally equivalent to* g*, and a smooth map* $f\in\mathscr{H}$ *such that* $f:(M,\tilde{g})\to(N,h)$ *is exponentially harmonic in the weak sense.*

For the proof, refer to [H].

We remark that *extremal maps* $f:(M,\tilde{g})\to(N,h)$ *in the weak sense* means that they satisfy

$$\int_M\left(A\tilde{g}^{ij}\frac{\partial f^\gamma}{\partial x^j}\frac{\partial\phi^\gamma}{\partial x^i}-A\tilde{g}^{ij N}\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\phi^\gamma\right)v_{\tilde{g}}=0$$

for any $\phi\in C_0^\infty(M,N)$. Here A stands for $A=e^{|df|_{\tilde{g}}^2/2}$ for exponentially harmonic and $A=(1+|df|_{\tilde{g}}^2)^{(p-2)/2}$ for $(1,p)$-harmonic.

Similarly to this theorem, we can get conformal equivalence and existence theorems concerned to extremal maps of various types in case $m=2$, as follows:

THEOREM 4.2. *Suppose $m=\dim M=2$ and $f:(M,g)\to(N,h)$ is exponentially harmonic. Then there exists a continuous metric $\tilde{g}$ conformally equivalent to g such that $f:(M,\tilde{g})\to(N,h)$ is $(1,p)$-harmonic $(p>2)$ in the weak sense.*

PROOF. Let $f:(M,g)\to(N,h)$ be an exponentially harmonic map. Putting $A=e^{|df|^2/2}$ and $\psi^{(2-m)/2}=1$ at (2.2), we have

$$-\frac{1}{\sqrt{\tilde{g}}}\frac{\partial}{\partial x^i}\left(e^{|df|^2/2}\sqrt{\tilde{g}}\,\tilde{g}^{ij}\frac{\partial f^\gamma}{\partial x^j}\right)=e^{|df|^2/2}\tilde{g}^{ij N}\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\qquad(1\le\gamma\le n)\,.$$

Putting

$$\psi:=\begin{cases}\dfrac{|df|^2}{e^{|df|^2/(p-2)}-1} & x\in M_+\\ p-2 & x\notin M_+\,,\end{cases}$$

we can prove immediately that ψ is a positive continuous function assuming $p>2$ and satisfies

$$(1+|df|_{\tilde{g}}^2)^{(p-2)/2}=(1+\psi^{-1}|df|^2)^{(p-2)/2}=e^{|df|^2/2}\,.$$

Substituting ψ obtained above and $A=e^{|df|^2/2}$ into (2.2), we have the Euler-Lagrange equation for $(1,p)$-harmonic maps with respect to $\tilde{g}$ and h. q.e.d.

THEOREM 4.3. *Suppose $m=\dim M=2$ and $f:(M,g)\to(N,h)$ is $(1,p)$-harmonic. Then there exists a smooth metric $\tilde{g}$ conformally equivalent to g on M_+ such that $f:(M_+,\tilde{g})\to(N,h)$ is p-harmonic $(p\neq2)$, where $M_+=\{x\in M;|df(x)|\neq0\}$.*

PROOF. Similarly to the proof of Theorem 4.2, let $f:(M,g)\to(N,h)$ be as in the assumption. Put $A=(1+|df|^2)^{(p-2)/2}$ and $\psi:=|df|^2/(1+|df|^2)$ at (2.2). It is easily proved that ψ is a positive smooth function of M_+ and satisfies

$$|df|_{\tilde{g}}^{p-2}=\psi^{-(p-2)/2}|df|^{p-2}=(1+|df|^2)^{(p-2)/2}\,.$$

So (2.2) implies $f:(M_+,\tilde{g})\to(N,h)$ is p-harmonic. q.e.d.

THEOREM 4.4. *Suppose $m=\dim M\ge2$. Then there exists a smooth metric $\tilde{g}$*

conformally equivalent to g on M_+, and a map $f \in \mathscr{H}$ such that $f:(M_+, \tilde{g}) \to (N, h)$ is p-harmonic $(p \neq m)$.

PROOF. In the case $m \geq 3$, combination of Proposition 3.5 and Theorem 3.8 putting $p'=2$ yields this theorem. In this process, $\tilde{g}$ is explicitly given by

$$\tilde{g}=\left(\frac{|df|^2}{1+|df|^2}\right)^{(2-p)/(m-p)} g .$$

In the case $m=2$, by the results of Sacks and Uhlenbeck in [S-U], there exists a map $f \in \mathscr{H}$ such that $f:(M, g) \to (N, h)$ is $(1, p)$-harmonic for $p>2$. Combination of the above results of Sacks and Uhlenbeck and Theorem 4.3 gives $\tilde{g}=(|df|^2/(1+|df|^2))g$ and yields Theorem 4.4. This $\tilde{g}$ coincides with the one in the case $m \geq 3$, putting $m=2$. Therefore Theorem 4.4 is valid in the case $m \geq 2$. q.e.d.

THEOREM 4.5. *Suppose $m=\dim M=2$. Then there exists a continuous metric $\tilde{g}$ conformally equivalent to g and a map $f \in \mathscr{H}$ such that $f:(M, \tilde{g}) \to (N, h)$ is $(1, p)$-harmonic for any p $(1<p<\infty)$ in the weak sense. (cf. see remark below Theorem 4.1.)*

PROOF. For $p>2$, by the results of Sacks and Uhlenbeck in [S-U], there exists a map $f \in \mathscr{H}$ such that $f:(M, g) \to (N, h)$ is $(1, p)$-harmonic.

We shall show that changing metric g to $\tilde{g}$ conformally makes $f:(M, \tilde{g}) \to (N, h)$ $(1, p')$-harmonic for any p' $(1<p'<\infty)$. We put $A=(1+|df|^2)^{(p-2)/2}$ and

$$\psi:=\begin{cases} \dfrac{|df|^2}{(1+|df|^2)^{(p-2)/2}-1} & x \in M_+ \\[2ex] \dfrac{2}{p-2} & x \notin M_+ \end{cases}$$

in the case $p'=2$ and

$$\psi:=\begin{cases} \dfrac{|df|^2}{(1+|df|^2)^{(p-2)/(p'-2)}-1} & x \in M_+ \\[2ex] \dfrac{p'-2}{p-2} & x \notin M_+ \end{cases}$$

in the case $p' \neq 2$ at (2.2).

It is verified that ψ is a positive continuous function and satisfies

$$(1+|df|_{\tilde{g}}^2)^{(p'-2)/2}=(1+|df|_g^2)^{(p-2)/2}$$

in each case. So $f:(M, \tilde{g}) \to (N, h)$ is $(1, p')$-harmonic for any p'.

Rewriting p' into p, we have proved Theorem 4.5. q.e.d.

THEOREM 4.6. *Suppose $m=\dim M=2$ and $f:(M, g) \to (N, h)$ is p-harmonic. Then there exists a smooth metric $\tilde{g}$ conformally equivalent to g on M_+ such that $f:(M_+, \tilde{g}) \to$*

(N, h) *is* $(1, p)$*-harmonic.*

PROOF. Putting $A=|df|^{p-2}$ and $\psi:=|df|^2/(|df|^2-1)$ at (2.2), we prove Theorem 4.6. q.e.d.

THEOREM 4.7. *Suppose* $m=\dim M=2$ *and* $f:(M, g)\to(N, h)$ *is exponentially harmonic. Then there exists a smooth metric* $\tilde{g}$ *conformally equivalent to* g *on* M_+ *such that* $f:(M_+, \tilde{g})\to(N, h)$ *is p-harmonic* $(p\neq 2)$.

PROOF. Putting $A=e^{|df|^2/2}$ and $\psi:=|df|^2/e^{|df|^2/(p-2)}$ at (2.2), we prove Theorem 4.7. q.e.d.

THEOREM 4.8. *Suppose* $m=\dim M=2$ *and* $f:(M, g)\to(N, h)$ *is p-harmonic* $(p\neq 2)$. *Then there exists a smooth metric* $\tilde{g}$ *conformally equivalent to* g *on* M_+ *such that* $f:(M_+, \tilde{g})\to(N, h)$ *is exponentially harmonic.*

PROOF. Putting $A=|df|^{p-2}$ and $\psi:=|df|^2/(\log|df|^2)$ at (2.2), we prove Theorem 4.8. q.e.d.

We have accomplished to show all our results in the diagram of the introduction about conformal equivalence and existence of extremal maps of various types in the remained case $m=2$.

5. Stability of extremal maps of various types.

In this section, we study stability of several extremal maps $f:(M, \tilde{g})\to(N, h)$. At first, we describe the second variation formulae of the extremal maps.

Let $f:(M, g)\to(N, h)$ be an extremal map and V and W be variation vector fields along f, that is, $V(x)$, $W(x)\in T_{f(x)}N$, $x\in M$. Let $f_{s,t}$ be a two-parameter variation of an extremal map $f=f_{0,0}$, $|s|, |t|<\varepsilon$ and $V=(\partial f_{s,t}/\partial s)|_{s,t=0}$, $W=(\partial f_{s,t}/\partial t)|_{s,t=0}$. Then a straightforward calculation gives the second variation formulae:

$$\dot{H}_f(V, W)=\int_M [B\{\langle\nabla V, \nabla W\rangle-\langle R^N(V, df\cdot e_i)df\cdot e_i, W\rangle\}+C\langle\nabla V, df\rangle\langle\nabla W, df\rangle]v_g,$$

where ∇ is the covariant derivative for the section of the pull back $f^{-1}TN$ induced from the Levi-Civita connection of (N, h) and R^N is the sectional curvature of N and $\{e_i\}$ is a local orthogonal frame field of M.

Here B stands for

$$1, \quad e^{|df|^2/2}, \quad p|df|^{p-2} \quad \text{and} \quad p(1+|df|^2)^{(p-2)/2}$$

and C stands for

$$0, \quad e^{|df|^2/2}, \quad p(p-2)|df|^{p-4} \quad \text{and} \quad p(p-2)(1+|df|^2)^{(p-4)/2}$$

in the cases the map f is harmonic, exponentially harmonic, p-harmonic and $(1, p)$-

harmonic, respectively.

DEFINITION 5.1. An extremal map f is called *stable* if $H_f(V, V) \geq 0$ for any variation vector field V along f.

Now let us study the following problem: Suppose that an extremal map $f: (M, g) \to (N, h)$ becomes another type of extremal map $f: (M, \tilde{g}) \to (N, h)$ by a conformal change of metric g, and that $f: (M, g) \to (N, h)$ is stable as an extremal map of the above type. Is $f: (M, \tilde{g}) \to (N, h)$ stable as another extremal map?

H. Takeuchi showed in [T, Proposition 2] that if $p \geq p'$ and $f: (M, g) \to (N, h)$ is stable as a p'-harmonic map, then $f: (M_+, \tilde{g}) \to (N, h)$ is also stable as a p-harmonic map.

We shall show the stability of conformal equivalence of harmonic map and the other extremal maps by extending the method employed by H. Takeuchi.

THEOREM 5.2. *Suppose $m = \dim M \geq 3$ and $f: (M, g) \to (N, h)$ is harmonic. Then there exists a smooth metric $\tilde{g}$ conformally equivalent to g such that $f: (M, \tilde{g}) \to (N, h)$ is exponentially harmonic, p-harmonic on $M_+ (p \neq m)$ and $(1, p)$-harmonic, respectively.*

Moreover, if $p \geq 2$ and $f: (M, g) \to (N, h)$ is stable as a harmonic map, then $f: (M, \tilde{g}) \to (N, h)$ is also stable as an exponentially harmonic map, a p-harmonic map on M_+ and a $(1, p)$-harmonic map, respectively.

PROOF. We have already proved the first part in Proposition 3.5, Theorem 4.4 and Theorem 3.1.

Let f : $(M, g) \to (N, h)$ be stable as a harmonic map. We have

$$H_f(V, V) = \int_M \{|\nabla V|^2 - \langle R^N(V, df \cdot e_i) df \cdot e_i, V \rangle\} v_g \geq 0 .$$

Substituting $g = \psi^{-1} \tilde{g}$ into the above inequality, we have

$$H_f(V, V) = \int_M \psi^{(2-m)/2} \{|\nabla V|_{\tilde{g}}^2 - \langle R^N(V, df \cdot e_i) df \cdot e_i, V \rangle_{\tilde{g}}\} v_{\tilde{g}} \geq 0 , \tag{5.1}$$

where $\langle\ ,\ \rangle_{\tilde{g}}$, $v_{\tilde{g}}$ is the inner product and volume element measured by $\tilde{g}$, respectively.

By [H, Theorem 3, p. 490], Theorem 3.8 putting $p' = 2$ and Lemma 3.4, $\psi^{(2-m)/2}$ is determined as

$$e^{|df|_{\tilde{g}}^2/2} , \quad |df|_{\tilde{g}}^{p-2} \quad \text{and} \quad (1 + |df|_{\tilde{g}}^2)^{(p-2)/2} ,$$

according to exponential harmonic, p-harmonic, $(1, p)$-harmonic, respectively.

On the other hand, the second variation formulae of exponentially harmonic map, p-harmonic map and $(1, p)$-harmonic map $f: (M, \tilde{g}) \to (N, h)$ are given as

$$H_f(V, V)_{\tilde{g}} = \int_M [B\{|\nabla V|_{\tilde{g}}^2 - \langle R^N(V, df \cdot e_i) df \cdot e_i, V \rangle_{\tilde{g}}\} + C \langle \nabla V, df \rangle_{\tilde{g}}^2] v_{\tilde{g}} ,$$

where B stands for

$$e^{|df|_{\bar g}^2/2}\,,\quad p|df|_{\bar g}^{p-2}\,,\quad \text{and}\quad p(1+|df|_{\bar g}^2)^{(p-2)/2}\,,$$

and C stands for

$$e^{|df|_{\bar g}^2/2}\,,\quad p(p-2)|df|_{\bar g}^{p-4}\,,\quad \text{and}\quad p(p-2)(1+|df|_{\bar g}^2)^{(p-4)/2}\,,$$

respectively.

For every extremal map, it is clear that $C\geq 0$ if $p\geq 2$ and that B in the second variational formulae coincides with $\psi^{(2-m)/2}$ in (5.1) up to a multiple by p.

So we consequently have

$$H_f(V, V)_{\bar g}\geq 0\,.$$

This proves Theorem 5.2. q.e.d.

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