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Some Characterizations of Bloch Functions on Strongly Pseudoconvex Domains

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Introduction.

The purpose of this paper is to characterize Bloch functions on smoothly bounded strongly pseudoconvex domains in terms of invariant geometry, Bergman-Carleson measures and Kähler diffusion processes.

We will begin with describing the motivation of this paper. As is well known, the class of Bloch functions on the unit disc can be characterized in many different ways, and therefore it arises in several different areas such as function theory, operator theory and harmonic analysis etc. For the case of several complex variables, Timoney [21] extended a number of characterizations of Bloch functions on the unit disc to bounded homogeneous domains (see [21]). Later, Krantz and Ma [10] studied systematically Bloch functions on strongly pseudoconvex domains, and characterized them in terms of Schlicht disks, BMOA functions and normal families of mappings.

Recently, two new characterizations of Bloch functions on the open unit ball in C^n were given by Choa, Kim and Park ([5]) and by Muramoto ([18]); first, a Bergman-Carleson measure characterization ([5, Main Theorem]), and secondly a characterization in terms of hyperbolic Brownian motion ([18, Theorem]). (In [18], Muramoto assumed that $n=1$.)

It must be noted that Lyons pointed out already in [13] a close connection between hyperbolic Brownian motion and Bloch functions on the unit disc. Furthermore, using this connection, Lyons [13] proved a certain law of the iterated logarithm for boundary behavior of Bloch functions on the unit disc, which is regarded as a probabilistic analogue of Makarov's celebrated law of the iterated logarithm for radial behavior of Bloch functions on the unit disc ([15]). Moreover, in [13] it was proved that his probabilistic analogue implies Makarov's law of the iterated logarithm.

Now we will explain our results. Our main theorem is Theorem 2 stated in Section 2, which extends the characterizations by Choa, Kim and Park [5] and Muramoto [18]

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to strongly pseudoconvex domains. Moreover, as an application of our main theorem, we will prove a several dimensional version, for Bloch functions on a certain strongly pseudoconvex domain, of Lyon's law of the iterated logarithm.

The plan of this paper is as follows: In Section 1 we will prepare some notation from function theory, invariant geometry and probability theory. Our main theorem is stated and proved in Section 2. In Section 3 we will prove that our main result includes the main theorem in Choa, Kim and Park [5]. Furthermore, using Theorem 2, we will give a several dimensional version of Lyons' law of the iterated logarithm.

1. Preliminaries.

Throughout this paper, we denote by D a smoothly bounded strongly pseudoconvex domain in C^n, $n \geq 2$, and by ∂D the boundary of D. For $z \in D$, let $\delta(z) := \inf\{|z-\zeta| : \zeta \in \partial D\}$. We can take a positive number r_0 such that for every $z \in D$ with $\delta(z) < r_0$, there exists a unique point $b(z) \in \partial D$ such that $|z - b(z)| = \delta(z)$. If $s > 0$, we denote by $D(s)$ the set $\{z \in D : \delta(z) < s\}$.

If $z \in D$ and $\xi \in T_z(D)$, then we denote by $F_K(z, \xi)$ the infinitesimal Kobayashi metric for D (cf [9]). A holomorphic function f on D is said to be a Bloch function $f \in \mathscr{B}(B)$ if

$$\|f\|_{\mathscr{B}} := \sup\{|f_*(p) \cdot \xi| / F_K(p, \xi)\} < \infty ,$$

where the sup is taken over all $p \in D$ and $0 \neq \xi \in T_p(D)$, and $f_*(p)$ is the mapping from $T_p(D)$ to $T_{f(p)}(C)$ induced by f. This definition was given in Krantz and Ma [10].

Let $H(D)$ be the set of all holomorphic functions on D and $A^2(D)$ the Bergman space, that is,

$$A^2(D) := \left\{ f \in H(D) : \int_D |f(z)|^2 dV(z) < \infty \right\},$$

where dV is the $2n$-dimensional Lebesgue measure on C^n.

Following Luecking [12], we call a positive measure μ on D is a Bergman-Carleson measure if there exists a positive constant C such that for all $f \in A^2(D)$,

$$\int_D |f(z)|^2 d\mu(z) \leq C \int_D |f(z)|^2 dV(z) . \tag{1}$$

We denote by $C(\mu)$ the infimum over the constants on the right-hand side of (1).

Let $(g_{i\bar{j}}(z))$ be the Bergman metric of D and $d_g(z, w)$ the distance function with respect to the Bergman metric of D. For $z \in D$ and $r > 0$, let $E(z, r) := \{w \in D : d_g(z, w) < r\}$.

If $f \in C^1(D)$ is holomorphic and $z \in D$, then we denote by $\|\tilde{\nabla} f(z)\|$ the norm of the gradient of f with respect to the Bergman metric of D, that is,

$$\|\tilde{\nabla} f(z)\|^2 := \sum_{j,k=1}^{n} g^{\bar{j}k}(z) \frac{\partial f(z)}{\partial z_j} \overline{\frac{\partial f(z)}{\partial z_k}} ,$$

where $(g^{\bar{j}k}(z))$ is the inverse matrix of the Bergman metric $(g_{j\bar{k}}(z))$.

Let L be the Laplace-Beltrami operator of the Bergman metric of D.

In order to describe our probabilistic characterization of Bloch functions, we will recall the definition and basic properties of Brownian motion on the Kähler manifold $(D, (g_{i\bar{j}}))$.

In [8] it was proved that the holomorphic sectional curvatures of the Bergman metric of D approach a negative constant near the boundary. Therefore by [23, Theorem 2 and 4] the operator $(1/2)L$ is the generator of an FD semigroup $(T_t)_t$ with an honest transition function $p(t, x, y)$ (see [22] for the definitions of FD semigroups and honest transition functions). As is well known, from this semigroup one can construct an honest FD diffusion on D (cf. [22, III, Theorem 28]). Since this diffusion has continuous paths in D and its life time is equal to ∞, we can represent the diffusion as follows: Denote by W the set of all continuous maps from $[0, \infty)$ to D and by Z_t the coordinate projection mapping from W to D via $Z_t(w)=w(t)$, $w\in W$. Let $\mathscr{B}$ (resp. $\mathscr{B}_t$) be the smallest σ-field for which all elements of $\{Z_s: 0\leq s\}$ (resp. $\{Z_s: 0\leq s\leq t\}$) are measurable. For a probability Borel measure ν on D, there exists a unique probability measure P^ν on $(W, \mathscr{B})$ such that for every $0<t_1<\cdots<t_m$ and for any finite Borel subsets $G_1, \cdots, G_m$ of D,

$$P^\nu(\{w\in W: w(t_j)\in G_j, j=1, \cdots, m\})=\int_D\int_{G_1}\int_{G_2}\cdots\int_{G_m} p(t_1, z_0, z_1)$$

$$\times p(t_2-t_1, z_2, z_1)\cdots p(t_m-t_{m-1}, z_{m-1}, z_m)dV_g(z_m)\cdots dV_g(z_2)dV_g(z_1)d\nu(z_0),$$

where dV_g is the Riemannian measure on $(D, (g_{j\bar{k}}))$. If ν is the unit mass at $a\in B_n$, then P^a denotes P^ν. Let $(W, \mathscr{F}^\nu, \mathscr{F}^\nu_t, P^\nu)$ be the usual P^ν augmentation of $(W, \mathscr{B}, \mathscr{B}_t, P^\nu)$ in the sense of [22, II, 40]. Put $\tilde{\mathscr{F}}=\bigcap\mathscr{F}^\nu$ and $\tilde{\mathscr{F}}_t=\bigcap\mathscr{F}^\nu_t$, where the intersections being taken over all probability Borel measures ν on B_n. Then $Z=(Z_t, W, \tilde{\mathscr{F}}, \tilde{\mathscr{F}}_t, P^a: a\in D)$ is an honest FD diffusion which is called Brownian motion on the manifold $(D, (g_{j\bar{k}}))$.

The following lemma is very close to Theorem 1 in Debiard and Gaveau [6].

LEMMA 1. *Let $a\in D$. Then for dP^a-almost every $w\in W$, a limit $Z_\infty(w)=\lim_{t\to\infty} Z_t(w)$ exists and $Z_\infty(w)\in\partial D$.*

A proof of this lemma will be given in Appendix.

Let $\mathscr{T}$ be the collection of all $(\mathscr{F}_t)$-stopping times. As usual, we denote by $E^a[\ ]$ the expectation associated with P^a.

In this paper we will use the notation $A_1\approx A_2$ to mean that there exists a positive constant c depending only on D such that $c^{-1}A_2\leq A_1\leq cA_2$. Moreover, we will use $C_1, C_2, C_3\cdots$ to denote positive constants depending only on n, the Bergman metric and the Kobayashi metric for D.

2. Bloch functions, Bergman-Carleson measures and Kähler diffusions.

In this section we will characterize Bloch functions on D in terms of invariant geometry, Bergman-Carleson measures and invariant diffusion processes.

Our main results is the following theorem.

THEOREM 2. *Let f be a holomorphic function in D. Then the following four conditions are mutually equivalent*:

(i) *f is a Bloch function.*

(ii) $\sup_{z\in D}\|\tilde{\nabla}f(z)\|<\infty$.

(iii) *The measure*

$$d\mu_f(z):=\|\tilde{\nabla}f(z)\|^2 dV(z)$$

is a Bergman-Carleson measure on D.

(iv) *The stochastic process $\{f(Z_t)\}_t$ satisfies that*

$$\|f\|^2_{B,prob}:=\sup_{z\in D}\left\{\sup\left\{\frac{E^z[|f(Z_T)-f(Z_0)|^2]}{E^z[T]}: T\in\mathscr{T}, E^z[T]>0\right\}\right\}<\infty .$$

Besides, four quantities $\|f\|_{\mathscr{B}}$, $\sup_{z\in D}\|\tilde{\nabla}f(z)\|$, $C(\nu_f)^{1/2}$ and $\|f\|_{B,prob}$ are equivalent.

The equivalence relations "(i)"$\Leftrightarrow$(ii)", "(i)$\Leftrightarrow$(iii)" and "(i)$\Leftrightarrow$(vi)" are invariant analogues, for strongly pseudoconvex domains, of Choa, Kim and Park [5, Theorem 2.4 (a)$\Leftrightarrow$(b)], [5, Main Theorem] and Muramoto [18, Theorem] respectively.

In order to prove these equivalent relations, we need to modify the original proofs in [5] and [18], because a generic smoothly bounded strongly pseudoconvex domain is not homogeneous. In particular, our proofs of "(i)$\Leftrightarrow$(ii)" and "(i)$\Leftrightarrow$(iii)" differ from the proofs for the case of the unit ball by Choa, Kim and Park ([5]) in detail and method. Indeed, the method in [5] is based on the transitivity of the group of automorphisms on the unit ball. To prove "(i)$\Leftrightarrow$(vi)", we will simplify and exploit the method in Muramoto [18] by combining an idea in Lyons [13].

PROOF OF THEOREM 1. (Proof of "(i)$\Rightarrow$(ii)".) Suppose that f is a Bloch function. Let $\nabla f=(\partial f/\partial z_1, \cdots, \partial f/\partial z_n)$. Denote by $\nabla_N f$ (resp. $\nabla_T f$) the complex normal piece of ∇f (resp. the complex tangential piece of ∇f). Then by the proof of [10, Theorem 2.1], $|\nabla_N f(z)|=O(\delta(z)^{-1})$ and $|\nabla_T f(z)|=O(\delta(z)^{-1/2})$. Let λ be a C^∞ strictly plurisubharmonic function on a neighborhood U of the closure $\bar{D}$ of D such that $D=\{w\in U: \lambda(w)<0\}$ and $d\lambda(\zeta)\neq 0$ for $\zeta\in\partial D$. Then $-\lambda(z)=O(\delta(z))$ when $z\in D$ is near to ∂D. Now we recall Fefferman's result about the Bergman kernel ([7, Corollary]). Let $K(z, w)$ be the Bergman kernel on D. Fefferman's theorem asserts that

$$K(z, z)=\Phi(z)(-\lambda(z))^{-(n+1)}+\Phi_1(z)\log(-\lambda(z)) ,$$

where Φ and Φ_1 are C^∞ functions on the intersection of $\bar{D}$ and a neighborhood of ∂D, and $\Phi \neq 0$ on ∂D (see Fefferman [7, p. 45]).

Let $c_1 := \Phi\{1 + \Phi^{-1}\Phi_1(-\lambda)^{n+1}\log(-\lambda)\}$. Then $K(z, z) = c_1(z)(-\lambda(z))^{-(n+1)}$ near ∂D. Since $K(z, z) > 0$, the definition of the function c_1 guarantees that there is an open set $W \subset \boldsymbol{C}^n$ such that $\partial D \subset W$, $c_1 \geq b$ on $W \cap D$ for some positive constant b, and c_1 has all derivatives of order $\leq n$ bounded continuous in $W \cap D$. Therefore, we get that there exists a positive C^∞ function c on D such that

$$K(z, z) = c(z) \times (-\lambda(z))^{-(n+1)}, \tag{2}$$

where $c \geq b$ on $W \cap D$ and c has all derivatives of order $\leq n$ bounded continuous on M. (Note that $n \geq 2$.) Therefore, by a similar argument as in [20, p. 57] we have that

$$(g^{\bar{i}j}) = \frac{1}{n+1}(-\lambda)^2 \left(\begin{array}{c|c} 4/d^2 + O(-\lambda) & O(1) \\ \hline O(1) & (-\lambda)^{-1}Q_2^{-1} + O(1) \end{array}\right),$$

where

$$Q_2 = \left(\frac{\partial^2 \lambda}{\partial z_i \partial \bar{z}_j}\right) \qquad \text{and} \qquad d = |\nabla\lambda(b(z))|\,.$$

Computing $(g_{i\bar{j}})$ we get that

$$\|\tilde{\nabla} f(z)\| \approx \delta(z)|\nabla_N f(z)| + \delta(z)^{1/2}|\nabla_T f(z)|$$

whenever $z \in D$ is near to ∂D. Consequently $\|\tilde{\nabla} f\|$ is bounded in D.

(Proof of "(ii) $\Rightarrow$ (iii)".) This part is obvious.

(Proof of "(iii) $\Rightarrow$ (i)".) Assume that μ_f is a Bergman-Carleson measure. We will prove that f is a Bloch function. By virtue of [10, Theorem 2.1] and of the maximum principle, it is sufficient to prove that there exists $r_1 > 0$ depending only on D such that

$$\sup_{z \in D(r_1)} |\nabla_N f(z)|\delta(z) \leq C_1 C(\mu_f)^{1/2}\,. \tag{3}$$

To prove this we use the following

LEMMA 3 ([11, Theorem 12]). *If a number $R > 0$ is given, then there are positive constants r_1, c_1, and c_2 depending only on R and D such that $r_1 < r_0$ and*

$$c_1\delta(z)^{-(n+1)} \leq |K(z, w)| \leq c_2\delta(z)^{-(n+1)},$$

whenever $z \in D(r_1)$ and $w \in E(z, R)$.

Now let a be an arbitrary point in $D(r_1)$. Denote by v the inward unit normal vector field on ∂D with respect to the Euclidean metric. By an appropriate coordinate change we may suppose that $v_{b(a)}=(1, 0, \cdots, 0)$. Then $a=(\delta(a), 0, \cdots, 0)$. Note that the ball $E(a, 1)$ is comparable to the following polydisc:

$$P(a):=\{(w_1, \cdots, w_n) : |w_1-\delta(a)|<\gamma_1\delta(z), |w_j|<\gamma_j\delta(a)^{1/2}, j=2, \cdots, n\},$$

where γ_j is a positive constant depending only on D. Since $\partial f/\partial z_1$ is holomorphic in D we get that

$$\nabla_N f(a)=\frac{\partial f}{\partial z_1}(a)=\frac{1}{|P(a)|}\int_{P(a)}\frac{\partial f}{\partial z_1}(w)dV(w),$$

where (and always from now on) for a measurable set $E\subset D$, we denote by $|E|$ the $2n$-dimensional Lebesgue measure of E. Therefore we have that

$$\delta(a)^2|\nabla_N f(a)|^2\le\delta(a)^2\frac{1}{|P(a)|}\int_{P(a)}\left|\frac{\partial f}{\partial z_1}(w)\right|^2 dV(w).$$

Note that $|a-w|\approx\delta(a)\approx\delta(w)$ whenever $w\in P(a)$, and hence for $w\in P(a)$,

$$\nabla_N f(w)=\frac{\partial f}{\partial z_1}(w)+O(\delta(w)|\nabla f(w)|).$$

Therefore if $w\in P(a)$, then

$$\delta(w)\left|\frac{\partial f}{\partial z_1}(w)\right|\le\delta(w)|\nabla_N f(w)|+O(\delta(w)^2|\nabla f(w)|)\le C_2\|\tilde{\nabla} f(w)\|.$$

Consequently we have that

$$\delta(a)^2|\nabla_N f(a)|^2\le\frac{C_3}{|P(a)|}\int_{P(a)}\|\tilde{\nabla} f(w)\|^2 dV(w).$$

Since $E(z, 1)$ is comparable to $P(z)$, we have the following inequalities by Lemma 3 and by (1) in Section 1:

$$\begin{aligned}\delta(a)^2|\nabla_N f(a)|^2&\le\frac{C_3}{|P(a)|}\int_{P(a)}\|\tilde{\nabla} f(w)\|^2 dV(w)\\&\le\frac{C_4}{|E(a, 1)|}\int_{E(a,1)}\|\tilde{\nabla} f(w)\|^2 dV(w)\\&\le\frac{C_5}{\delta(a)^{n+1}}\int_{E(a,1)}|K(a, w)|^2\delta(a)^{2(n+1)}\|\tilde{\nabla} f(w)\|^2 dV(w)\\&\le C_5\delta(a)^{n+1}\int_D|K(a, w)|^2\|\tilde{\nabla} f(w)\|^2 dV(w)\end{aligned}$$

$$\leq C_5 C(\mu_f)\delta(a)^{n+1}\int_D |K(a, w)|^2 dV(w)$$
$$= C_5 C(\mu_f)\delta(a)^{n+1} K(a, a) \leq C_6 C(\mu_f) .$$

Therefore we find that f is a Bloch function.

(Proof of "(ii) $\Rightarrow$ (vi)".) The following proof, which is inspired from Lyons [13], simplifies one in [18]. Assume that $\sup_{z\in D}\|\tilde{\nabla} f(z)\| < \infty$. By Ito's formula we get that for every $z \in D$ and $T \in \mathscr{T}$,

$$E^z[|f(Z_T)-f(Z_0)|^2] = \frac{c}{2} E^z\left[\int_0^T \|\tilde{\nabla} f(Z_s)\|^2 ds\right] \leq C_7 \sup_{z\in D} \|\tilde{\nabla} f(z)\| E^z[T] ,$$

where c is the constant depending only on n. Therefore

$$\|f\|_{B,prob} \leq C_7 \sup_{z\in D} \|\tilde{\nabla} f(z)\| < \infty .$$

(Proof of "(vi) $\Rightarrow$ (ii)".) To prove this we use an idea in Muramoto [18]. Suppose that $\|f\|_{B,prob} < \infty$. We will estimate $\|f\|_B := \sup_{z\in D}\|\tilde{\nabla} f(z)\|$.

Take an arbitrary number $0 < \alpha < \|f\|_B$. Then there exist $\eta \in D$ and a positive number ε such that $\alpha \leq \|\tilde{\nabla} f(z)\|$, whenever $z \in E(\eta, \varepsilon)$. Let

$$T(\varepsilon) := \inf\{t \geq 0 : d_g(Z_t, Z_0) \geq \varepsilon\} .$$

Then we get that

$$\alpha^2 E^\eta[T(\varepsilon)] = E^\eta\left[\int_0^{T(\varepsilon)} \alpha^2 ds\right] \leq E^\eta\left[\int_0^{T(\varepsilon)} \|\tilde{\nabla} f(Z_s)\|^2 ds\right]$$
$$= E^\eta[|f(Z_{T(\varepsilon)}) - f(Z_0)|^2] \leq \|f\|^2_{B,prob} E^\eta[T(\varepsilon)] .$$

This implies that $\alpha \leq \|f\|_{B,prob}$, and therefore we obtain that $\|f\|_B \leq \|f\|_{B,prob}$. □

3. Applications.

Let $B_n = \{z \in \mathbf{C}^n : |z| < 1\}$. By an easy calculation we have that for every holomorphic function f on B_n,

$$\|\tilde{\nabla} f\|^2 = 4(1-|z|^2)(|\nabla f(z)|^2 - |\mathscr{R} f(z)|^2) ,$$

where $\mathscr{R} f = \sum_{j=1}^n z_j(\partial f)/(\partial z_j)$. Therefore as a corollary of Theorem 2 "(i) $\Leftrightarrow$ (iii)" we get the following

COROLLARY 4 (Choa, Kim and Park [5]). *Let f be a holomorphic function on B_n. Then f is a Bloch function if and only if the measure defined by*

$$(1-|z|^2)(|\nabla f(z)|^2 - |\mathscr{R} f(z)|^2) dV(z)$$

is a Bergman-Carleson measure.

As we noted in Section 1, Lyons obtained in [13] a probabilistic analogue of the Makarov's law of the iterated logarithm. By virtue of Theorem 2, his method is applicable to our setting, and we have a several dimensional version of Lyons' law of the iterated logarithm:

COROLLARY 5. *Suppose that the Kähler manifold $(D, (g_{i\bar{j}}))$ is negatively curved and simply connected. Fix a point $a \in D$. Then for every Bloch function f on D,*

$$\limsup_{t\to\infty} \frac{|f(Z_t)|}{(\log(\delta(Z_t)^{-1})\log\log\log(\delta(Z_t)^{-1}))^{1/2}} \le C_8 \|f\|_{\mathscr{B}} \qquad a.s.\ dP^a .$$

PROOF OF COROLLARY 5. Since the geometry of strongly pseudoconvex domains is more subtle than that of the unit disc, we need extra arguments in order to apply Lyons' arguments in [13] to our context.

By Ito's formula we have that $\{f(Z_t)\}_t$ is a conformal local martingale with respect to P^a in the sense of Getoor and Sharpe (see [19, p. 177] for the definition of conformal local martingales). Hence by Theorem 2.4 in [19, p. 178], there exists (possibly on an enlargement for the usual P^a augmentation of $(W, \mathscr{B}, \mathscr{B}_t, P^a)$) a complex Brownian motion X_t such that $f(Z_t) - f(a) = X_{\tau(t)}$, where $\tau(t)$ is the quadratic variational process to $f(z)$, that is, $\tau(t) = \langle f(z), \overline{f(z)} \rangle_t / 2$. Recall the following Khintchine's law of the iterated logarithm for complex Brownian motion (cf. [13, p. 160]):

$$\limsup_{t\to\infty} \frac{|X_{\tau(t)}|}{(\tau(t)\log\log\tau(t))^{1/2}} = 1 \qquad \text{a.s. } dP^a .$$

Now Theorem 2 "(i) $\Rightarrow$ (vi)" yields that $\tau(t) \le C_9 \|f\|_{\mathscr{B}}^2 t$. Accordingly we get

$$\limsup_{t\to\infty} \frac{|f(Z_t)|}{(t\log\log t)^{1/2}} \le C_{10} \|f\|_{\mathscr{B}} \qquad \text{a.s. } dP^a .$$

Let $d(t) = d_g(a, Z_t)$. Note that the sectional curvatures of M is bounded by two negative constants because of our hypothesis. Therefore $d(t) \approx t$ as $t \to \infty$ (cf. (3.2) in [17, p. 254]). Consequently we have that

$$\limsup_{t\to\infty} \frac{|f(Z_t)|}{(d(t)\log\log d(t))^{1/2}} \le C_{11} \|f\|_{\mathscr{B}} \qquad \text{a.s. } dP^a . \tag{4}$$

Since it is plain that $d(t) \approx \log(\delta(Z_t)^{-1})$ as $t \to \infty$, the inequality (4) yields Corollary 5. □

REMARK 1. In [13] Lyons showed that Makarov's law of the iterated logarithm is obtained as a direct consequence of both his probabilistic analogue of the Makarov's law and a result in [14] for a symmetric space of rank one.

REMARK 2. It should be noted that Makarov studied dyadic Bloch martingales in [16].

4. Appendix (Proof of Lemma 1).

Lemma 1 is proved by a similar way as the proof of [6, Theorem 1]. However, it is also a consequence of a result in [3] that the boundary ∂D consists of the minimal Martin boundary points with respect to L. In order to make the paper reasonably self-contained, we will mention the proof of Lemma 1 given in [3].

PROOF OF LEMMA 1 (cf. [3]). We will use essentially the potential theory for coercive operators developed by Ancona ([1], [2]). It is assumed the reader is familiar with this theory. Our starting point is [1, Theorem 8]. We will recall it. Let (M, h) be a complete Riemannian manifold with bounded geometry property such that the Laplacian on M is coercive. Ancona's result is that if a compactification of M satisfies the (G.A.) condition in the sense of [1], then it is homeomorphic to the Martin compactification of (M, h) and its Martin boundary consists of minimal Martin boundary points (see [1, Theorem 8]). Here we will prove that L is coercive and that the Euclidean compactification $\bar{D}$ of D satisfies the (G.A.) condition. By the boundary behavior of $(g^{i\bar{j}})$ mentioned in the proof of Theorem 2 "(i) $\Rightarrow$ (ii)" we have that for every $0<\gamma<\infty$,

$$L((-\lambda)^{n\gamma})=\frac{4}{n+1}\{n^2\gamma(\gamma-1)+O(-\lambda)\}(-\lambda)^{n\gamma},$$

where λ is a C^∞ strictly plurisubharmonic defining function of D. Therefore if $0<\gamma<1$, then there exist a compact subset $K\subset\Omega$ and a positive number ε such that $(-\lambda)^{n\gamma}$ is a positive $(L+\varepsilon I)$-superharmonic function on $\Omega\setminus K$. Hence by [1, Lemma 2 and the proof of Lemma 21] we have that the operator L is coercive on Ω.

To prove that ∂D has (G.A.) condition, we prepare some notation. Let v be the inward unit normal vector field on ∂D with respect to the Euclidean metric. For $\zeta\in\partial D$ and $r>0$, let $\zeta(r)=\zeta+rv_\zeta$. We set

$$Q_\alpha(\zeta, r):=\left\{z\in\Omega : \delta(z)<r, \sum_{j,k=1}^{n} g_{j\bar{k}}(\zeta(r))(b(z)_j-\zeta_j)\overline{(b(z)_k-\zeta_k)}<\alpha\right\},$$

for $\zeta\in\partial D$, $\alpha>0$ and $0<r<r_0$. Let $\Pi_{b(z)}$ (resp. $\Pi_{b(z)}^{\perp}$) be the orthogonal projection of $\boldsymbol{C}^n$ to the complex linear space spanned by the vector $v_{b(z)}$ (resp. $\{w\in\boldsymbol{C}^n : w\cdot\overline{v_{b(z)}}=0\}$). Since the equation (2) in the proof of Theorem 2 "(i) $\Rightarrow$ (ii)" guarantees that

$$\left(\sum_{j,k=1}^{n} g_{j\bar{k}}(z)\xi_j\overline{\xi_k}\right)\approx\frac{1}{\delta(z)}|\Pi_{b(z)}\xi|+\frac{1}{\delta(z)^{1/2}}|\Pi_{b(z)}^{\perp}\xi|,$$

for all $\xi\in\boldsymbol{C}^n$ and z near to ∂D, we have by the same way as in the proofs of [1, Theorem 11 and Lemma 26], that there exists a positive constant $0<c'<1$ depending only on the Bergman metric g such that two points $y \in \partial Q_\alpha(\zeta, r)$ and $y' \in \partial Q_\alpha(\zeta, (c')^2 r)$ can be joined by a Φ-chain through $\zeta(c'r)$ (for the definition of Φ-chain, see Definition 14 in [1]). Therefore by [1, Theorem 8] we obtain that $\bar{D}$ is homeomorphic to the Martin compactification of D with respect to L, and that ∂D consists of minimal Martin boundary points.

Consequently, from [2, Theorem 3.1] it follows that for $\alpha \in D$, a limit $Z_\infty(w) = \lim_{t\to\infty} Z_t(w)$ exists and $Z_\infty \in \partial D$ for dP^a-almost every $w \in W$. Furthermore we have that for a continuous function F on ∂D,

$$PI[F](a) := E^a[F(Z_\infty)], \qquad a \in D$$

is the unique L-harmonic function on D such that $\lim_{a\to\zeta} PI[F](a) = F(\zeta)$ for every $\zeta \in \partial D$. □

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