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A Dehn Surgery Formula for Walker Invariant on a Link

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0. Introduction.

In 1985, Andrew Casson defined an integer valued invariant $\lambda(M)$ for any oriented integral homology 3-sphere M, which counts the "signed" irreducible representations of the fundamental group $\pi_1(M)$ into $SU(2)$ [1]. In 1989, Kevin Walker extended the Casson's invariant to rational homology 3-spheres, by taking into account the reducible representations of $\pi_1(M)$ coming from torsion [6]. In this paper we give a formula for Walker's invariant in the case where a rational homology 3-sphere H is obtained by Dehn surgery on a link L in a rational homology 3-sphere M, and furthermore the linking number between every pair of components of L is zero. In this case the Walker's invariant, $\lambda(H)$, can be expressed in terms of $\lambda(M)$, the surgery coefficients of L, a certain coefficient from each of the Conway polynomials of L and all its sublinks, and a certain function τ which was introduced by Walker. In the case of original Casson's invariant, a formula for Dehn surgery on a link in an integral homology 3-sphere was given by Jim Hoste [3]. We adapt his method to the case of the Walker's invariant and obtain a formula.

Suppose $L=\{K_1, \cdots, K_n\}$ is a link in a rational homology sphere M. Let $N(K_i)$ be a tubular neighborhood of K_i. Let $x_i \in H_1(\partial N(K_i); \boldsymbol{Z})$ be a primitive homology class. We call pairs $\{(K_1, x_1), \cdots, (K_n, x_n)\}$ *a framed link* and denote by $\chi((K_1, x_1), \cdots, (K_n, x_n); M)$, or simply by $\chi(L; M)$, the manifold obtained from M by Dehn surgery along L according to the given framings x_is. Let $\langle \cdot, \cdot \rangle$ denote the intersection pairing on $H_1(\partial N(K_i); \boldsymbol{Z})$. (The orientation of $\partial N(K_i)=\partial(M-N(K_i))$ is induced from that of $M-N(K_i)$ via the "inward normal last" convention.) Let m_i and l_i be the meridian and longitude of K_i respectively. Walker gives the following formula for Dehn surgery on a knot K (i.e. one component link):

$$\lambda(\chi(K; M))=\lambda(M)+\tau(m, x; l)+\frac{\langle m, x\rangle}{\langle m, l\rangle\langle x, l\rangle}\Gamma(K; M).$$

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Here, m and l are the meridian and longitude of K respectively, $x \in H_1(\partial N(K); \boldsymbol{Z})$ is a primitive homology class which gives framing, τ is a function which depends on m, x and l, and $\Gamma(K; M)$ is the second derivative of the symmetrized Alexander polynomial of K evaluated at 1. We will extend this surgery formula to a link of n components, all of whose linking numbers are zero. For computative reasons, we will use the Conway polynomial instead of the Alexander polynomial. If L bounds a Seifert surface F with $\partial F = K_1 \cup \cdots \cup K_n$, then it can be shown that the Conway polynomial of L, $\nabla_{L;M}(z)$, has the form $\nabla_{L;M}(z) = z^{n-1}(a_0 + a_1 z^2 + \cdots + a_k z^{2k})$, where $a_i \in \boldsymbol{Q}$ and k is some positive integer. Let $\varphi_i(L; M) = a_i$. Suppose that each component of L is null-homologous. Then we will show that

$$\lambda(\chi(L; M)) = \lambda(M) + \sum_{i=1}^{n} \tau(m_i, x_i; l_i) + 2 \sum_{L' \subset L} \left(\prod_{i \in L'} \frac{\langle m_i, x_i \rangle}{\langle x_i, l_i \rangle} \right) \varphi_1(L'; M) .$$

Here the sum is taken over all sublinks L' of L and the product over all i for which K_i is a component of L'. We have abbreviated this as $i \in L'$. Actually, the sum need only be taken over all sublinks having less than four components as $\varphi_1(L'; M) = 0$ otherwise. We will also show a similar formula in the case that some components of L are not null-homologous.

In Section 1, we will state Walker's theorem (including the Dehn surgery formula on a knot) with the definition of τ. In Section 2, we will establish some facts for the Conway polynomial and the Alexander polynomial. The only difficulty in deriving the formula from Walker's Dehn surgery formula is in computing $\Gamma(K_n; \chi(K_1, \cdots, K_{n-1}; M))$ in terms of original link data. Section 3 is devoted to doing this. Then in Section 4 we obtain the formula for $\lambda(\chi(L; M))$ (Theorem 4.1 and Theorem 4.2).

1. Theorem of Walker.

In order to state Walker's theorem, we need to introduce two functions, Γ and τ.

Let K be a knot in M and $\Delta_{K;M}(t)$ be the Alexander polynomial of K. Normalize $\Delta_{K;M}(t)$ so that $\Delta_{K;M}(1) = 1$ and $\Delta_{K;M}(t^{-1}) = \Delta_{K;M}(t)$. Let $\Gamma(K; M) \in \boldsymbol{Q}$ denote the second derivative of $\Delta_K(t)$ evaluated at $t = 1$.

The definition of τ is more complicated. Let $N(K)$ be a tubular neighborhood of K and let l be a longitude of K. Let $\langle \cdot , \cdot \rangle$ denote the intersection pairing on $H_1(\partial N(K); \boldsymbol{Z})$. (The orientation of $\partial N(K) = \partial(M - N(K))$ is induced from that of $M - N(K)$ via the "inward normal last" convention.) Let $a, b \in H_1(\partial N(K); \boldsymbol{Z})$ be primitive homology classes such that $\langle a, l \rangle \neq 0$ and $\langle b, l \rangle \neq 0$. Choose a basis v, w of $H_1(\partial N(K); \boldsymbol{Z})$ such that $\langle v, w \rangle = 1$ and $l = dw$ for some $d \in \boldsymbol{Z}$. Define

$$\tau(a, b; l) \stackrel{\text{def}}{=} -s(\langle v, a \rangle, \langle w, a \rangle) + s(\langle v, b \rangle, \langle w, b \rangle) + \frac{1}{12}\left(1 - \frac{1}{d^2}\right)\left(\frac{\langle v, a \rangle}{\langle w, a \rangle} - \frac{\langle v, b \rangle}{\langle w, b \rangle}\right),$$

where $s(q, p)$ denotes the Dedekind sum

$$s(q, p) \stackrel{\text{def}}{=} \operatorname{sign}(p) \sum_{k=1}^{|p|} ((k/p))((kq/p))$$

$$((x)) \stackrel{\text{def}}{=} \begin{cases} 0\,, & x \in \boldsymbol{Z} \\ x-[x]-1/2\,, & \text{otherwise}\,. \end{cases}$$

Note that $\tau(a, b; l)$ depends only on a, b, l and $\langle \cdot, \cdot \rangle$, not on v, w.

THEOREM (Walker). 1. *There is a unique function* λ: {*rational homology spheres*} $\to \boldsymbol{Q}$ *such that*

(a) $\lambda(S^3)=0$, *and*

(b) *Dehn surgery formula: Let K be a knot in a rational homology sphere M, l be a longitude of K and $N=M-N(K)$. Then*

$$\lambda(N_b)=\lambda(N_a)+\tau(a, b; l)+\frac{\langle a, b\rangle}{\langle a, l\rangle\langle b, l\rangle}\Gamma(K; M)$$

for all primitive $a, b \in H_1(\partial N(K); \boldsymbol{Z})$, $\langle a, l\rangle \neq 0$, $\langle b, l\rangle \neq 0$. Here $N_x=N \cup_f (D^2 \times S^1)$, and $f: \partial D^2 \times S^1 \to \partial N=\partial N(K)$ maps $\partial D^2 \times \{\theta\}$ to a curve representing x.

2. *The λ invariant has the following properties*:

(a) *Let $-M$ denote M with the opposite orientation. Then*

$$\lambda(-M)=-\lambda(M)\,.$$

(b) *Let M_1 and M_2 be rational homology spheres. Then*

$$\lambda(M_1 \# M_2)=\lambda(M_1)+\lambda(M_2)\,.$$

2. Some properties of the Conway polynomial and the Alexander polynomial.

Let $lk(K, J; M)$ denote the linking number of K and J in a rational homology sphere M. Let $L=\{K_1, \cdots, K_n\}$ be an oriented link in M. Suppose that L bounds a Seifert surface F with $\partial F=K_1 \cup \cdots \cup K_n$. Let $\{e_1, \cdots, e_r\}$ be a basis for $H_1(F; \boldsymbol{Z})$. Let $V(L; M)=(v_{ij})$ be a matrix given by $v_{ij}=lk(e_i^+, e_j; M)$. This is called a Seifert matrix. In this case, the Conway polynomial of L, $\nabla_{L;M}$, is

$$\nabla_{L;M}(z)=\det(tV(L; M)-t^{-1}V(L; M)^T)\,,$$

where $z=t-t^{-1}$.

PROPOSITION 2.1. *Let $L=\{K_1, \cdots, K_n\}$ be an oriented link in M such that there is a Seifert surface F for L with $\partial F=K_1 \cup \cdots \cup K_n$. Let $\nabla_{L;M}(z)$ be the Conway polynomial of L. Then $\nabla_{L;M}$ has the form*

$$\nabla_{L;M}(z)=z^{n-1}(a_0+a_1z^2+\cdots+a_mz^{2m})\,, \qquad a_i \in \boldsymbol{Q}\,,$$

where m is some positive integer.

PROOF. Simply let V denote the Seifert matrix $V(L; M)$. The Alexander polynomial of L, $\Delta_{L;M}(t)$ is given by

$$\Delta_{L;M}(t)=t^{-r/2}(\det(tV-V^T)),$$

where r is the rank of $H_1(F; \boldsymbol{Z})$. Note that if $n=1$, then this is the symmetric normal form of $\Delta_{K;M}(t)$ of a null-homologous knot K (i.e. which satisfies $\Delta_{K;M}(t)=\Delta_{K;M}(t^{-1})$ and $\Delta_{K;M}(1)=1$). Now

$$\begin{aligned}\Delta_{L;M}(t^2)&=t^{-r}(\det(t^2V-V^T))=t^{-r}\det(t(tV-t^{-1}V^T))\\&=\det(tV-t^{-1}V^T)=\nabla_{L;M}(z).\end{aligned}$$

Note that $\Delta_{L;M}(t^{-1})=(-1)^r\Delta_{L;M}(t)$. Hence, if r is even (i.e. n is odd), then $\Delta_{L;M}(t)$ has the form

$$\Delta_{L;M}(t)=c_0+c_1(t+t^{-1})+c_2(t^2+t^{-2})+\cdots+c_{r/2}(t^{r/2}+t^{-r/2}), \qquad c_i\in\boldsymbol{Q}$$

and if r is odd (i.e. n is even), then $\Delta_{L;M}(t)$ has the form

$$\Delta_{L;M}(t)=c_0(t^{1/2}-t^{-1/2})+c_1(t^{3/2}-t^{-3/2})+\cdots+c_{(r-1)/2}(t^{r/2}-t^{-r/2}), \qquad c_i\in\boldsymbol{Q}.$$

So, if r is even, then

$$\Delta_{L;M}(t^2)=c_0+c_1(t^2+t^{-2})+c_2(t^4+t^{-4})+\cdots+c_{r/2}(t^r+t^{-r}), \qquad c_i\in\boldsymbol{Q}$$

and if r is odd, then

$$\Delta_{L;M}(t^2)=c_0(t-t^{-1})+c_1(t^3-t^{-3})+\cdots+c_{(r-1)/2}(t^r-t^{-r}), \qquad c_i\in\boldsymbol{Q}.$$

But it can be shown that $t^{2m}+t^{-2m}$ has the form

$$t^{2m}+t^{-2m}=d_0+d_1(t-t^{-1})^2+d_2(t-t^{-1})^4+\cdots+d_m(t-t^{-1})^{2m}, \qquad d_i\in\boldsymbol{Z}$$

and $t^{2m+1}-t^{-(2m+1)}$ has the form

$$t^{2m+1}-t^{-(2m+1)}=d_0(t-t^{-1})+d_1(t-t^{-1})^3+\cdots+d_m(t-t^{-1})^{2m+1}, \qquad d_i\in\boldsymbol{Z}.$$

Then it follows that $\nabla_{L;M}(z)$ has the form

$$\begin{cases}\nabla_{L;M}(z)=b_0+b_1z^2+b_2z^4+\cdots+b_lz^{2l}, & b_i\in\boldsymbol{Q}, \quad r\text{: even } (n\text{: odd})\\ \nabla_{L;M}(z)=b_0z+b_1z^3+\cdots+b_lz^{2l-1}, & b_i\in\boldsymbol{Q}, \quad r\text{: odd } (n\text{: even}).\end{cases}$$

Now, since F is a surface with n boundary components, we may assume that the basis of $H_1(F; \boldsymbol{Z})$ has been chosen so that V has the form

$$V=\begin{pmatrix} A & B\\ B^T & C\end{pmatrix},$$

where A is a $2g\times 2g$ matrix, B is a $2g\times(n-1)$ matrix, and C is an $(n-1)\times(n-1)$

symmetric matrix. Then

$$\nabla_{L;M}(z)=\det\begin{pmatrix} tA-t^{-1}A^T & zB \\ zB^T & zC \end{pmatrix}.$$

Hence $\nabla_{L;M}(z)$ is divisible by z^{n-1}. This completes the proof. □

Let $\varphi_i(L; M)$ denote the coefficient of z^{n+i} in $\nabla_{L;M}(z)$ (i.e. $\varphi_i(L; M)=a_i$ in Proposition 2.1). Let $\Delta_{K;M}(t)$ be the symmetrized normalized Alexander polynomial of a knot K in M (i.e. which satisfies $\Delta_{K;M}(t)=\Delta_{K;M}(t^{-1})$ and $\Delta_{K;M}(1)=1$). Let $\Gamma(K; M)$ denote the second derivative of $\Delta_{K;M}(t)$ evaluated at $t=1$.

PROPOSITION 2.2. *Let K be a null-homologous knot in M. Then*

$$\Gamma(K; M)=2\varphi_1(K; M).$$

PROOF. In the proof of Proposition 2.1, we have shown that $\Delta_{K;M}(t^2)=\nabla_{K;M}(z)$, where $z=t-t^{-1}$. Then we can conclude that $\Delta_{K;M}(t)=\nabla_{K;M}(t^{1/2}-t^{-1/2})$. From this and Proposition 2.1, it follows that

$$\Gamma(K; M)=\left[\frac{d^2}{dt^2}\nabla_{K;M}(t^{1/2}-t^{-1/2})\right]_{t=1}=2\varphi_1(K; M). \qquad \square$$

Suppose that a knot K in M is not null-homologous. Let F be a Seifert surface for K. We can assume that the longitude of K, l is represented by d parallel curves on $\partial N(K)$. Consider the surface $F-N(K)$. We also denote this surface by F. Let $\alpha_1, \cdots, \alpha_{2g}, \gamma_1, \cdots, \gamma_{d-1}$ be simple closed curves representing a basis of $H_1(F; \mathbf{Q})$ as shown in Figure 1. Orient the α_is so that

$$\langle\alpha_i, \alpha_j\rangle=\begin{cases} 1, & i\text{: odd}, \quad j=i+1 \\ -1, & i\text{: even}, \quad j=i-1 \\ 0, & \text{otherwise}. \end{cases}$$

Let $V=(v_{ij})$ be a matrix given by $v_{ij}=lk(\alpha_i^+, \alpha_j; M)$. Then it can be shown that

$$\Gamma(K; M)=\left[\frac{d^2}{dt^2}t^{-g}\det(tV-V^T)\right]_{t=1}+\frac{d^2-1}{12},$$

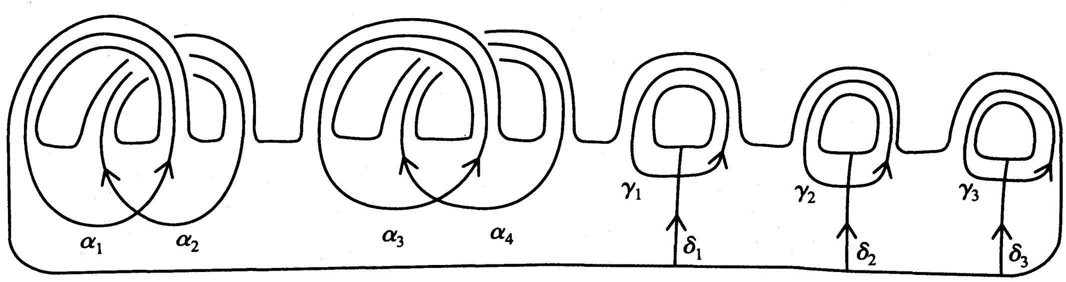

FIGURE 1

where g is genus of F (see [6]). By cutting F along δ_is (shown in Figure 1), we get a surface F'. Then we obtain a knot K' spanned by F'. We call this knot *an associate of* K. This is null-homologous and it's Seifert matrix is V. Hence $[(d^2/dt^2)t^{-g}\det(tV-V^T)]_{t=1}$ is equal to $2\varphi_1(K'; M)$. So we have shown the next proposition.

PROPOSITION 2.3. *Let K be a knot in M and K' be an associate of K. If a longitude of K is represented by d parallel curves, then*

$$\Gamma(K; M)=2\varphi_1(K'; M)+\frac{d^2-1}{12}.$$

3. Linking in $\chi(L; M)$.

Suppose that $L=\{(K_1, x_1), \cdots, (K_n, x_n)\}$ is a framed oriented link in a rational homology sphere M. Let m_i be a meridian of K_i and l_i be a longitude of K_i. Hereafter we choose m_i and l_i so that $\langle m_i, l_i\rangle>0$. Throughout the rest of this paper we consider the case that $lk(K_i, K_j; M)=0$ for all $i\neq j$. Then $\chi(L; M)$ is a rational homology sphere iff $\langle x_i, l_i\rangle\neq 0$ for all i. Suppose this is the case. Now if J_1 and J_2 are two knots in $M-L$, then we may think of them either as knots in M or as in $\chi(L; M)$. In either case they have a well-defined linking number.

LEMMA 3.1. *Suppose J_1 and J_2 are two knots in $M-L$. Then*

$$lk(J_1, J_2; \chi(L; M))=lk(J_1, J_2; M)-\sum_{i=1}^{n} lk(J_1, K_i; M)\frac{\langle m_i, l_i\rangle\langle m_i, x_i\rangle}{\langle x_i, l_i\rangle} lk(J_2, K_i; M).$$

PROOF. Suppose first that $lk(J_1, K_i; M)=0$ for all i. Then J_1 bounds a Seifert surface F in $M-L$. Hence $lk(J_1, J_2; \chi(L; M))=lk(J_1, J_2; M)$.

If $lk(J_1, K_i; M)\neq 0$ for some i, then we will proceed as follows. First, consider a band connected sum of $\prod_{i=1}^{n}\langle x_i, l_i\rangle$ copies of J_1. Here, we choose bands so that the band connected sum respects the orientations. We denote this knot by J'_1. Next we will "slide" J'_1 over the components of L until the linking number becomes zero. Here, "slide" J'_1 over K_i means following move. Let X_i be an oriented simple closed curve on $\partial N(K_i)$ representing x_i. Replace J'_1 with a band connected sum of J'_1 and X_i. Note that the band connected sum may either respect or disrespect the orientations of two curves. We determine the orientation of the band connected sum by that of J'_1.

Slide J'_1 over each K_i s_i times. We denote this knot by J''_1. Choose s_i to be positive if the band connected sum respects the orientations and choose s_i to be negative if it disrespects the orientations. Since $lk(K_k, K_l; M)=0$ for all $k\neq l$,

$$lk(J''_1, K_i; M)=lk(J'_1, K_i; M)+s_i\frac{\langle x_i, l_i\rangle}{\langle m_i, l_i\rangle}.$$

Suppose that $lk(J''_1, K_i; M)=0$. Then

$$s_i = -\frac{\langle m_i, l_i\rangle}{\langle x_i, l_i\rangle} lk(J'_1, K_i; M) = -\frac{\langle m_i, l_i\rangle}{\langle x_i, l_i\rangle}\left(\prod \langle x_j, l_j\rangle\right)\frac{\langle F_i, J_1\rangle}{\langle m_i, l_i\rangle}$$

$$= -\frac{1}{\langle x_i, l_i\rangle}\left(\prod \langle x_j, l_j\rangle\right)\langle F_i, J_1\rangle \in \boldsymbol{Z},$$

where F_i is a Seifert surface for K_i.

So, we can make the linking number of J''_1 and K_i zero for all i. Then

$$lk(J''_1, J_2; \chi(L; M)) = lk(J''_1, J_2; M) = lk(J'_1, J_2; M) + \sum_{i=1}^{n} s_i lk(X_i, J_2; M)$$

$$= lk(J'_1, J_2; M) - \left(\prod \langle x_j, l_j\rangle\right)\sum_{i=1}^{n}\frac{\langle m_i, l_i\rangle}{\langle x_i, l_i\rangle} lk(J_1, K_i; M)\langle m_i, x_i\rangle lk(J_2, K_i; M).$$

On the other hand

$$lk(J''_1, J_2; \chi(L; M)) = lk(J'_1, J_2; \chi(L; M)).$$

Since J'_1 is a band connected sum of $\prod\langle x_j, l_j\rangle$ copies of J_1, we get from two equations above

$$lk(J_1, J_2; \chi(L; M)) = lk(J_1, J_2; M) - \sum_{i=1}^{n} lk(J_1, K_i; M)\frac{\langle m_i, l_i\rangle\langle m_i, x_i\rangle}{\langle x_i, l_i\rangle} lk(J_2, K_i; M).$$

□

Now suppose that K is a null-homologous knot in $M-L$ such that K bounds a Seifert surface F in $M-L$. Let $\{e_1, \cdots, e_n\}$ be a basis for $H_1(F; \boldsymbol{Z})$. Now F, together with the choice of basis $\{e_i\}$, gives rise to two Seifert matrices: one for K considered as a knot in M, the other for K considered as a knot in $\chi(L; M)$. The (i, j) entry of the first matrix is given by $lk(e_i^+, e_j; M)$, and for the second by $lk(e_i^+, e_j; \chi(L; M))$. It follows easily from Lemma 3.1 that the two Seifert matrices are related as follows.

LEMMA 3.2. *Let M, L, K, F, and $\{e_i\}$ be given as above. Then*

$$V(K; \chi(L; M)) = V(K; M) - E\begin{pmatrix}\frac{\langle m_1, l_1\rangle\langle m_1, x_1\rangle}{\langle x_1, l_1\rangle} & & \\ & \ddots & \\ & & \frac{\langle m_n, l_n\rangle\langle m_n, x_n\rangle}{\langle x_n, l_n\rangle}\end{pmatrix}E^T,$$

where $E=(e_{ij})$ is given by $e_{ij}=lk(e_i, K_j; M)$.

LEMMA 3.3. *Suppose $\{K_1, \cdots, K_n\}$ is an oriented link in a rational homology sphere M with $lk(K_i, K_j; M)=0$ for all $i\neq j$ and each K_i is null-homologous. Then there exist Seifert surfaces F_1 and F_2 such that $\partial F_1 = K_1$, $\partial F_2 = K_2 \cup \cdots \cup K_n$, and $F_1 \cap F_2$ is* *either empty or consists of a single ribbon intersection. Furthermore, in the latter case,* $F_1 \cap F_2 \subset \mathrm{int} F_1$, $F_1 \cap \partial F_2 \subset K_2$, *and* $F_1 \cap F_2$ *does not separate* F_2.

PROOF. See [3]. The proof given there can be adapted to this case. □

LEMMA 3.4. *Let* $L=\{(K_1, x_1), \cdots, (K_n, x_n)\}$ *be a framed link in a rational homology sphere* M *with* $lk(K_i, K_j; M)=0$ *for all* $i \neq j$ *and each* K_i *is null homologous. Then for each* $1 \leq s \leq n$ *we have*

$$\varphi_1(K_n, \cdots, K_s; \chi(K_1, \cdots, K_{s-1}; M))$$
$$= \sum_{L' \subset K_1, \cdots, K_{s-1}} \left(\prod_{i \in L'} \frac{\langle m_i, x_i \rangle}{\langle x_i, l_i \rangle} \right) \varphi_1(L', K_s, \cdots, K_n; M).$$

Here the sum is taken over all sublinks of $\{K_1, \cdots, K_{s-1}\}$ *including the empty sublink. The product is over all* i *such that* $K_i \subset L'$, *which we have abbreviated as* $i \in L'$. *If* L' *is empty we interpret the product as* 1.

PROOF. We proceed by induction on n. If $n=1$, then the formula is trivially true. So suppose that L is a link of n components but that the lemma is true for any link of $n-1$ or fewer components.

If $s=1$, then again, the lemma is trivially true. So we shall begin with the case $s=2$. Thus we seek to prove that

$$\varphi_1(K_2, \cdots, K_n; \chi(K_1; M)) = \varphi_1(K_2, \cdots, K_n; M) + \frac{\langle m_1, x_1 \rangle}{\langle x_1, l_1 \rangle} \varphi_1(K_1, \cdots, K_n; M).$$

Now by Lemma 3.3 there exist Seifert surfaces F_1 and F_2 such that $\partial F_1 = K_1$, $\partial F_2 = K_2 \cup \cdots \cup K_n$, and either $F_1 \cap F_2$ is empty or consists of a single ribbon intersection. If the intersection is empty, then $\varphi_1(K_1, \cdots, K_n; M)=0$. (This is well known if $M=S^3$. See for example [4], and notice that the argument given there will work in the more general setting of an arbitrary rational homology sphere.) But by Lemma 3.2, $\varphi_1(K_2, \cdots, K_n; \chi(K_1; M)) = \varphi_1(K_2, \cdots, K_n; M)$ since, for any choice of basis of $H_1(F_2; \boldsymbol{Z})$, $E=0$. Hence the lemma is true.

Now suppose that $F_1 \cap F_2$ is a single ribbon intersection as described in Lemma 3.3. Let $\{e_i\}$ be a basis for $H_1(F_2; \boldsymbol{Z})$ such that e_1 meets F_1 transversely in a single point and $e_i \cap F_1 = \emptyset$ for $i>1$. Hence $E^T = (\pm 1\, 0 \cdots 0)$ and by Lemma 3.2 we have

$$W = V(K_2, \cdots, K_n; \chi(K_1; M)) = V(K_2, \cdots, K_n; M) - E\left(\frac{\langle m_1, x_1 \rangle}{\langle x_1, l_1 \rangle}\right) E^T.$$

By definition, $\nabla_{K_2, \cdots, K_n; \chi(K_1; M)}(z) = \det(tW - t^{-1} W^T)$, where $z = t - t^{-1}$. This gives

$$\nabla_{K_2, \cdots, K_n; \chi(K_1; M)}(z) = \det(tV - t^{-1} V^T) - \frac{\langle m_1, x_1 \rangle}{\langle x_1, l_1 \rangle} z \det(tV_{11} - t^{-1} V_{11}^T)$$

$$=\nabla_{K_2,\cdots,K_n;M}(z)-\frac{\langle m_1, x_1\rangle}{\langle x_1, l_1\rangle}\, z\nabla_{L';M}(z)\,,$$

where $V=V(K_2, \cdots, K_n; M)$, V_{11} is the (1, 1) minor of V, and L' is the n component link that is spanned by the Seifert surface obtained by cutting F_2 along F_1. Hence we have

$$\varphi_1(K_2, \cdots, K_n; \chi(K_1; M))=\varphi_1(K_2, \cdots, K_n; M)-\frac{\langle m_1, x_1\rangle}{\langle x_1, l_1\rangle}\,\varphi_0(L'; M)\,.$$

Thus it only remains to show that $\varphi_1(K_1, \cdots, K_n; M)=-\varphi_0(L'; M)$.

Let F be a Seifert surface for the link L obtained from F_1 and F_2 as follows. Away from $F_1 \cap F_2$ let F be $F_1 \cup F_2$ and near the intersection let F appear as in Figure 2. Let $\{d_j\}$ be a basis for $H_1(F_1; \boldsymbol{Z})$ so that $\{c, \{d_j\}, \{e_i\}\}$ is a basis for $H_1(F; \boldsymbol{Z})$, where c is the curve shown in the figure.

If $V'=V(K_1, M)$ is the Seifert matrix determined by $\{d_j\}$, then a Seifert matrix for L in M has the form

$$\left(\begin{array}{c|c|c} 0 & 0 & 1 \quad 0\cdots 0 \\ \hline 0 & V' & A \\ \hline \begin{matrix}1\\0\\ \vdots\\ 0\end{matrix} & A^T & V \end{array}\right).$$

Hence we have

$$\nabla_{L;M}(z)=\det\left(\begin{array}{c|cc} 0 & 0 & z \quad 0\cdots 0 \\ \hline 0 & tV'-t^{-1}V'^T & zA \\ \hline \begin{matrix}z\\0\\ \vdots\\ 0\end{matrix} & zA^T & tV-t^{-1}V^T \end{array}\right)$$

$$=-z^2\det\begin{pmatrix} tV'-t^{-1}V'^T & zA' \\ zA'^T & tV_{11}-t^{-1}V_{11}^T \end{pmatrix},$$

where A' is obtained from A by removing the first column.

But since F_2 a surface with $n-1$ boundary components, we may assume that the

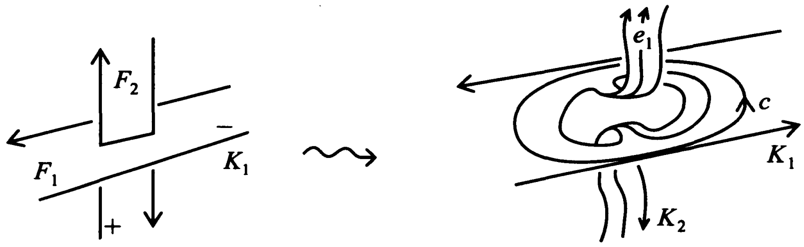

FIGURE 2

$\{e_i\}$ has been chosen so that V has the form

$$V=\begin{pmatrix} B & C \\ C^T & D \end{pmatrix},$$

where B is a $2h\times 2h$ matrix, C is a $2h\times n-2$ matrix, and D is an $n-2\times n-2$ symmetric matrix. This additional information gives

$$\nabla_{L;M}(z)=-z^2\det\left(\begin{array}{c|cc} tV'-t^{-1}V'^T & \multicolumn{2}{c}{zA'} \\ \hline zA'^T & tB_{11}-t^{-1}B_{11}^T & zC' \\ & zC'^T & zD \end{array}\right),$$

where C' is obtained from C by deleting the first row.

Now $\varphi_1(L;M)$ is the coefficient of z^{n+1} in $\nabla_{L;M}(z)$. This is actually the smallest power of z to appear since $lk(K_i, K_j; M)=0$ for all $i\neq j$ implies that $\varphi_0(L;M)=0$. Hence $\nabla_{L;M}(z)/z^{n+1}=\varphi_1(L;M)+\varphi_2(L;M)z^2+\cdots$, and $\varphi_1(L;M)=\lim_{z\to 0}\nabla_{L;M}(z)/z^{n+1}$. But

$$\nabla_{L;M}(z)/z^{n+1}=-\frac{1}{z}\det\left(\begin{array}{c|cc} tV'-t^{-1}V'^T & \multicolumn{2}{c}{zA'} \\ \hline zA_1'^T & tB_{11}-t^{-1}B_{11}^T & zC' \\ A_2'^T & C'^T & D \end{array}\right),$$

where A_1' is the first $2h-1$ columns of A' and A_2' is the last $n-2$ columns. And we may assume that $lk(e_2^+, e_i; M)=lk(e_i^+, e_2;, M)$ for all $i\geq 2$, hence every entry of the first row of $tB_{11}-t^{-1}B_{11}^T$ is divisible by z. Then we have

$$\begin{aligned}\varphi_1(L;M)&=-\left(\lim_{z\to 0}\det(tV'-t^{-1}V'^T)\right)\left(\lim_{z\to 0}\frac{1}{z}\det\begin{pmatrix} tB_{11}-t^{-1}B_{11}^T & zC' \\ C'^T & D\end{pmatrix}\right)\\ &=-\nabla_{K_1;M}(0)\varphi_0(L';M)=-1\cdot\varphi_0(L';M).\end{aligned}$$

This completes the proof for $s=2$.

Now assume that $s>2$. We have, using our inductive hypothesis, that

$$\begin{aligned}&\varphi_1(K_n, \cdots, K_s; \chi(K_{s-1}, \cdots, K_1; M))\\&=\varphi_1(K_n, \cdots, K_s; \chi(K_{s-1}, \cdots, K_2; \chi(K_1; M)))\\&=\sum_{L''\subset K_2,\cdots,K_{s-1}}\left(\prod_{i\in L''}\frac{\langle m_i, x_i\rangle}{\langle x_i, l_i\rangle}\right)\varphi_1(L'', K_s, \cdots, K_n; \chi(K_1; M))\,.\end{aligned}$$

Now, using the inductive hypothesis if $L''\neq\{K_2, \cdots, K_{s-1}\}$ and the result for $s=2$ otherwise, we have

$$\begin{aligned}&\varphi_1(K_n, \cdots, K_s; \chi(K_{s-1}, \cdots, K_1; M))\\&=\sum_{L''\subset K_2,\cdots,K_{s-1}}\left(\prod_{i\in L''}\frac{\langle m_i, x_i\rangle}{\langle x_i, l_i\rangle}\right)\Bigg[\varphi_1(L'', K_s, \cdots, K_n; M)\\&\qquad+\frac{\langle m_1, x_1\rangle}{\langle x_1, l_1\rangle}\varphi_1(K_1, L'', K_s, \cdots, K_n; M)\Bigg]\\&=\sum_{L''\subset K_1,\cdots,K_{s-1}}\left(\prod_{i\in L''}\frac{\langle m_i, x_i\rangle}{\langle x_i, l_i\rangle}\right)\varphi_1(L'', K_s, \cdots, K_n; M)\,.\end{aligned}$$ □

Next we consider the case that some components of L are not null-homologous. In Section 2, we constructed a null-homologous knot K' from a knot K which is not null homologous. We called this a knot associated to K. (See the description before Proposition 2.2.) By considering K_i' for each K_i, we obtain a link $L'=\{K_1', \cdots, K_n'\}$ such that each component of L' is null-homologous. We call this link *a link associated to L.* Note that $lk(K_i', K_j'; M)=0$ for all $i\neq j$. Then the next lemma holds.

LEMMA 3.5. *Let $L=\{(K_1, x_1), \cdots, (K_n, x_n)\}$ be a framed link in a rational homology sphere M with $lk(K_i, K_j; M)=0$ for all $i\neq j$ and $L'=\{K_1', \cdots, K_n'\}$ be as above. Then for each $1\leq s\leq n$ we have*

$$\begin{aligned}&\varphi_1(K_n', \cdots, K_s'; \chi(K_1, \cdots, K_{s-1}; M))\\&=\sum_{L''\subset K_1',\cdots,K_{s-1}'}\left(\prod_{i\in L''}\frac{\langle m_i, x_i\rangle}{\langle m_i, l_i\rangle\langle x_i, l_i\rangle}\right)\varphi_1(L'', K_s', \cdots, K_n'; M)\,.\end{aligned}$$

Here the sum is taken over all sublinks of $\{K_1', \cdots, K_{s-1}'\}$ including the empty sublink. The product is over all i such that $K_i'\subset L''$, which we have abbreviated as $i\in L''$. If L'' is empty we interpret the product as 1.

PROOF. Adapt the proof of Lemma 3.4 directly. In this case we get

$$\varphi_1(K_2', \cdots, K_n'; \chi(K_1; M))=\varphi_1(K_2', \cdots, K_n'; M)+\frac{\langle m_1, x_1\rangle}{\langle m_1, l_1\rangle\langle x_1, l_1\rangle}\varphi_1(K_1', \cdots, K_n'; M)$$

since, $E^T=(\pm 1/\langle m_1, l_1\rangle, 0, \cdots, 0)$ and by Lemma 3.2 we have

$$W=V(K_2', \cdots, K_n'; \chi(K_1; M))=V(K_2', \cdots, K_n')-E\left(\frac{\langle m_1, l_1\rangle\langle m_1, x_1\rangle}{\langle x_1, l_1\rangle}\right)E^T.$$

Then the argument in the proof of Lemma 3.4 shows the conclusion. □

Actually, many terms in the sum given in Lemmas 3.4 and 3.5 are zero. This follows from the following lemma.

LEMMA 3.6. *Suppose $L=\{K_1, \cdots, K_n\}$ is a link in a rational homology sphere M with $lk(K_i, K_j; M)=0$ for all $i\neq j$ and each K_i is null homologous, and furthermore $n>3$. Then $\varphi_1(L; M)=0$.*

PROOF. See [3]. The proof given there can be adapted to this case. □

Here the sum given in Lemmas 3.4 and 3.5 may actually just taken over all 1, 2, and 3-component sublinks.

4. A formula for λ.

In this section we will establish a formula for λ. It is derived from Walker's Dehn surgery formula, Lemma 3.4 and Lemma 3.5. First we consider the case that each component of a link L is null-homologous.

THEOREM 4.1. *Let $L=\{(K_1, x_1), \cdots, (K_n, x_n)\}$ be a framed oriented link in a rational homology sphere M with $lk(K_i, K_j; M)=0$ for all $i\neq j$ and each K_i is null-homologous. Let m_i be a meridian of K_i and l_i be a longitude of K_i for each i. Then the Walker invariant of $\chi(L; M)$ is given by*

$$\lambda(\chi(L; M))=\lambda(M)+\sum_{i=1}^{n}\tau(m_i, x_i; l_i)+2\sum_{L'\subset L}\left(\prod_{i\in L'}\frac{\langle m_i, x_i\rangle}{\langle x_i, l_i\rangle}\right)\varphi_1(L'; M).$$

Actually, the sum need only be taken over those sublinks of L having less than four components.

PROOF. We proceed by induction on n. If $n=1$, then by Walker's theorem as mentioned in section 1 and Proposition 2.2, it follows that

$$\lambda(\chi(L; M))=\lambda(M)+\tau(m_1, x_1; l_1)+2\frac{\langle m_1, x_1\rangle}{\langle x_1, l_1\rangle}\varphi_1(K_1; M).$$

Hence the theorem is true for $n=1$.

Now assume that $n>1$. Then

$$\begin{aligned}\lambda(\chi(L; M))&=\lambda(\chi(K_n; \chi(K_1, \cdots, K_{n-1}; M)))\\&=\lambda(\chi(K_1, \cdots, K_{n-1}; M))+\tau(m_n, x_n; l_n)\end{aligned}$$

$$+2\frac{\langle m_n, x_n\rangle}{\langle x_n, l_n\rangle}\varphi_1(K_n; \chi(K_1, \cdots, K_{n-1}; M)).$$

By the inductive hypothesis and Lemma 3.4, we have

$$\begin{aligned}\lambda(\chi(L; M)) &= \lambda(M) + \sum_{i=1}^{n-1}\tau(m_i, x_i; l_i) + 2\sum_{L'\subset K_1,\cdots,K_{n-1}}\left(\prod_{i\in L'}\frac{\langle m_i, x_i\rangle}{\langle x_i, l_i\rangle}\right)\varphi_1(L'; M)\\ &\quad + \tau(m_n, x_n; l_n) + 2\frac{\langle m_n, x_n\rangle}{\langle x_n, l_n\rangle}\left(\sum_{L'\subset K_1,\cdots,K_{n-1}}\left(\prod_{i\in L'}\frac{\langle m_i, x_i\rangle}{\langle x_i, l_i\rangle}\right)\varphi_1(L', K_n; M)\right)\\ &= \lambda(M) + \sum_{i=1}^{n}\tau(m_i, x_i; l_i) + 2\sum_{L'\subset L}\left(\prod_{i\in L'}\frac{\langle m_i, l_i\rangle}{\langle x_i, l_i\rangle}\right)\varphi_1(L'; M).\end{aligned}$$

Finally, using Lemma 3.6, we see that only sublinks having less than four components will contribute to the sum. □

Suppose that some components of L are not null-homologous. In Section 3, we considered a link $L'=\{K_1', \cdots, K_n'\}$ such that each component of L' is null-homologous. (See the description before Proposition 3.5.) We called this *a link associated to L*. We can assume that the longitude l_i of K_i consists of d_i parallel curves on $\partial N(K_i)$. Then using Proposition 2.3 and Lemma 3.5, and proceeding the same as the proof of Theorem 4.1, we obtain the next theorem.

THEOREM 4.2. *Let $L=\{(K_1, x_1), \cdots, (K_n, x_n)\}$ be a framed oriented link in a rational homology sphere M with $lk(K_i, K_j; M)=0$ for all $i\neq j$. Let $L'=\{K_1', \cdots, K_n'\}$ be a link associated to L. Let m_i be a meridian of K_i and l_i be a longitude of K_i. If l_i is represented by d_i parallel curves, then the Walker invariant of $\chi(L; M)$ is given by*

$$\begin{aligned}\lambda(\chi(L; M)) &= \lambda(M) + \sum_{i=1}^{n}\tau(m_i, x_i; l_i) + 2\sum_{L''\subset L'}\left(\prod_{i\in L''}\frac{\langle m_i, x_i\rangle}{\langle m_i, l_i\rangle\langle x_i, l_i\rangle}\right)\varphi_1(L''; M)\\ &\quad + \frac{1}{12}\sum_{i=1}^{n}\frac{\langle m_i, x_i\rangle}{\langle m_i, l_i\rangle\langle x_i, l_i\rangle}(d_i^2-1).\end{aligned}$$

Actually, the sum need only be taken over those sublinks of L' having less than four components.

PROOF. We can prove this theorem as same as Theorem 4.1. □

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