TOKYO J. MATH.
VOL. 17, NO. 2, 1994

The Condition of Beineke and Harary on Edge-Disjoint Paths Some of Which are Openly Disjoint

Hikoe ENOMOTO and Atsushi KANEKO

Keio University

Abstract. A pair (t, s) of nonnegative integers is said to be a connectivity pair for distinct vertices x and y of a graph G if it satisfies the following conditions which were introduced by Beineke and Harary:

(1) For any subset $T \subseteq V(G)-\{x, y\}$ and any subset $S \subseteq E(G)$ with $|T| \leqq t$, $|S| \leqq s$ and $|T|+|S|<t+s$, $G-(T \cup S)$ still contains an x-y path.

(2) There exist a subset $T' \subseteq V(G)-\{x, y\}$ and a subset $S' \subseteq E(G)$ with $|T'|=t$ and $|S'|=s$ such that $G-(T' \cup S')$ contains no x-y path.

Let q, r, s and t be integers with $t \geqq 0$ and $s \geqq 1$ such that $t+s=q(t+1)+r$, $1 \leqq r \leqq t+1$, and let x and y be distinct vertices of a graph G. It is shown that if $q+r>t$ and if (t, s) is a connectivity pair for x and y, then G contains $t+s$ edge-disjoint x-y paths $t+1$ of which are openly disjoint.

1. Introduction.

In this note, G will denote a finite undirected graph with no loops but with multiple edges being allowed. Let x and y be distinct vertices of G. Denote the set of edges joining x to y by $E_G(x, y)$. ($E_G(x, y)$ may be empty.) A set of x-y paths $P_1, P_2, \cdots, P_n$ is said to be *openly* disjoint if $V(P_i) \cap V(P_j)=\{x, y\}$ for all $i \neq j$, $1 \leqq i, j \leqq n$. An (x, y)-*n*-*fan* F means a union of n openly disjoint x-y paths. Notice that an edge xy is also an x-y path, so that in the above definition, some x-y paths in F may be elements of $E_G(x, y)$. (For terms not defined here, see [2].) A pair (t, s) of nonnegative integers is said to be a *connectivity pair* for distinct vertices x and y of G if it satisfies the following conditions:

(1) For any subset $T \subseteq V(G)-\{x, y\}$ and any subset $S \subseteq E(G)$ with $|T| \leqq t$, $|S| \leqq s$ and $|T|+|S|<t+s$, $G-(T \cup S)$ still contains an x-y path.

(2) There exist a subset $T' \subseteq V(G)-\{x, y\}$ and a subset $S' \subseteq E(G)$ with $|T'|=t$ and $|S'|=s$ such that $G-(T' \cup S')$ contains no x-y path.

The following "theorem" was given in [1].

THEOREM A. *If (t, s) is a connectivity pair for distinct vertices x and y of G, then G contains $t+s$ edge-disjoint x-y paths t of which are openly disjoint.*

Received March 3, 1993

Unfortunately, the proof given in [1] seems to be erroneous. With the authors' best knowledge, no alternative proof for this "theorem" has been found. As noted by Mader in [4], we also feel that under the condition in "Theorem A" G may contain more than t openly disjoint paths. In fact, although there remain some restrictions, we prove the following theorem.

THEOREM 1. *Let q, r, s and t be integers with $t \geqq 0$ and $s \geqq 1$ such that $t+s=q(t+1)+r$, $1 \leqq r \leqq t+1$, and let x and y be distinct vertices of a graph G. If $q+r>t$ and if (t, s) is a connectivity pair for x and y, then G contains $t+s$ edge-disjoint x-y paths $t+1$ of which are openly disjoint.*

For positive integers λ and n, and for a graph G with distinct vertices x and y, we consider the following property $F(\lambda, n)$ for x and y:

For any subset $T \subseteq V(G)-\{x, y\}$ and any subset $S \subseteq E(G)$ with $|T| \leqq n-1$ and $\lambda|T|+|S|<\lambda n$, $G-(T \cup S)$ still contains an x-y path.

This definition sounds complicated, as does that of the connectivity pair. However, this property is characterized in terms of edge-disjoint n-fans in [3].

THEOREM B. *Let λ and n be positive integers, and let x and y be distinct vertices of a graph G. A graph G has the property $F(\lambda, n)$ for x and y if and only if G contains λ edge-disjoint (x, y)-n-fans.*

In order to prove Theorem 1, we use this necessary and sufficient condition for a graph to have a given number of edge-disjoint n-fans.

2. Proof of Theorem 1.

By the definition (2) of the connectivity pair we see that $|E_G(x, y)| \leqq s$. We claim that in the definition (1) of the connectivity pair the restriction that $|S| \leqq s$ is meaningless. For suppose that there exist a subset $S'=\{e_1, e_2, \cdots, e_{s+k}\} \subseteq E(G)$ and a subset $T' \subseteq V(G)-\{x, y\}$ with $|S'|=s+k$, $1 \leqq k \leqq t-1$, and $|T'|+|S'|<t+s$ such that $G-(T' \cup S')$ contains no x-y path. Then we see that $E_G(x, y) \subseteq S'$. Since $|E_G(x, y)| \leqq s$, we may assume that $E_G(x, y) \subseteq \{e_1, e_2, \cdots, e_s\}$. Put $S=\{e_1, e_2, \cdots, e_s\}$ and $T=T' \cup \{v_1, v_2, \cdots, v_k\}$ where v_i, $1 \leqq i \leqq k$, is an end of e_{s+i} not contained in $\{x, y\}$. $G-(S \cup T)$ contains no x-y path and moreover we know that $|T| \leqq t$, $|S| \leqq s$ and $|S|+|T|=|S'|+|T'|<t+s$. This contradicts the definition of the connectivity pair, implying that we can take no account of the restriction that $|S| \leqq s$, as claimed. Let H be a graph obtained from G by adding new $t+1-r$ multiple edges joining x to y. Denote this set of new multiple edges by E_{xy}. Put $p=t+1-r=|E_{xy}|$. Since (t, s) is a connectivity pair for x and y of G, $(t, s+p)$ is a connectivity pair for x and y of H. Note that $t+s+p=(q+1)(t+1)$. Thus since $q+1 \geqq 1$, it follows from our claim that H has the property $F(q+1, t+1)$ for x and y. The graph H therefore contains $q+1$

edge-disjoint (x, y)-$t+1$-fans $F_1, F_2, \cdots, F_{q+1}$ by Theorem B. Since $(q+1)-p=q+r-t>0$, we may assume that the paths in F_1 contain no edge from E_{xy}. Put $F_1=\bigcup_{i=1}^{t+1} P_i$ and $\bigcup_{j=2}^{q+1}(F_j-E_{xy})=\bigcup_{k=t+2}^{t+s} P_k$, where the paths P_i, $1\leqq i\leqq t+1$, and the paths P_k, $t+2\leqq k\leqq t+s$, are x-y paths of which F_1 and the F_j, $2\leqq j\leqq q+1$, consist respectively, and which are contained in G. We see that $P_1, P_2, \cdots, P_{t+s}$ are $t+s$ edge-disjoint x-y paths in G and $P_1, P_2, \cdots, P_{t+1}$ are openly disjoint. □

From our theorem the following conjecture may hold.

CONJECTURE. Let t and s be integers with $t\geqq 0$ and $s\geqq 1$, and let x and y be distinct vertices of a graph G. If (t, s) is a connectivity pair for x and y, then G contains $t+s$ edge-disjoint x-y paths $t+1$ of which are openly disjoint.

References

[1] L. W. BEINEKE and F. HARARY, The connectivity function of a graph, Mathematika **14** (1967), 197–202.

[2] G. CHARTRAND and L. LESNIAK, *Graphs & Digraphs*, 2nd ed., Wadsworth (1986).

[3] Y. EGAWA, A. KANEKO and M. MATSUMOTO, A mixed version of Menger's theorem, Combinatorica **11** (1991), 71–74.

[4] W. MADER, Connectivity and edge-connectivity in finite graphs, *Survey in Combinatorics* (ed. B. Bollobás), Cambridge Univ. Press (1979), 66–95.

Present Addresses:

HIKOE ENOMOTO
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY,
HIYOSHI, KOHOKU-KU, YOKOHAMA, 223 JAPAN.

ATSUSHI KANEKO
DEPARTMENT OF ELECTRONIC ENGINEERING, KOGAKUIN UNIVERSITY,
NISHI-SHINJUKU, SHINJUKU-KU, TOKYO, 163–91 JAPAN.