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Moduli of Stable 2-Bundles with $\Delta=-5$ on a Non-Singular Cubic Surface

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1. Introduction.

Let S be a non-singular cubic surface in the projective 3-space $\boldsymbol{P}^3$ defined over an algebraically closed field of arbitrary characteristic. In [H2], we have constructed universal families of moduli of semi-stable 2-bundles with $\Delta=C_1^2-4C_2=-4$ on S. In this article, we will treat the case of $\Delta=-5$. For a torsion free sheaf, the semi-stability may vary by twisting line bundles. In fact, for a modulo 2 class in $\mathrm{Pic}\,S$, we had to consider 243 species of moduli of semi-stable 2-bundles with $\Delta=-4$. On the other hand, a semi-stable sheaf of rank 2 with $\Delta=-5$ is always a μ-stable bundle (Proposition 2.2). We can normalize the first Chern class to reduce two cases: (i) $C_1=e$, $C_2=1$ and (ii) $C_1=h$, $C_2=2$ where e is an exceptional class and h is the hyperplane class. The moduli of the first case is obtained by contracting the line, corresponding to the first Chern class C_1, on the cubic surface S (Theorem 3.4). The moduli of the second case is the cubic surface S itself (Theorem 6.3). In each case, we will construct a universal family. It is worth while to mention that the universal family of the first case which we will construct is obtained by modification of the family considered in the case of $\Delta=-4$. For the second case, to construct a universal family, we will use the method introduced in [H2, §3 Limits of extensions]. We shall use notation and terminology in [H1, H2] freely.

2. Stability and locally freeness.

LEMMA 2.1. *Let $\mathscr{E}$ be a 2-bundle with $\Delta(\mathscr{E})=C_1(\mathscr{E})^2-4C_2(\mathscr{E})\equiv-1 \pmod 4$ on S. If $\Delta(\mathscr{E})\geq-1$, $\mathscr{E}$ is not μ-stable.*

PROOF. By [H2, Lemma 2.1 (2)], there is a line bundle $\mathscr{L}$ such that for $\mathscr{E}'=\mathscr{E}\otimes\mathscr{L}$, $C_1(\mathscr{E}')=e$ for some exceptional class e or $C_1(\mathscr{E}')=h$. We first consider the case $C_1(\mathscr{E}')=e$.

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Since $C_2(\mathscr{E}')\leq 0$, $\chi(\mathscr{E}')=2-C_2(\mathscr{E}')\geq 2$. There is a non-zero section s of $\mathscr{E}'$. If $\mathscr{E}'$ is μ-stable, the scheme of zeroes of s has codimension ≥ 2. Therefore $C_2(\mathscr{E}')=0$ and there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\ s\ } \mathscr{E}' \longrightarrow \mathcal{O}_S(e) \longrightarrow 0\,.$$

This must split, because of $h^1(S, \mathcal{O}_S(-e))=0$. Thus $\mathscr{E}'$ is not μ-stable. Now consider the case $C_1(\mathscr{E}')=h$. Since $C_2(\mathscr{E}')\leq 1$, $\chi(\mathscr{E}')=5-C_2(\mathscr{E}')\geq 4$. There is a non-zero section s of $\mathscr{E}'$. Let Z be the scheme of zeroes of s. Now assume that $\mathscr{E}'$ is μ-stable. If Z has a codimension one component, it must be a line E by the μ-stability. But $C_1(\mathscr{E}'(-E))^2=-5$ and $C_2(\mathscr{E}'(-E))<0$, this is absurd. If Z has codimension ≥ 2, $\operatorname{length} Z=C_2(\mathscr{E}')=1$ or 0. Since $\operatorname{Ext}^1(\mathcal{O}_S(h)\otimes \mathscr{I}_Z, \mathcal{O}_S)=(0)$, $\mathscr{E}'\cong \mathcal{O}_S\oplus \mathcal{O}_S(h)\otimes \mathscr{I}_Z$. This is a contradiction. □

PROPOSITION 2.2. *Let $\mathscr{E}$ be a semi-stable sheaf of rank 2 on S with $\Delta=-5$, then $\mathscr{E}$ is a μ-stable vector bundle.*

PROOF. Since $C_1(\mathscr{E})\equiv e$ (mod 2) for some exceptional class e or $C_1(\mathscr{E})\equiv h$ (mod 2), $C_1(\mathscr{E})\cdot h$ and 2 are coprime. Therefore $\mathscr{E}$ is μ-stable. If $\mathscr{E}$ is not locally free, the quotient $\mathscr{T}=\mathscr{E}^{\vee\vee}/\mathscr{E}$ is not zero and there is a short exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}^{\vee\vee} \longrightarrow \mathscr{T} \longrightarrow 0\,.$$

Since $\dim \operatorname{Supp}\mathscr{T}=0$, $C_1(\mathscr{E}^{\vee\vee})=C_1(\mathscr{E})$ and $C_2(\mathscr{E}^{\vee\vee})=C_2(\mathscr{E})-\operatorname{length}\mathscr{T}$. Therefore $\mathscr{E}^{\vee\vee}$ is a μ-stable vector bundle with $\Delta(\mathscr{E}^{\vee\vee})\equiv -1$ (mod 4) and $\Delta(\mathscr{E}^{\vee\vee})\geq -1$. This is a contradiction. □

In the following sections, we consider the two cases (i) $C_1=e$, $C_2=1$, (ii) $C_1=h$, $C_2=2$ separately.

3. Moduli of μ-stable 2-bundles with $C_1=e$ and $C_2=1$.

Let E be a line on S and e be the exceptional class corresponding to E. Fix a line E in this section.

LEMMA 3.1. *Let $\mathscr{E}$ be a μ-stable 2-bundle with $C_1(\mathscr{E})=e$ and $C_2(\mathscr{E})=1$. Then there is a short exact sequence*

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathscr{E} \longrightarrow O_S(e)\otimes \mathscr{M}_x \longrightarrow 0$$

for some closed point x in S.

PROOF. Since $\chi(\mathscr{E})=1$, $\mathscr{E}$ has a non-zero section s. The scheme of zeroes of s has codimension ≥ 2, by the μ-stability. We have a desired short exact sequence. □

For any closed point x in S, since $\mathscr{E}xt^1(\mathcal{O}_S(e)\otimes \mathscr{M}_x, \mathcal{O}_S)\cong k(x)$, there is a unique

2-bundle $\mathscr{E}_x$ which is defined by the following non-trivial extension

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathscr{E}_x \longrightarrow \mathcal{O}_S(e) \otimes \mathscr{M}_x \longrightarrow 0 .$$

Remark that for any closed point x in S, $\mathscr{E}_x$ is μ-stable.

LEMMA 3.2. *For any closed points x and y in S, $\mathscr{E}_x \cong \mathscr{E}_y$ if and only if $x = y$ or $x, y \in E$.*

PROOF. Now assume $\mathscr{E}_x \cong \mathscr{E}_y$ and $x \neq y$. Then $h^0(S, \mathscr{E}_x) = h^0(S, \mathscr{E}_y) \geq 2$. Thus $h^0(S, \mathcal{O}_S(e) \otimes \mathscr{M}_x) \neq 0$ and $h^0(S, \mathcal{O}_S(e) \otimes \mathscr{M}_y) \neq 0$. Therefore $x, y \in E$. Conversely, assume that $x, y \in E$. There are short exact sequences

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(e) \otimes \mathscr{M}_x \xrightarrow{\ i\ } \mathcal{O}_E(-2) \longrightarrow 0 ,$$

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(e) \otimes \mathscr{M}_y \xrightarrow{\ j\ } \mathcal{O}_E(-2) \longrightarrow 0 .$$

If $x = y$, there are nothing to prove. Assume that $x \neq y$. Let $\mathscr{E}$ be the kernel of the homomorphism

$$i + j : \quad \mathcal{O}_S(e) \otimes \mathscr{M}_x \oplus \mathcal{O}_S(e) \otimes \mathscr{M}_y \longrightarrow \mathcal{O}_E(-2) .$$

There is the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_S(e) \otimes \mathscr{M}_y & \longrightarrow & \mathcal{O}_E(-2) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathscr{E} & \longrightarrow & \mathcal{O}_S(e) \otimes \mathscr{M}_x & \longrightarrow & 0 . \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_S & = & \mathcal{O}_S & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Hence $\mathscr{E} \cong \mathscr{E}_x \cong \mathscr{E}_y$. $\square$

Let $\tilde{\mathscr{E}}$ be the 2-bundle on $S \times S$, obtained in [H2, Proposition 2.8], defined by the following short exact sequence

$$0 \longrightarrow O_{S \times S} \longrightarrow \tilde{\mathscr{E}} \longrightarrow (\mathcal{O}_S(h-e) \boxtimes \mathcal{O}_S(e)) \otimes \mathscr{I}_{\Delta_S} \longrightarrow 0 .$$

PROPOSITION 3.3. (1) *For any closed point x in S,*

$$\tilde{\mathscr{E}}\,|_{\{x\}\times S}\cong\mathscr{E}_x\,.$$

(2) *For any closed point y in S,*

$$\tilde{\mathscr{E}}\,|_{E\times\{y\}}\cong\mathcal{O}_E(1)\oplus\mathcal{O}_E(1)\,.$$

PROOF. (1) is obvious (cf. [H2, Proposition 2.8]). (2) For any closed point y in S, the restriction $\tilde{\mathscr{E}}\,|_{S\times\{y\}}$ of $\tilde{\mathscr{E}}$ to $S\times\{y\}$ is contained in the following short exact sequence

$$0\longrightarrow\mathcal{O}_S\xrightarrow{\ s\ }\tilde{\mathscr{E}}\,|_{S\times\{y\}}\longrightarrow\mathcal{O}_S(h-e)\otimes\mathscr{M}_y\longrightarrow 0\,. \tag{3.3.1}$$

If the closed point y is contained in a reducible member $F+G\in|H-E|$, the restriction $\tilde{\mathscr{E}}\,|_{E\times\{y\}}\cong\mathcal{O}_E(f)\oplus\mathcal{O}_E(g)\cong\mathcal{O}_E(1)\oplus\mathcal{O}_E(1)$ by [H2, Lemma 2.6 (1), (2)]. If the closed point y is contained in an irreducible member $D\in|H-E|$, we may assume $y\in D\cap E$ by [H2, Lemma 2.6 (4)]. The section s of $\tilde{\mathscr{E}}\,|_{S\times\{y\}}$ given by the sequence (3.3.1), restricting to E, gives $\mathcal{O}_E(1)$ as a subbundle of $\tilde{\mathscr{E}}\,|_{E\times\{y\}}$. Thus we have $\tilde{\mathscr{E}}\,|_{E\times\{y\}}\cong\mathcal{O}_E(1)\oplus\mathcal{O}_E(1)$. □

Let $\Phi_E=\Phi_{|h+e|}:S\to S_E$ be the contraction of the line E. For the morphism $\phi_E=\Phi_E\times\mathrm{id}_S: S\times S\to S_E\times S$, $\mathscr{F}_{S_E}:=\phi_{E*}(\tilde{\mathscr{E}}\otimes p_1^*\mathcal{O}_S(e))$ is a vector bundle on $S_E\times S$ and $\phi_E^*\mathscr{F}_{S_E}\cong\tilde{\mathscr{E}}\otimes p_1^*\mathcal{O}_S(e)$ by virtue of a generalization of a theorem of Schwarzenberger by S. Ishimura [Is] and Proposition 3.3 (2). By Lemma 3.2 and Proposition 3.3 (1), we obtain:

THEOREM 3.4. *The family $(S_E,\mathscr{F}_{S_E})$ is a universal family of the moduli of μ-stable 2-bundles with $C_1=e$ and $C_2=1$ on the cubic surface S.*

4. Stable 2-bunbles with $C_1=h$ and $C_2=2$.

Let $M(h,2)$ be the set of isomorphism classes of μ-stable 2-bundles with $C_1=h$ and $C_2=2$ on S and, for a line E on S, $M(h,2)_E$ be the subset of $M(h,2)$ consisting of vector bundles which contain $\mathcal{O}_S(e)$ as a subsheaf. The purpose of the remaining sections is to construct a universal family of $M(h,2)$ over S and to prove that under the isomorphism $M(h,2)\cong S$, $M(h,2)_E$ is isomorphic to E.

From now on, fix a Schläfli's double-six $(\{E_i\},\{F_i\})$ and denote by $l=(h+e_1+\cdots+e_6)/3$ the contraction of $\{E_i\}$ [H2, §1].

LEMMA 4.1. *Let $\mathscr{E}$ be a μ-stable 2-bundle with $C_1(\mathscr{E})=h$ and $C_2(\mathscr{E})=2$ on S. Then either* (1) *there is a non-trivial extension*

$$0\longrightarrow\mathcal{O}_S(h-l)\longrightarrow\mathscr{E}\longrightarrow\mathcal{O}_S(l)\longrightarrow 0$$

or (2) *there is a non-trivial extension*

$$0\longrightarrow\mathcal{O}_S(f_i)\longrightarrow\mathscr{E}\longrightarrow\mathcal{O}_S(l-e_i)\longrightarrow 0$$

for some i.

PROOF. Since $\chi(\mathcal{E}(l-h))=1$ and $C_1(\mathcal{E}(l-h))\cdot h=3$, $\mathcal{E}(l-h)$ has a non-zero section s. If the scheme of zeroes of s has codimension ≥ 2, we have the case (1) since $C_2(\mathcal{E}(l-h))=0$. Assume that the scheme of zeroes of s has a codimension one component D. By the μ-stability, D is a line. Since $C_2(\mathcal{E}(l-h-d))=-2l\cdot d\geq 0$, $l\cdot d=0$. Therefore $D=E_i$ for some i and we have the case (2). □

LEMMA 4.2. (1) *For a non-trivial extension*

$$0\longrightarrow \mathcal{O}_S(h-l)\longrightarrow \mathcal{E}\longrightarrow \mathcal{O}_S(l)\longrightarrow 0\,,$$

if $\mathcal{E}$ is not μ-stable, there is a non-trivial extension

$$0\longrightarrow \mathcal{O}_S(l-e_i)\longrightarrow \mathcal{E}\longrightarrow \mathcal{O}_S(f_i)\longrightarrow 0$$

for some i.

(2) *For a non-trivial extension*

$$0\longrightarrow \mathcal{O}_S(f_i)\longrightarrow \mathcal{E}\longrightarrow \mathcal{O}_S(l-e_i)\longrightarrow 0\,,$$

$\mathcal{E}$ is μ-stable.

PROOF. (1) If $\mathcal{E}$ is not μ-stable, there is a class $d\in \operatorname{Pic} S$ of degree two such that $\mathcal{E}(-d)$ has a non-zero section. Since $l-d$ is of degree one, the complete linear system $|l-d|$ consists of a line E. Now consider the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_E(h-2l+e) & \longrightarrow & \mathcal{E}(e-l)\otimes\mathcal{O}_E & \longrightarrow & \mathcal{O}_E(-1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S(h-2l+e) & \longrightarrow & \mathcal{E}(e-l) & \longrightarrow & \mathcal{O}_S(e) & \longrightarrow & 0 \\
 & & {\scriptstyle a}\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S(h-2l) & \longrightarrow & \mathcal{E}(-l) & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0\,. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Let ξ be the extension class in $\operatorname{Ext}^1(\mathcal{O}_S(l), \mathcal{O}_S(h-l))\cong H^1(S, \mathcal{O}_S(h-2l))$ corresponding to the given extension. By the diagram chasing, $H^0(S, \mathcal{E}(e-l))\neq(0)$ if and only if $H^1(a)(\xi)=0$. Thus $H^0(E, \mathcal{O}_E(h-2l+e))\neq(0)$ and $e\cdot(h-2l+e)=-2e\cdot l\geq 0$. Therefore $l\cdot e=0$ so that $E=E_i$ for some i. Then we get a desired extension. (2) is obvious. □

LEMMA 4.3. *Let $\mathcal{E}$ be a μ-stable 2-bundle with $C_1(\mathcal{E})=h$ and $C_2(\mathcal{E})=2$ on S. If $\mathcal{E}$*

contains a line bundle of degree one, there is a non-trivial extension

$$0 \longrightarrow \mathcal{O}_S(e) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_S(h-e) \longrightarrow 0$$

for some exceptional class e.

PROOF. Let $\mathcal{O}_S(d)$ be a subsheaf of degree one. Since $\mathcal{E}(-d)$ has a non-zero section whose scheme of zeroes has codimension ≥ 2, $C_2(\mathcal{E}(-d)) = 1 + d^2 \geq 0$. Therefore $\chi(\mathcal{O}_S(d)) = 1 + (1+d^2)/2 \geq 1$ so that d is an effective class. Thus we have a desired extension. □

Put

$$P = P(\mathrm{Ext}^1(\mathcal{O}_S(l), \mathcal{O}_S(h-l))^\vee)$$
$$P_i = P(\mathrm{Ext}^1(\mathcal{O}_S(l-e_i), \mathcal{O}_S(f_i))^\vee)$$
$$Q_i = P(\mathrm{Ext}^1(\mathcal{O}_S(f_i), \mathcal{O}_S(l-e_i))^\vee) .$$

Since

$$\dim \mathrm{Ext}^1(\mathcal{O}_S(l), \mathcal{O}_S(h-l)) = h^1(\mathcal{O}_S(h-2l)) = 3$$
$$\dim \mathrm{Ext}^1(\mathcal{O}_S(l-e_i), \mathcal{O}_S(f_i)) = h^1(\mathcal{O}_S(2f_i-h)) = 2$$
$$\dim \mathrm{Ext}^1(\mathcal{O}_S(f_i), \mathcal{O}_S(l-e_i)) = h^1(\mathcal{O}_S(h-2f_i)) = 1 ,$$

we have isomorphisms $P \cong P^2$, $P_i \cong P^1$ and Q_i is a point. By the proof of Lemma 4.2 (1), we can regard Q_i as a point in P. And also by Lemma 4.2 and Lemma 4.3, $M(h, 2) = (P \setminus \{Q_1, \cdots, Q_6\}) \cup P_1 \cup \cdots \cup P_6$ and $M(h, 2)_{F_i} = P_i$ set-theoretically. Let

$$\Phi : S' \to P$$

be the blowing up of P at $Q_1, \cdots, Q_6$. In the following section, we will construct a universal family of $M(h, 2)$ over S' and prove that the restriction of the family to the exceptional curve $\Phi^{-1}(Q_i)$ is a universal family of $M(h, 2)_{F_i}$. In the next to the following section, we will prove that the moduli of $M(h, 2)$ is isomorphic to S itself and that under the isomorphism, the moduli of $M(h, 2)_E$ is isomorphic to E for any line E on S.

5. Universal family of $M(h, 2)$.

In this section, to construct a universal family of $M(h, 2)$, we will use the method introduced in [H1, §3 Limits of extensions] and use notation and terminology in it.

LEMMA 5.1. *For an unstable extension*

$$0 \longrightarrow \mathcal{O}_S(h-l) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_S(l) \longrightarrow 0 ,$$

the destabilizing diagram is

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S(h-l) & \longrightarrow & \mathcal{O}_S(f_i) & \longrightarrow & \mathcal{O}_{E_i} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_S(h-l) & \longrightarrow & \mathscr{E} & \longrightarrow & \mathcal{O}_S(l) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathcal{O}_S(l-e_i) & = & \mathcal{O}_S(l-e_i) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

for some i. And the homomorphism

$$c:\quad \mathrm{Ext}^1(\mathcal{O}_S(l), \mathcal{O}_S(h-l)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_S(l-e_i), \mathcal{O}_S(f_i))$$

is surjective.

PROOF. By Lemma 4.2 (1), the former part is clear. The second part is obvious. □

There is a universal family $\tilde{\mathscr{E}}_{\boldsymbol{P}}$ of extensions in $\mathrm{Ext}^1(\mathcal{O}_S(l), \mathcal{O}_S(h-l))$ such that $\tilde{\mathscr{E}}_{\boldsymbol{P}}$ is given by a non-trivial extension

$$0 \longrightarrow \mathcal{O}_S(h-l) \boxtimes \mathcal{O}_{\boldsymbol{P}}(1) \longrightarrow \tilde{\mathscr{E}}_{\boldsymbol{P}} \longrightarrow \mathcal{O}_S(l) \boxtimes \mathcal{O}_{\boldsymbol{P}} \longrightarrow 0,$$

over $S \times \boldsymbol{P}$ (cf. [HI, Theorem 2.2]). For the morphism $\phi = \mathrm{id}_S \times \Phi : S \times S' \to S \times \boldsymbol{P}$, put $\tilde{\mathscr{E}}_{S'} := \phi^* \tilde{\mathscr{E}}_{\boldsymbol{P}}$ and $\tilde{\mathscr{E}}_{\Phi^{-1}(Q_i)} := \tilde{\mathscr{E}}_{S'}|_{S \times \Phi^{-1}(Q_i)}$. By the surjective homomorphisms

$$\tilde{\mathscr{E}}_{S'} \longrightarrow \tilde{\mathscr{E}}_{\Phi^{-1}(Q_i)} \longrightarrow \mathcal{O}_S(f_i) \boxtimes \mathcal{O}_{\Phi^{-1}(Q_i)},$$

define the vector bundle $\mathscr{F}_{S'}$ by

$$\mathscr{F}_{S'} := \mathrm{Ker}\left(\tilde{\mathscr{E}}_{S'} \to \bigoplus_{i=1}^{6} \mathcal{O}_S(f_i) \boxtimes \mathcal{O}_{\Phi^{-1}(Q_i)} \right).$$

We also put

$$\mathscr{F}_{\Phi^{-1}(Q_i)} := \mathscr{F}_{S'}|_{S \times \Phi^{-1}(Q_i)}.$$

PROPOSITION 5.2. *The family $(S', \mathscr{F}_{S'})$ is a universal family of $M(h, 2)$ and the family $(\Phi^{-1}(Q_i), \mathscr{F}_{\Phi^{-1}(Q_i)})$ is a universal family of $M(h, 2)_{F_i}$.*

PROOF. By Lemma 5.1 and [H1, Proposition 3.3]. □

6. Moduli of μ-stable 2-bundles with $C_1=h$ and $C_2=2$.

For a non-zero element ξ in $\mathrm{Ext}^1(\mathcal{O}_S(l), \mathcal{O}_S(h-l)) \cong H^1(S, \mathcal{O}_S(h-2l))$, let

$$0 \longrightarrow \mathcal{O}_S(h-l) \longrightarrow \mathcal{E}_\xi \longrightarrow \mathcal{O}_S(l) \longrightarrow 0,$$

be the corresponding non-trivial extension. For the line E_i, there is the following short exact sequence

$$0 \longrightarrow \mathcal{O}_S(h-2l) \longrightarrow \mathcal{O}_S(h-2l+e_i) \longrightarrow \mathcal{O}_{E_i} \longrightarrow 0.$$

By the proof of Lemma 4.2 (1), $H^0(S, \mathcal{E}_\xi(e_i-l)) \neq (0)$ if and only if $\xi \in \mathrm{Im}(H^0(E_i, \mathcal{O}_{E_i}) \to H^1(S, \mathcal{O}_S(h-2l)))$. Remark that $\mathcal{O}_S(h-2l)=\mathcal{O}_S(l-\sum_{i=1}^6 e_i)$. Now consider the following short exact sequence

$$0 \longrightarrow \mathcal{O}_S\left(l-\sum_{i=1}^6 e_i\right) \longrightarrow \mathcal{O}_S(l) \longrightarrow \bigoplus_{i=1}^6 \mathcal{O}_{E_i} \longrightarrow 0.$$

Taking long exact sequence of cohomology groups, we get

$$0 \longrightarrow H^0(S, \mathcal{O}_S(l)) \xrightarrow{A} \bigoplus_{i=1}^6 H^0(E_i, \mathcal{O}_{E_i}) \xrightarrow{B} H^1\left(S, \mathcal{O}_S\left(l-\sum_{i=1}^6 e_i\right)\right) \longrightarrow 0.$$

Let

$$\Psi = \Psi_{|l|}: \quad S \longrightarrow \boldsymbol{P}^2$$

be the contraction morphism of $\{E_i\}$. Put $\Psi(E_i)=P_i$ $(i=1, \cdots, 6)$. Taking an appropriate coordinate system of $\boldsymbol{P}^2$, we may assume $P_1=(1,0,0)$, $P_2=(0,1,0)$, $P_3=(0,0,1)$, $P_4=(1,1,1)$, $P_5=(1,a,b)$ and $P_6=(1,c,d)$. Then the homomorphism A is given by the matrix

$$^t\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & a & c \\ 0 & 0 & 1 & 1 & b & d \end{pmatrix}.$$

Since the homomorphism B is surjective, B is given by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & a & b & 0 & -1 & 0 \\ 1 & c & d & 0 & 0 & -1 \end{pmatrix}$$

for an appropriate coordinate system of $H^1(S, \mathcal{O}_S(l-\sum_{i=1}^6 e_i))$. Since the point Q_i corresponds to the homomorphism $H^0(E_i, \mathcal{O}_{E_i}) \to H^1(S, \mathcal{O}_S(l-\sum_{i=1}^6 e_i))$, the coordinates of Q_i's are given by $Q_1=(1,1,1)$, $Q_2=(1,a,c)$, $Q_3=(1,b,d)$, $Q_4=(1,0,0)$, $Q_5=(0,1,0)$ and $Q_6=(0,0,1)$ for the coordinate system of $\boldsymbol{P}$ induced by that of $H^1(S, \mathcal{O}_S(l-\sum_{i=1}^6 e_i))$. Since P_i's are in general position, so are Q_i's. Let C_j be the non-singular conic passing through all P_i's but P_j and D_j be the non-singular conic passing through all Q_i's but Q_j.

LEMMA 6.1. *There is a birational mapping*

$$\phi: \ \boldsymbol{P}^2 \cdots \rightarrow \boldsymbol{P}$$

such that the fundamental points of ϕ *and* ϕ^{-1} *are precisely all* P_i*'s and* Q_i*'s respectively and such that* $\phi(P_i)=D_i$ *and* $\phi^{-1}(Q_i)=C_i$.

PROOF. For homogeneous polynomials

$$\begin{aligned}
l_{45} &= (a-b)x_0+(b-1)x_1-(a-1)x_2 \\
l_{46} &= -(c-d)x_0-(d-1)x_1+(c-1)x_2 \\
l_{56} &= (ad-bc)x_0+(b-d)x_1-(a-c)x_2 \\
c_4 &= (a-c)bdx_0x_1-(ad-bc)x_1x_2-(b-d)acx_0x_2 \\
c_5 &= -(c-1)dx_0x_1+(c-d)x_1x_2+c(d-1)x_0x_2 \\
c_6 &= (a-1)bx_0x_1-(a-b)x_1x_2-a(b-1)x_0x_2 ,
\end{aligned}$$

define $\phi=(c_5c_6l_{56}, c_4c_6l_{46}, c_4c_5l_{45})$. Since the line (l_{ij}) passes through the points P_i and P_j, and $(c_i)=C_i$, the fundamental points of ϕ is precisely all P_i's and the restriction $\phi|_{C_i}$ is constant. It is also clear that the strict transform $\phi[C_i]=Q_i$ for $i=4, 5, 6$. Now we construct the inverse of ϕ. For homogeneous polynomials

$$\begin{aligned}
m_{12} &= (a-c)y_0+(c-1)y_1-(a-1)y_2 \\
m_{13} &= -(b-d)y_0-(d-1)y_1+(b-1)y_2 \\
m_{23} &= (ad-bc)y_0+(c-d)y_1-(a-b)y_2 \\
d_1 &= (a-c)cdy_0y_1-(ad-bc)y_1y_2-(c-d)aby_0y_2 \\
d_2 &= -(b-1)dy_0y_1+(b-d)y_1y_2+b(d-1)y_0y_2 \\
d_3 &= (a-1)cy_0y_1-(a-c)y_1y_2-a(c-1)y_0y_2 ,
\end{aligned}$$

define $\psi=(d_2d_3m_{23}, d_1d_3m_{13}, d_1d_2m_{12})$. The fundamental points of ψ is precisely all Q_i's and the restriction $\psi|_{D_i}$ is constant and also the strict transform $\psi[D_i]=P_i$ for $i=1, 2, 3$ as above. It can be checked that $\phi^{-1}=\psi$ and that $\phi[C_i]=Q_i$ and $\psi[D_i]=P_i$ for $i=1, \cdots, 6$. It is not advisable to calculate by hand. □

PROPOSITION 6.2. *S' is isomorphic to S and under the isomorphism, $\Phi^{-1}(Q_i)$ is isomorphic to F_i.*

PROOF. Consider the graph Γ of the birational mapping ϕ in Lemma 6.1. One sees that $\Gamma\cong S\cong S'$ and $\Phi^{-1}(Q_i)=F_i$ in Γ. □

THEOREM 6.3. *There is a vector bundle $\mathscr{F}_S$ over $S\times S$ such that $\mathscr{F}_S$ is a universal* *family of* $M(h, 2)$ *and the restriction* $\mathscr{F}_S|_{S\times E}$ *is a universal family of* $M(h, 2)_E$ *for every line* E *on* S.

PROOF. By Propositions 5.2 and 6.2, there is a vector bundle $\mathscr{F}_S$ over $S\times S$ such that $\mathscr{F}_S$ is a universal family of $M(h, 2)$ and the restriction $\mathscr{F}_S|_{S\times F_i}$ is a universal family of $M(h, 2)_{F_i}$. To construct a universal family, we can start from any double six. So the last assertion follows. □

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