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Stability of the Identity Map of $SU(3)/T(k, l)$

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Abstract. In this paper, we get the fact that the identity map of $SU(3)/T(k,l)$ which is a 7-dimensional non-symmetric normal homogeneous space is stable.

1. Introduction and the main result.

It is interesting whether the identity map of a given Riemannian manifold (M, g) is stable or not. Y. Ohnita [3] obtained the complete stability results about identity maps of compact irreducible simply connected Riemannian symmetric spaces. It is known [7] that the identity maps of every closed Riemannian manifold of constant curvature (positive, zero or negative) are stable except the standard unit spheres (S^n, can), $n \geq 3$. To show stability of the identity map of positively curved homogeneous spaces which are not symmetric and have non-constant curvatures seems to be difficult.

In this paper, we consider the stability of the non-symmetric 7 dimensional homogeneous space $SU(3)/T(k, l)$, admitting positively curved Riemannian metrics, which was discovered by S. Aloff and N. R. Wallach (cf. [1]). Here $T(k, l) = \{\mathrm{diag}[e^{2\pi ik\theta}, e^{2\pi il\theta}, e^{-2\pi i(k+l)\theta}]\,;\ \theta \in \boldsymbol{R}\}, |k|+|l| \neq 0$ $(k, l \in \boldsymbol{Z})$, $i=\sqrt{-1}$. We fix an $\mathrm{Ad}(SU(3))$-invariant inner product $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{su}(3)$ of $SU(3)$. Let g be the $SU(3)$-invariant Rimannian metric on $SU(3)/T(k, l)$ induced from this inner product $(\cdot, \cdot)$.

In this paper, we have the following:

THEOREM. *Let $SU(3)/T(k, l)$ have the $SU(3)$-invariant metric g which is canonically induced from an* $\mathrm{Ad}(SU(3))$*-invariant inner product on the Lie algebra* $\mathfrak{su}(3)$. *Assume that k, l are relatively prime. Then, the identity map of* $(SU(3)/T(k, l), g)$ *is stable.*

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2. Proof of the main theorem.

In this section, we use the following notations.

$G=SU(3)$, $\mathfrak{g}$: the Lie algebra of $SU(3)$, $i=\sqrt{-1}$,
$\mathfrak{t}=\{\mathrm{diag}[ix_1, ix_2, ix_3] \mid x_1+x_2+x_3=0$, each $x_j \in \boldsymbol{R}\}$,
$T=T(k, l)=\{\mathrm{diag}[e^{2\pi ik\theta}, e^{2\pi il\theta}, e^{-2\pi i(k+l)\theta}] \mid \theta \in \boldsymbol{R}\}$, $|k|+|l| \neq 0$ $(k, l \in \boldsymbol{Z})$,
$\mathfrak{t}(k, l)$: the Lie algebra of $T(k, l)$,
$B(X, Y)=6\mathrm{Trace}(XY)$, $X, Y \in \mathfrak{g}$: the Killing form of $\mathfrak{g}$,
$M=G/T=SU(3)/T(k, l)$,
$\Gamma(G)=\{H \in \mathfrak{t} \mid \exp H=e\}$: the unit lattice,
$I=\{\lambda \in \sqrt{-1}\,\mathfrak{t}^* \mid \lambda(H) \in \sqrt{-1}\,2\pi \boldsymbol{Z}$ for all $H \in \Gamma(G)\}$:
the set of all G-integral forms on $\mathfrak{t}$.

We give an Ad(G)-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$ by

(1) $$(X, Y)=-B(X, Y)=-6\mathrm{Trace}(X, Y), \qquad (X, Y \in \mathfrak{g}).$$

Let g be the $SU(3)$-invariant Riemannian metric on $SU(3)/T(k, l)$ induced from this inner product $(\cdot, \cdot)$. We denote by $e_j \in \sqrt{-1}\,\mathfrak{t}^*$ $(j=1, 2, 3)$ the linear map

$$\mathfrak{t}^c \ni \mathrm{diag}[x_1, x_2, x_3] \mapsto x_j .$$

Put $\alpha=e_1-e_2$, $\beta=e_2-e_3$ and $\gamma=e_1-e_3$. We fix a lexicographic order $<$ on $\sqrt{-1}\,\mathfrak{t}^*$ in such a way that $0<\beta<\alpha$. Then the set P of all positive roots of $\mathfrak{g}^c$ relative to $\mathfrak{t}^c$ and half the sum δ of all elements in P are given by

(2) $$P=\{\alpha, \beta, \gamma\} \quad \text{and} \quad \delta=2e_1+e_2 .$$

On the other hand, the elements $H_{e_i-e_j} \in \sqrt{-1}\,\mathfrak{t}$ such that $(e_i-e_j)(H)=B(H_{e_i-e_j}, H)$ for all $H \in \mathfrak{t}^c$ and (e_i, e_j) are given as follows:

(3) $$\begin{cases} H_\alpha=\frac{1}{6}\mathrm{diag}[1, -1, 0], \quad H_\beta=\frac{1}{6}\mathrm{diag}[0, 1, -1], \quad H_\gamma=\frac{1}{6}\mathrm{diag}[1, 0, -1], \\ (e_1, e_1)=(e_2, e_2)=1/9, \quad (e_1, e_2)=-1/18 . \end{cases}$$

Then the set $D(G)=\{\lambda \in I \mid (\lambda, \alpha) \geq 0$ for all $\alpha \in P\}$ of all dominant integral forms relative to $\mathfrak{t}$ is given by

(4) $$D(G)=\{\lambda=m_1e_1+m_2e_2 \mid m_1 \geq m_2 \geq 0, \ m_j \in \boldsymbol{Z}\ (j=1, 2)\} .$$

There exists a natural bijection from $D(G)$ onto the set of all non-equivalent finite dimensional irreducible unitary representation (V_λ, π_λ) having highest weight λ. For $\lambda \in D(G)$, put $d(\lambda)$ the dimension of the representation V_λ. $d(\lambda)$ is given by

$$d(\lambda)=\prod_{\alpha\in P}\frac{(\lambda+\alpha,\alpha)}{(\delta,\alpha)}. \tag{5}$$

Therefore, we have for $\lambda=m_1e_1+m_2e_2$

$$d(\lambda)=\tfrac{1}{2}(m_1-m_2+1)(m_1+2)(m_2+1). \tag{6}$$

Let $\mathfrak{X}(M)$ be the set of all C^∞-vector fields on M. We identify $\mathfrak{X}(M)$ with the following $C_T^\infty(G,\mathfrak{m})$ (cf. [4, 7]). Here $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{t}(k,l)$ in $\mathfrak{g}$.

DEFINITION 2.1. Let $C^\infty(G,\mathfrak{m})$ be the space of all smooth maps of G into $\mathfrak{m}$. We define the subspace $C_T^\infty(G,\mathfrak{m})$ of $C^\infty(G,\mathfrak{m})$ by

$$C_T^\infty(G,\mathfrak{m}):=\{f\in C^\infty(G,\mathfrak{m})\mid f(xh)=\mathrm{Ad}(h^{-1})f(x),\ x\in G,\ h\in T\}.$$

The identification Φ of $\mathfrak{X}(M)$ with $C_T^\infty(G,\mathfrak{m})$, $\Phi: C_T^\infty(G,\mathfrak{m})\to\mathfrak{X}(M)$, is given by

$$\Phi(f)(\bar{x}):=(\tau_x)_*(f(x))_o, \qquad x\in G. \tag{7}$$

Here X_o, $(X\in\mathfrak{m})$, is the tangent vector of M at the origin $\{T\}$ corresponding to $f(x)\in\mathfrak{m}$, and $(\tau_x)_*$ is the differantial of the translation $\tau_x: M\ni\bar{y}\mapsto\overline{xy}\in M$. Then it turns out that Φ is an isomorphism of $C_T^\infty(G,\mathfrak{m})$ onto $\mathfrak{X}(M)$. Under the G-actions on $\mathfrak{X}(M)$ or $C_T^\infty(G,\mathfrak{m})$ defined by

$$\begin{cases}((\tau_x)_*V)_{\bar{y}}:=(\tau_x)_*V_{\overline{x^{-1}y}}, & x,y\in G,\quad V\in\mathfrak{X}(M),\\ (\tau_xf)(y):=f(x^{-1}y), & x,y\in G,\quad f\in C_T^\infty(G,\mathfrak{m}),\end{cases} \tag{8}$$

Φ is a G-isomorphism, that is,

$$\Phi\circ\tau_xf=(\tau_x)_*\circ\Phi(f), \qquad x\in G,\quad f\in C_T^\infty(G,\mathfrak{m}). \tag{9}$$

The Jacobi operator $J_{id}:\mathfrak{X}(M)\to\mathfrak{X}(M)$ of the identity map of M is G-invariant (cf. [7, p. 580]), that is,

$$J_{id}((\tau_x)_*V)=(\tau_x)_*(J_{id}V), \qquad V\in\mathfrak{X}(M). \tag{10}$$

Furthermore $C_T^\infty(G,\mathfrak{m})$ is identified with the subspace $(C^\infty(G)\otimes\mathfrak{m})_T$ of the tensor product $C^\infty(G)\otimes\mathfrak{m}$.

DEFINITION 2.2. $(C^\infty(G)\otimes\mathfrak{m})_T$ is defined by the subspace of all elements $\sum_{i=1}^l f_i\otimes X_i\in C^\infty(G)\otimes\mathfrak{m}$ satisfying

$$\sum_{i=1}^{l}R_hf_i\otimes\mathrm{Ad}(h)X_i=\sum_{i=1}^{l}f_i\otimes X_i$$

for all $h\in T$. Here $(R_hf)(x):=f(xh)$, $h\in T$, $x\in G$, $f\in C^\infty(G)$.

Under the G-actions on $C^\infty(G)\otimes\mathfrak{m}$ or $C^\infty(G)$ defined by

$$(11)\qquad \begin{cases} (\tau_x f)(y) := f(x^{-1}y)\,, & x, y \in G\,,\quad f \in C^\infty(G)\,,\\ \tau_x(f \otimes X) := \tau_x f \otimes X\,, & X \in \mathfrak{m}\,,\end{cases}$$

the $(C^\infty(G)\otimes\mathfrak{m})_T$ is a G-submodule. The identification Ψ of $C_T^\infty(G, \mathfrak{m})$ with $(C^\infty(G)\otimes m)_T$ is given by

$$(12)\qquad \Psi(f) := \sum_{j=1}^{7} f_j \otimes X_j\,,\qquad f \in C_T^\infty(G, \mathfrak{m})\,,$$

where $f(x)=\sum_{j=1}^{7} f_j(x)X_j$, $x\in G$, and $\{X_j\}_{j=1}^{7}$ is a fixed orthonormal basis of $\mathfrak{m}$ with respect to $(\cdot,\cdot)$. Then Ψ is a G-isomorphism of $C_T^\infty(G, \mathfrak{m})$ onto $(C^\infty(G)\otimes\mathfrak{m})_T$ with

$$(13)\qquad \Psi\circ\tau_x=\tau_x\circ\Psi\,,\qquad x\in G\,.$$

DEFINITION 2.3. Via Φ and Ψ, a G-invariant operator $\tilde{J}$ on $(C^\infty(G)\otimes\mathfrak{m})_T$ is defined from the Jacobi operator J_{id} in such a way that the following diagram is commutative:

$$\begin{array}{ccccc} \mathfrak{X}(M) & \xrightarrow{\Phi^{-1}} & C_T^\infty(G,\mathfrak{m}) & \xrightarrow{\Psi} & (C^\infty(G)\otimes\mathfrak{m})_T \\ {\scriptstyle J_{id}}\downarrow & & & & \downarrow{\scriptstyle \tilde{J}} \\ \mathfrak{X}(M) & \xrightarrow{\Phi^{-1}} & C_T^\infty(G,\mathfrak{m}) & \xrightarrow{\Psi} & (C^\infty(G)\otimes\mathfrak{m})_T\,. \end{array}$$

By (9), (10) and (13), the operator $\tilde{J}$ is G-invariant, that is,

$$(14)\qquad \tilde{J}\circ\tau_x=\tau_x\circ\tilde{J}\,,\qquad x\in G\,.$$

DEFINITION 2.4. The operators D_i, $i=0, 1, 2, 3$, acting on $C^\infty(G)\otimes\mathfrak{m}$ are given by

$$D_0 := \sum_{k=1}^{8} X_k^2\otimes I\,,$$

$$D_1 := \sum_{k=1}^{7} X_k\otimes P_{\mathfrak{m}}\circ \mathrm{ad}(X_k)\,,$$

$$D_2 := I\otimes\sum_{j=1}^{7}\mathrm{ad}(X_j)\circ P_{\mathfrak{t}(k,l)}\circ\mathrm{ad}(X_j)\,,$$

$$D_3 := I\otimes \mathrm{ad}(X_8)^2\,,$$

where $P_{\mathfrak{m}}$ and $P_{\mathfrak{t}(k,l)}$ are the projections of $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{t}(k,l)$ onto $\mathfrak{m}$ and $\mathfrak{t}(k,l)$, respectively, $\{X_k\}_{k=1}^{8}$ is an orthonormal basis of $(\mathfrak{g},(\cdot,\cdot))$ such that $\{X_i\}_{i=1}^{7}$ (resp. $\{X_8\}$) is a basis of $\mathfrak{m}$ (resp. $\mathfrak{t}(k,l)$), I is the identity operator of $C^\infty(G)$ or $\mathfrak{m}$, $(Xf)(x) := (d/dt)f(x\exp(tX))|_{t=0}$ for $X\in\mathfrak{g}$, $f\in C^\infty(G)$ and $x\in G$.

All D_i, $i=0, 1, 2, 3$, are independent of the choice of the above basis $\{X_k\}_{k=1}^{8}$. Thus since $R_h\circ Xf=(\mathrm{Ad}(h)X)(R_h f)$, for $f\in C^\infty(G)$, $h\in T$, and $X\in\mathfrak{g}$, all D_i keep the subspace $(C^\infty(G)\otimes\mathfrak{m})_T$ invariant. The Urakawa's theorem can be stated as follows in our case:

THEOREM 2.5 (cf. [7, p. 586]). *The operator* $\tilde{J}$ *of* $(C^\infty(G)\otimes \mathfrak{m})_T$ *corresponding to the Jacobi operator* J_{id} *of the identity map* id_M *coincides with the operator*

$$D := -D_0 - D_1 + D_2 + D_3 ,$$

where all D_i *are defined in Definition* 2.4.

Let E_{ij} denote a square matrix of order 3 with the (i, j)-entry being 1, and all the other entries being 0. Then we put

$$X_1 := \frac{1}{\sqrt{12}}(E_{12}-E_{21}) , \qquad X_2 := \frac{\sqrt{-1}}{\sqrt{12}}(E_{12}+E_{21}) ,$$

$$X_3 := \frac{1}{\sqrt{12}}(E_{13}-E_{31}) , \qquad X_4 := \frac{\sqrt{-1}}{\sqrt{12}}(E_{13}+E_{31}) ,$$

$$X_5 := \frac{1}{\sqrt{12}}(E_{23}-E_{32}) , \qquad X_6 := \frac{\sqrt{-1}}{\sqrt{12}}(E_{23}+E_{32}) ,$$

$$X_7 := \frac{\sqrt{-1}}{6\sqrt{r}}\,\mathrm{diag}[(k+2l), -(2k+l), (k-l)] ,$$

$$X_8 := \frac{\sqrt{-1}}{\sqrt{12r}}\,\mathrm{diag}[k, l, -(k+l)] ,$$

where $r := k^2 + kl + l^2$. Then

$$\{X_1, X_2, \cdots, X_7\} \qquad (\text{resp. } \{X_8\}) \tag{15}$$

is an orthonormal basis of $\mathfrak{m}$ (resp. $\mathfrak{t}(k, l)$) with respect to $(\cdot, \cdot)$.

We define an inner product $((\cdot, \cdot))$ on $\mathfrak{X}(M)$ by

$$((V, W)) := \int_M g(V, W)v_g , \qquad (V, W \in \mathfrak{X}(M)) , \tag{16}$$

and similarly define the Hermitian inner product $((\cdot, \cdot))$ on the complexification $\mathfrak{X}^c(M)$ of $\mathfrak{X}(M)$. Then the representation $(\tau, \mathfrak{X}^c(M))$ of G which is defined by (8) is a unitary representation with respect to $((\cdot, \cdot))$.

Frobenius' reciprocity theorem can be stated as follows:

THEOREM 2.6 (cf. [2, 9]). *For the decomposition* $(\tau, \mathfrak{X}^c(M)) = \sum_{\lambda\in D(G)} m(\lambda)V_\lambda$ *of* $\mathfrak{X}^c(M)$ *into irreducible unitary representations of* G, *the multiplicity* $m(\lambda)$ *of* V_λ, $\lambda \in D(G)$, *in* $\mathfrak{X}^c(M)$ *or* $C_T^\infty(G, \mathfrak{m}^c)$ *is*

$$\dim \mathrm{Hom}_G(V_\lambda, C_T^\infty(G, \mathfrak{m}^c)) = \dim \mathrm{Hom}_T(V_\lambda, \mathfrak{m}^c) ,$$

where $\mathfrak{m}^c$ *is an* $\mathrm{Ad}(T)$*-module.*

To evaluate $m(\lambda)$ in Theorem 2.6, we apply the following Urakawa's proposition:

PROPOSITION 2.7 (cf. [8]). *Assume k and l are relatively prime. Let* (V_λ, π_λ) *be an irreducible unitary representation of G with the highest weight* $\lambda = m_1e_1 + m_2e_2 \in D(G)$. *Then, as a representation of T,* V_λ *is decomposed into T-irreducible submodules as follows*:

$$V_\lambda = \sum_{p=m_2+1}^{m_1+1} \sum_{q=0}^{m_2} \sum_{d=0}^{p-q-1} W_{k(m_1+m_2+2-2p-q+d)+l(1-p+q+2d)}, \tag{17}$$

where W_m $(m \in Z)$ *is the* 1-*dimensional irreducible T-submodule of* V_λ *with the character* $\chi_m : T(k, l) \ni \mathrm{diag}[e^{2\pi ik\theta}, e^{2\pi il\theta}, e^{-2\pi i(k+l)\theta}] \mapsto e^{2\pi im\theta}$, $i = \sqrt{-1}$.

By Theorem 2.6 and Proposition 2.7, we get for $\lambda \in D(G)$

(18) *$m(\lambda)$ is the number of elements m, (m in W_m of the right side of* (17)), *which belong to* $\{\pm(k-l), \pm(2k+l), 0, \pm(k+2l)\}$.

We get for later use

LEMMA 2.8. *Let* $(\tau, \mathfrak{X}^c(M)) = \sum_{\lambda \in D(G)} m(\lambda) V_\lambda$ *be the decomposition of* $\mathfrak{X}^c(M)$ *into irreducible unitary representations of G. Assume k and l are relatively prime. Then*

(a) $m(\lambda) = 0$ *for* $\lambda = e_1, e_1 + e_2 \in D(G)$,

(b) $(\lambda + 2\delta, \lambda) \geq 1$ *for* $\lambda \in D(G) - \{0, e_1, e_1 + e_2\}$.

PROOF. From (17),

$$V_{e_1} = W_k \oplus W_{-k-l} \oplus W_l \quad \text{and} \quad V_{e_1+e_2} = W_{-k} \oplus W_{k+l} \oplus W_{-l}.$$

Hence, from these decompositions and (18) we obtain (a). (b) follows from (1)~(4).

REMARK. It's very difficult to obtain $m(\lambda)$ for each $\lambda \in D(G)$ in Lemma 2.8 because the number of elements m (m in W_m of the right side of (17)) which become 0 is dependent on k, l (cf. [8]).

For $\sum_{i=1}^7 f_i \otimes X_i \in V_\lambda \subset (C_c^\infty(G) \otimes \mathfrak{m})_T$ $(\lambda = m_1e_1 + m_2e_2 \in D(G))$, we have

$$\begin{aligned} (19)\quad & D_1\left(\sum_{i=1}^7 f_i \otimes X_i\right) \\ &= \left\{\frac{1}{\sqrt{12}}(X_5f_3 - X_3f_5 + X_6f_4 - X_4f_6) + \frac{(k+l)}{2\sqrt{r}}(X_2f_7 - X_7f_2)\right\} \otimes X_1 \\ &+ \left\{\frac{1}{\sqrt{12}}(X_5f_4 - X_4f_5 + X_3f_6 - X_6f_3) + \frac{(k+l)}{2\sqrt{r}}(X_7f_1 - X_1f_7)\right\} \otimes X_2 \\ &+ \left\{\frac{1}{\sqrt{12}}(X_1f_5 - X_5f_1 + X_6f_2 - X_2f_6) + \frac{l}{2\sqrt{r}}(X_4f_7 - X_7f_4)\right\} \otimes X_3 \end{aligned}$$

$$+\left\{\frac{1}{\sqrt{12}}(X_1f_6-X_6f_1+X_2f_5-X_5f_2)+\frac{l}{2\sqrt{r}}(X_7f_3-X_3f_7)\right\}\otimes X_4$$
$$+\left\{\frac{1}{\sqrt{12}}(X_3f_1-X_1f_3+X_4f_2-X_2f_4)+\frac{k}{2\sqrt{r}}(X_7f_6-X_6f_7)\right\}\otimes X_5$$
$$+\left\{\frac{1}{\sqrt{12}}(X_4f_1-X_1f_4+X_2f_3-X_3f_2)+\frac{k}{2\sqrt{r}}(X_5f_7-X_7f_5)\right\}\otimes X_6$$
$$+\left\{\frac{(k+l)}{2\sqrt{r}}(X_1f_2-X_2f_1)+\frac{k}{2\sqrt{r}}(X_6f_5-X_5f_6)+\frac{l}{2\sqrt{r}}(X_3f_4-X_4f_3)\right\}\otimes X_7\,,$$

(20)
$$D_2\left(\sum_{i=1}^{7} f_i\otimes X_i\right)=D_3\left(\sum_{i=1}^{7} f_i\otimes X_i\right)$$
$$=\frac{-(k-l)^2}{12r}(f_1\otimes X_1+f_2\otimes X_2)-\frac{(2k+l)^2}{12r}(f_3\otimes X_3+f_4\otimes X_4)$$
$$-\frac{(k+2l)^2}{12r}(f_5\otimes X_5+f_6\otimes X_6)\,,$$

where all X_i are defined in (15). Since all D_i, $i=0, 1, 2, 3$, are G-invariant, i.e., $D_i\circ\tau_x=\tau_x\circ D_i$ for all $x\in G$, all D_i preserve the subspaces V_λ invariant. By Schur's lemma, there exist constants $c_i(\lambda)$ such that $D_i=c_i(\lambda)I$ on V_λ $(i=0, 1, 2, 3)$. Here, I is the identity operator of V_λ. Then we get

LEMMA 2.9. *$c_1(\lambda)=0$ on V_λ.*

PROOF. Since k and l are relatively prime, we have from (20)

(21)
$$c_2(\lambda)=c_3(\lambda)\,,$$

(22)
$$c_2(\lambda)=\frac{-(k-l)^2}{12r}\quad\text{or}\quad\frac{-(2k+l)^2}{12r}\quad\text{or}\quad\frac{-(k+2l)^2}{12r}\quad\text{or}\quad 0\,.$$

If $\sum_{i=1}^{7} f_i\otimes X_i\in V_\lambda$, then we obtain from (22)

(23)
$$\begin{cases}\text{(a)}\ \ f_3=f_4=f_5=f_6=f_7=0 \text{ on } G, \text{ or}\\ \text{(b)}\ \ f_1=f_2=f_5=f_6=f_7=0 \text{ on } G, \text{ or}\\ \text{(c)}\ \ f_1=f_2=f_3=f_4=f_7=0 \text{ on } G, \text{ or}\\ \text{(d)}\ \ f_1=f_2=f_3=f_4=f_5=f_6=0 \text{ on } G.\end{cases}$$

Let $v_\lambda=\sum_{i=1}^{7} f_i\otimes X_i\in V_\lambda$ be the highest weight vector with $((v_\lambda, v_\lambda))=(\| v_\lambda \|_2)^2=1$. All D_i $(i=0, 1, 2, 3)$ keep $\{v_\lambda\}^c$ invariant. We define an inner product of $f, f'(\in C_c^\infty(G))$ by $\int_G f(x)f'(x)v_g$. Let

$$(\tau, C_c^\infty(G)) = \sum_{\lambda \in D(G)} n(\lambda) U_\lambda$$

be the decomposition of $C_c^\infty(G)$ into irreducible unitary representations of G. Here $n(\lambda)$ is the multiple of U_λ in $C_c^\infty(G)$ and the action of G on $C_c^\infty(G)$ is defined by (11). Classifying the highest unit vector

$$v_\lambda = \sum_{i=1}^{7} f_i \otimes X_i$$

into 4-cases of (23), we prove this lemma.

The case of (a) of (23); $v_\lambda = f_1 \otimes X_1 + f_2 \otimes X_2$. Since the coefficient functions f_1, f_2 $(\in C_c^\infty(G))$ in v_λ are highest weight vectors in the irreducible unitary representation space (τ, U_λ) of G, there exists constant c such that $f_1 = cf_2$. Now, if $c=0, f_1=0$ on G. Since $v_\lambda = f_2 \otimes X_2 = (R_{\exp tX_8} f_2) \otimes \mathrm{Ad}(\exp tX_8)X_2$ for any $t \in \boldsymbol{R}$, $f_2 = 0$. This fact results in wrong conclusion to the light of $\| v_\lambda \|_2 = 1$. Thus, $c \neq 0$. Then,

$$D_1 v_\lambda = \frac{-k-l}{2\sqrt{r}} \left\{ \frac{1}{c}(X_7 f_1) \otimes X_1 - (X_7 f_1) \otimes X_2 \right\} = c_1(\lambda) f_1 \otimes X_1 + \frac{c_1(\lambda)}{c} f_1 \otimes X_2 ,$$

by the help of (19). From this fact, $(1+c^2)c_1(\lambda)f_1 = 0$, i.e., $c = \pm\sqrt{-1}$ or $c_1(\lambda) = 0$. Assume $c = \pm\sqrt{-1}$. Then $v_\lambda = \pm\sqrt{-1} f_2 \otimes X_1 + f_2 \otimes X_2 = R_{\exp tX_8}(\pm\sqrt{-1} f_2) \otimes \mathrm{Ad}(\exp tX_8)X_1 + R_{\exp tX_8} f_2 \otimes \mathrm{Ad}(\exp tX_8)X_2$ for any $t \in \boldsymbol{R}$. From this equality, $f_1 = f_2 = 0$. This contradicts $\| v_\lambda \|_2 = 1$. Therefore $c_1(\lambda) = 0$.

The cases of (b), (c) of (23); These cases are proved in the same way as the above proof.

The case of (d) of (23); $v_\lambda = f_7 \otimes X_7$. Then $c_1(\lambda) = 0$ with the help of (19).

Thus the proof of Lemma 2.9 is completed.

LEMMA 2.10. $-1/2 < c_2(\lambda) \leq 0$.

PROOF. For k, l $(\in \boldsymbol{Z})$ satisfying the conditions in $T(k, l)$,

$$\frac{(k-l)^2}{12r} < \frac{1}{2}, \quad \frac{(2k+l)^2}{12r} < \frac{1}{2} \quad \text{and} \quad \frac{(k+2l)^2}{12r} < \frac{1}{2} .$$

Hence the proof of this lemma is completed by (22).

LEMMA 2.11. *For* $v = \sum_{i=1}^{7} f_i \otimes X_i \in V_\lambda$,

(a) $-D_0 v = (\lambda + 2\delta, \lambda) v$,

(b) *when* $\lambda = 0 \in D(G)$, $-D_i v = 0$ $(i = 1, 2, 3)$, *i.e.*, $\tilde{J} v = 0$.

PROOF. $-D_0$ is the Casimir operator of irreducible representation (π^λ, V_λ), $(V_\lambda \subset C_c^\infty(G))$, of G which is defined by $(\pi^\lambda(y)f)(x) := f(xy)$, $x, y \in G, f \in C_c^\infty(G)$. In general, the Casimir operator, of G, acting on $C_c^\infty(G)$ which is dependent on $(\cdot, \cdot)$ of (1) is $\sum_{i=1}^{8} X_i^2 = \sum_{i=1}^{8} \tilde{X}_i^2$, where $(X_i)_{i=1}^{8}$ is the above orthonormal basis of $\mathfrak{g}$ and each $\tilde{X}_i$ is a right invariant vector field satisfying $(\tilde{X}_i)_e = X_i$, (cf. [9, p. 51]). From these facts, we easily obtain (a) (cf. [5, 6]).

Furthermore, when $\lambda=0$ ($\in D(G)$), V_λ contained in $(C^\infty(G)\otimes \mathfrak{m}^c)_T$ is generated by $f_7\otimes X_7$, where f_7 is a constant function on G. Hence, $\tilde{J}v=0$.

Accordingly, the proof of Lemma 2.11 is completed.

Now, from Theorem 2.5 and Lemmas 2.8–2.11, we obtain the following.

$$((\tilde{J}v, v))\geq 0 \qquad \text{for} \quad v\in(C^\infty(G)\otimes \mathfrak{m})_T(=\mathfrak{X}^c(M))\,.$$

Thus, we get the main theorem.

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