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Some Isometric Minimal Immersions of the Three-Dimensional Sphere into Spheres

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Abstract. In the present paper we extend the study in [3]. Let $\psi(\xi, \eta, \zeta)$ be a harmonic homogeneous polynomial of degree $s=2\sigma\geq 4$ in three variables ξ, η, ζ. Then the bi-symmetric tensor C of bi-degree (s, s) satisfying

$$\psi(\langle J_1 w, v\rangle, \langle J_2 w, v\rangle, \langle J_3 w, v\rangle)=C(v, \cdots, v; w, \cdots, w)$$

identically belongs to the linear space $W(3, s)$ of isometric minimal immersions of the three-sphere into spheres. The purpose of the present paper is to study such tensors C and to state some related topics.

1. Introduction.

Isometric minimal immersions of spheres into spheres were studied by M. do Carmo and N. Wallach [1]. They established a theorem which is fundamental in the study of such immersions. In [1] we can see that such immersions can be regarded as $f: S^m(1)\to S^{n-1}(r)$ where n and r depend on m and a natural number s which is the order of the spherical harmonics on $S^m(1)$ inducing f, thus

$$n=n(m, s)=(2s+m-1)(s+m-2)!/(s!(m-1)!)\,,$$
$$r^2=(r(m, s))^2=m/(s(s+m-1)\,.$$

In the present paper the set of such isometric minimal immersions is denoted by $IMI(m, s)$. From an immersion $f\in IMI(m, s)$ we get a set of immersions by the action of the group of isometries of $S^{n-1}(r)$. This set is called the equivalence class of f and is denoted by $eq(f)$. There exists a linear space $W(m, s)$ with a compact convex body $L(m, s)$ such that to any set $eq(f)$ there corresponds just one point of $L(m, s)$. We consider only cases $m\geq 3$, $s\geq 4$ since $W(m, s)$ is nothing but one point only if $m<3$ or $s<4$.

We can regard $W(m, s)$ as a linear space of some tensors [2], [4], [6]. Any point C of $W(m, s)$ is a harmonic bi-symmetric tensor of bi-degree (s, s), namely C is a tensor of degree $2s$ on $\boldsymbol{R}^{m+1}$ satisfying the following conditions (i), (ii), (iii). In addition C

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satisfies the condition (iv).

(i) $C(v_1, \cdots, v_s; v_{1+s}, \cdots, v_{2s})$ is symmetric both in $v_1, \cdots, v_s$ and in $v_{1+s}, \cdots, v_{2s}$,

(ii) $C(v, \cdots, v; w, \cdots, w)=C(w, \cdots, w; v, \cdots, v)$,

(iii) $\sum_{i=1}^{m+1} C(e_i, e_i, v, \cdots, v; w, \cdots, w)=0$,

(iv) $C(w, w, v, \cdots, v; v, \cdots, v)=0$.

Here $v_1, \cdots, v_{2s}, v, w$ are arbitrary vectors of $\boldsymbol{R}^{m+1}$ in which $S^m(1)$ is embedded as the unit sphere and $\{e_1, \cdots, e_{m+1}\}$ is an orthonormal basis of $\boldsymbol{R}^{m+1}$. (iv) is equivalent to $C(w, v, \cdots, v; w, v, \cdots, v)=0$.

In §2 various formulas used later are explained. The isometric minimal immersions considered here are general ones, restricted somewhere only by $s=2\sigma$. In §3 we consider a homogeneous polynomial $\psi(\xi, \eta, \zeta)$ and try to find the condition for ψ to be related to an element of $W(3, s)$. Thus we find a mapping of $V(2, s)$ into $W(3, s)$, where $V(2, s)$ is the linear space of harmonic homogeneous polynomials of degree s in three variables. Here s must be even and this mapping is denoted by $J^{\#}$. In §4 contraction of the elements of $W(3, s)$ obtained in §3 is discussed. In §5 a mapping $I^{\#}$ having almost the same property as $J^{\#}$ is defined and some properties of them are studied. In §6 inner products are defined for the elements of $V(2, s)$ and those of $W(3, s)$ respectively and their relations to $J^{\#}$ and $I^{\#}$ are studied. §7 is devoted to elevation of elements of $J^{\#}V(2, s)$. In §8 some properties of geodesics of isometric minimal immersions associated with $J^{\#}V(2, s)$ are studied.

2. Preliminaries.

Let C be a point of $L(m, s)$ and an isometric minimal immersion $f_{m,s}$ associated with C be expressed by

$$f_{m,s}(u)=\sum_{A=1}^{n} f^A(u)\tilde{e}_A ,$$

where $u=\sum_{i=1}^{m+1} u^i e_i$ is the unit vector of $\boldsymbol{R}^{m+1}$ indicating a point of $S^m(1)$ and $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ is a fixed orthonormal basis of $\boldsymbol{R}^n$ in which $S^{n-1}(r)$ is embedded with center at the origin of $\boldsymbol{R}^n$. Then f^A $(A=1, \cdots, n)$ are spherical harmonics of degree s and there exists a set of symmetric harmonic tensors F^A of degree s such that

$$F^A(u, \cdots, u)=f^A(u) .$$

This means that, to an isometric minimal immersion $f_{m,s}$ there corresponds some set of n symmetric harmonic tensors $\{F^1, \cdots, F^n\}$ satisfying

$$f_{m,s}(u)=\sum_{A=1}^{n} F^A(u)\tilde{e}_A . \tag{2.1}$$

Let $V(m, s)$ denote the linear space of symmetric harmonic tensors of degree s on $\boldsymbol{R}^{m+1}$. Then $\dim V(m, s)=n(m, s)$ is the number already mentioned. Let $\{H^1, \cdots, H^n\}$

be any orthonormal basis of $V(m, s)$. Then there exists an isometric minimal immersions $h_{m,s}$ such that

$$h_{m,s}(u)=\sum_{A=1}^{n} H^A(u)\tilde{e}_A\,. \tag{2.2}$$

This immersion is called a standard minimal immersion.

REMARK. That $\{H^1, \cdots, H^n\}$ is an orthonormal basis of $V(m, s)$ may be interpreted as follows,

$$\int_M H^A(u)H^B(u)d\omega_m = c\delta^{AB}\,,$$

where M is $S^m(1)$, $d\omega_m$ is the volume element of $S^m(1)$ and

$$c=(r(m, s))^2 \operatorname{Vol}(S^m(1))/n(m, s)\,.$$

At the same time $H^A(u)$ satisfy

$$\sum_{A=1}^{n} (H^A(u))^2 = r^2\,,$$

when u is any unit vector of $\boldsymbol{R}^{m+1}$. For the details see, for example, [2] §5.

From tensors F^A and H^A we can construct tensors

$$\sum_A F^A \otimes F^A\,, \qquad \sum_A H^A \otimes H^A\,.$$

Here and in the sequel $\sum_A$ is the abbreviation of $\sum_{A=1}^{n}$. These tensors are harmonic bi-symmetric tensors of bi-degree (s, s). As it is easy to see, these tensors do not change if $f_{m,s}$ and $h_{m,s}$ are replaced by $\tilde{f}\in eq(f_{m,s})$ and $\tilde{h}\in eq(h_{m,s})$ respectively. The harmonic bi-symmetric tensor defined by

$$C=\sum_A F^A \otimes F^A - \sum_A H^A \otimes H^A \tag{2.3}$$

satisfies the isometry condition (iv).

For harmonic bi-symmetric tensors see §2 of [7]. The set of harmonic bi-symmetric tensors of bi-degree (s, s) is denoted by $B(m, s)$. If $s=2\sigma$, the unit tensor $U\in B(m, s)$ is defined by

$$U(v; w)=\sum_{p=0}^{\sigma} u_p\langle v, w\rangle^{s-2p}\langle v, v\rangle^p\langle w, w\rangle^p\,, \tag{2.4}$$

where $u_0=1$ and $u_1, \cdots, u_\sigma$ satisfy

$$(s-2p+2)(s-2p+1)u_{p-1}+2p(2s+m-2p-1)u_p=0\,.$$

The following relation between U and any standard minimal immersion is often used, (2.5) $$c'U=\sum_A H^A \otimes H^A\,,$$

where c' is the number such that

$$ac'=r^2\,, \qquad a=\sum_{p=0}^{\sigma} u_p\,.$$

When B is a bi-symmetric tensor of bi-degree (s, s), $B_{p,q}(v, w)$ is defined by

(2.6) $$B_{p,q}(v, w)=B(v, \cdots, v, w, \cdots, w; v, \cdots, v, w, \cdots, w)\,,$$

where in the right hand side w appears p times before the semicolon and q times after the semicolon. If we consider only orthonormal pairs $\{v, w\}$, $U_{p,q}(v, w)$ does not depend on the choice of the pair. Then we can write

(2.7) $$U_{p,q}(v, w)=u_{p,q}\,.$$

3. Some elements of $W(3, s)$ obtained from a polynomial $\psi(\xi, \eta, \zeta)$ and J_1, J_2, J_3.

First we define orthogonal transformations J_λ $(\lambda=1, 2, 3)$. Let us fix an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\boldsymbol{R}^4$. For $a=(a^1, a^2, a^3)\in \boldsymbol{R}^3$, linear transformations $J_a=a^1J_1+a^2J_2+a^3J_3$ are defined by

$$\begin{aligned}
&J_ae_1=-a^1e_2+a^2e_3-a^3e_4\,,\\
&J_ae_2=a^1e_1-a^2e_4-a^3e_3\,,\\
&J_ae_3=-a^1e_4-a^2e_1+a^3e_2\,,\\
&J_ae_4=a^1e_3+a^2e_2+a^3e_1\,.
\end{aligned}$$

J_λ satisfy $J_2J_3=-J_3J_2=J_1$, $J_3J_1=-J_1J_3=J_2$, $J_1J_2=-J_2J_1=J_3$, and $J_\lambda^2=-1$ ($\lambda=1, 2, 3$).

Let $\psi(\xi, \eta, \zeta)$ be a homogeneous polynomial of degree s in three variables ξ, η, ζ and let us suppose that there exists a harmonic bi-symmetric tensor B of bi-degree (s, s) satisfying

(3.1) $$\psi(\langle J_1\omega, v\rangle, \langle J_2w, v\rangle, \langle J_3w, v\rangle)=B(v; w)$$

for arbitrary vectors v, w of $\boldsymbol{R}^4$. Here $B(v; w)$ is the abbreviation of $B(v, \cdots, v; w, \cdots, w)$.

As we have $\langle J_\lambda w, v\rangle=-\langle J_\lambda v, w\rangle$, s must be even and we put $s=2\sigma$.

From (3.1) we get

$$sB(e_i, v, \cdots, v; w, \cdots, w)=\frac{\partial\psi}{\partial\xi}\langle J_1w, e_i\rangle+\frac{\partial\psi}{\partial\eta}\langle J_2w, e_i\rangle+\frac{\partial\psi}{\partial\zeta}\langle J_3w, e_i\rangle\,.$$

This formula is the result of replacing v by $v+te_i$, $t\in\boldsymbol{R}$, and differentiating the resulting formula with respect to t at $t=0$. Such means are often used hereafter. For the sake of convenience we put $\xi=\xi_1$, $\eta=\xi_2$, $\zeta=\xi_3$ and rewrite the above formula as follows,

$$sB(e_i, v, \cdots, v; w, \cdots, w)=\sum_\lambda \frac{\partial\psi}{\partial\xi_\lambda}\langle J_\lambda\omega, e_i\rangle . \tag{3.2}$$

Then following a similar process we get

$$s(s-1)B(e_i, e_j, v, \cdots, v; w, \cdots, w)=\sum_{\lambda,\mu}\frac{\partial^2\psi}{\partial\xi_\lambda\partial\xi_\mu}\langle J_\lambda w, e_i\rangle\langle J_\mu w, e_j\rangle .$$

Putting $j=i$ and summing for $i=1, 2, 3, 4$, we get

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}+\frac{\partial^2\psi}{\partial\zeta^2}=0 ,$$

since we have

$$\sum_i\langle J_\lambda w, e_i\rangle\langle J_\mu w, e_i\rangle=\langle J_\lambda w, J_\mu w\rangle=\delta_{\lambda\mu}\langle w, w\rangle$$

and B is harmonic.

On the other hand the tensor B satisfying (3.1) satisfies $B(w, w, v, \cdots, v; v, \cdots, v)=0$ if $\sigma\geq 2$. Hence we get the following theorem.

THEOREM 3.1. *Let $\psi(\xi, \eta, \zeta)$ be a homogeneous polynomial of degree $s=2\sigma\geq 4$. The bi-symmetric tensor B of bi-degree (s, s) satisfying* (3.1) *is a harmonic one if and only if ψ is a harmonic homogeneous polynomial. Moreover in this case B belongs to the linear space $W(3, s)$.*

This theorem states that there exists a mapping $J^\#$ of $V(2, s)$ into $W(3, s)$.

DEFINITION 3.2. $J^\#: V(2, s)\to W(3, s)$ is the mapping such that

$$\psi(\langle J_1 w, v\rangle, \langle J_2 w, v\rangle, \langle J_3 w, v\rangle)=(J^\#\psi)(v; w) . \tag{3.3}$$

4. Contraction.

DEFINITION 4.1. Let C be an element of $W(m, s)$. There exists an element C^1 of $W(m, s-1)$ such that

$$C^1(v; w)=\sum_i C(e_i, v, \cdots, v; w, \cdots, w, e_i) .$$

C^1 is called the contraction (reduction) of C [5].

We consider $C\in W(3, s)$ stated in Theorem 3.1, hence

$$C(v; w)=\psi(\langle J_1 w, v\rangle, \langle J_2 w, v\rangle, \langle J_3 w, v\rangle) .$$

From this we get

$$sC(e_i, v, \cdots, v; w, \cdots, w) = \sum_\lambda \frac{\partial\psi}{\partial\xi_\lambda} \langle J_\lambda w, e_i\rangle .$$

Replacing w by $w+te_j$ and differentiating with respect to t, we get

$$s^2C(e_i, v, \cdots, v; w, \cdots, w, e_j) = \sum_{\lambda,\mu} \frac{\partial^2\psi}{\partial\xi_\lambda\partial\xi_\mu} \langle J_\lambda w, e_i\rangle\langle J_\mu e_j, v\rangle + \sum_\lambda \frac{\partial\psi}{\partial\xi_\lambda}\langle J_\lambda e_j, e_i\rangle$$

and then

$$s^2\sum_i C(e_i, v, \cdots, v; w, \cdots, w, e_i) = -\sum_{\lambda,\mu}\left(\frac{\partial^2\psi}{\partial\xi_\lambda\partial\xi_\mu}\sum_i \langle J_\lambda w, e_i\rangle\langle J_\mu v, e_i\rangle\right).$$

Since we have

$$\sum_i (\langle J_\lambda w, e_i\rangle\langle J_\mu v, e_i\rangle + \langle J_\mu w, e_i\rangle\langle J_\lambda v, e_i\rangle) = \langle J_\lambda w, J_\mu v\rangle + \langle J_\mu w, J_\lambda v\rangle = 2\langle v, w\rangle\delta_{\lambda\mu} ,$$

we get

$$\sum_i C(e_i, v, \cdots, v; w, \cdots, w, e_i) = 0 .$$

Thus we have the following theorem.

THEOREM 4.2. *Any element of $J^\# V(2, s)$ vanishes by contraction.*

5. The mappings $J^\#$ and $I^\#$.

Let $M_4 : \boldsymbol{R}^4 \to \boldsymbol{R}^4$ be the linear transformation such that $M_4e_i = e_i$ for $i = 1, 2, 3$ and $M_4e_4 = -e_4$. Then I_λ $(\lambda = 1, 2, 3)$ defined by

$$I_\lambda = M_4J_\lambda M_4$$

satisfy $I_2I_3 = -I_3I_2 = I_1$, $I_3I_1 = -I_1I_3 = I_2$, $I_1I_2 = -I_2I_1 = I_3$, $I_\lambda^2 = -1$, and $J_\lambda I_\mu = I_\mu I_\lambda$.

It is easy to prove the following theorem.

THEOREM 5.1. *There exists a mapping $I^\# : V(2, s) \to W(3, s)$, $s = 2\sigma \geq 4$, such that*

$$\psi(\langle I_1w, v\rangle, \langle I_2w, v\rangle, \langle I_3w, v\rangle) = (I^\#\psi)(v; w)$$

holds with every $\psi \in V(2, s)$ and $I^\#\psi \in W(3, s)$.

The contraction of $I^\#\psi$ also vanishes.

Let $I_0 = J_0$ be the identity transformation and $a = (a^0, a^1, a^2, a^3) \in \boldsymbol{R}^4$ be such that $(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2 = 1$. Then the sets $\{I_a = a^0I_0 + a^1I_1 + a^2I_2 + a^3I_3\}$ and $\{J_a = a^0J_0 + a^1J_1 + a^2J_2 + a^3J_3\}$ are subgroups of $SO(4)$, which are denoted by O_I and O_J, respectively. We get, in view of $J_\lambda I_a = I_a J_\lambda$,

$$\langle J_\lambda I_a w, I_a v\rangle = \langle (I_a)^{-1} J_\lambda I_a w, v\rangle = \langle J_\lambda w, v\rangle .$$

This proves that every element of $J^\# V(2, s)$ is invariant by the action of O_I [5]. In such a way we get the following theorem.

THEOREM 5.2. *Every element of $J^\# V(2, s)$ (resp. $I^\# V(2, s)$) is invariant by the action of O_I (resp. O_J).*

Clearly the mapping $J^\#$ is linear. In order to show that $J^\#$ is an injection, it is sufficient to prove that the $\psi \in V(2, s)$ satisfying

$$\psi(\langle J_1 w, v\rangle, \langle J_2 w, v\rangle, \langle J_3 w, v\rangle) = 0$$

identically is nothing but $\psi \equiv 0$. To prove this, let us take v and w such that, for example,

$$v = -\xi e_2 + \eta e_3 - \zeta e_4 + a e_1 , \qquad w = e_1 ,$$

where ξ, η, ζ are considered to be arbitrary. Then we have

$$\psi(\langle J_1 w, v\rangle, \langle J_2 w, v\rangle, \langle J_3 w, v\rangle) = \psi(\xi, \eta, \zeta)$$

and hence $\psi = 0$ identically, which proves that $J^\#$ is an injection.

DEFINITION 5.3. $W_0(m, s)$ is the linear subspace of $W(m, s)$ consisting of all elements of $W(m, s)$ with vanishing contraction.

Then we get the following theorem.

THEOREM 5.4. *$J^\#$ and $I^\#$ are injective linear mappings of $V(2, s)$ into $W_0(3, s)$ respectively.*

6. Inner products and $J^\#$, $I^\#$.

Let ψ_1 and ψ_2 be elements of $V(2, s)$, $s = 2\sigma$, where

$$\psi_1(\xi_1, \xi_2, \xi_3) = A^{\lambda_1 \cdots \lambda_s} \xi_{\lambda_1} \cdots \xi_{\lambda_s} , \qquad \psi_2(\xi_1, \xi_2, \xi_3) = B^{\lambda_1 \cdots \lambda_s} \xi_{\lambda_1} \cdots \xi_{\lambda_s} .$$

DEFINITION 6.1. The inner product $\langle \psi_1, \psi_2\rangle$ of the polynomials ψ_1 and ψ_2 is defined by

$$\langle \psi_1, \psi_2\rangle = A^{\lambda_1 \cdots \lambda_s} B_{\lambda_1 \cdots \lambda_s}$$

where $B_{\lambda_1 \cdots \lambda_s} = B^{\lambda_1 \cdots \lambda_s}$.

DEFINITION 6.2. Let C_1 and C_2 be elements of $W(m, s)$. The inner product $\langle C_1, C_2\rangle$ is defined by

$$\langle C_1, C_2\rangle = \sum_i{}^* \sum_j{}^* C_1^{i_1 \cdots i_s, j_1 \cdots j_s} C_2^{i_1 \cdots i_s, j_1 \cdots j_s} ,$$

namely

$$\langle C_1, C_2\rangle = \sum_i{}^* \sum_j{}^* C_1(e_{i_1}, \cdots, e_{i_s}; e_{j_1}, \cdots, e_{j_s}) \times C_2(e_{i_1}, \cdots, e_{i_s}; e_{j_1}, \cdots, e_{j_s}),$$

where

$$\sum_k{}^* = \sum_{k_1=1}^{m+1} \cdots \sum_{k_s=1}^{m+1}.$$

Now we consider the case $m=3$, $s=2\sigma$ and inquire into the relation between $\langle \psi_1, \psi_2\rangle$ and $\langle C_1, C_2\rangle$ when

$$\psi_a(\langle J_1 w, v\rangle, \langle J_2 w, v\rangle, \langle J_3 w, v\rangle) = C_a(v; w),$$

where $a=1, 2$. In order to get the formulas of $\langle \psi_1, \psi_2\rangle$ and $\langle C_1, C_2\rangle$ it is desirable to rewrite the formula written above, namely

$$\psi_a^{\lambda_1\cdots\lambda_s}\langle J_{\lambda_1} w, v\rangle \cdots \langle J_{\lambda_s} w, v\rangle = C_a(v; w)$$

in a precise form

$$(s!)^{-2}\sum_P \sum_Q \psi_a^{\lambda_1\cdots\lambda_s}\langle J_{\lambda_1} w_{Q(1)}, v_{P(1)}\rangle \cdots \langle J_{\lambda_s} w_{Q(s)}, v_{P(s)}\rangle = C_a(v_1, \cdots, v_s; w_1, \cdots, w_s),$$

where each of P and Q is a permutation of $1, \cdots, s$. Since $\psi_a^{\lambda_1\cdots\lambda_s}$ is symmetric with respect to $\lambda_1, \cdots, \lambda_s$, the left hand side can be rewritten

$$(s!)^{-1}\sum_Q \psi_a^{\lambda_1\cdots\lambda_s}\langle J_{\lambda_1} w_{Q(1)}, v_1\rangle \cdots \langle J_{\lambda_s} w_{Q(s)}, v_s\rangle.$$

In order to get $\langle C_1, C_2\rangle$, we must replace v_r by e_{i_r} and w_r by e_{j_r} where $r=1, \cdots, s$. Now we can find the relation between $\langle C_1, C_2\rangle$ and $\langle \psi_1, \psi_2\rangle$ following the method used in [3] pages 344, 345. Though s is an even number ≥ 4, the way of deduction is similar to the case $s=4$. Thus we get for each permutation P the number c_P satisfying

$$c_P = \gamma_P \langle \psi_1, \psi_2\rangle,$$

where γ_P is a number depending only on P. Hence we get

$$\langle C_1, C_2\rangle = (s!)^{-1}\Big(\sum_P \gamma_P\Big)\langle \psi_1, \psi_2\rangle,$$

and this proves

$$\langle J^*\psi_1, J^*\psi_2\rangle = (s!)^{-1}\Big(\sum_P \gamma_P\Big)\langle \psi_1, \psi_2\rangle, \tag{6.1}$$

$$\langle I^*\psi_1, I^*\psi_2\rangle = (s!)^{-1}\Big(\sum_P \gamma_P\Big)\langle \psi_1, \psi_2\rangle. \tag{6.2}$$

On the other hand we have $\langle J^{\#}\psi_1, I^{\#}\psi_2\rangle = 0$. The proof is almost the same as that in the case $s=4$ (see [3] pages 345, 346).

Thus we get the following theorem.

THEOREM 6.3. *Let ψ_1 and ψ_2 be elements of $V(2, s)$. Then we have*

$$\langle J^{\#}\psi_1, J^{\#}\psi_2\rangle = (s!)^{-1}\Big(\sum_P \gamma_P\Big)\langle \psi_1, \psi_2\rangle\,,$$

$$\langle I^{\#}\psi_1, I^{\#}\psi_2\rangle = (s!)^{-1}\Big(\sum_P \gamma_P\Big)\langle \psi_1, \psi_2\rangle\,,$$

$$\langle J^{\#}\psi_1, I^{\#}\psi_2\rangle = 0\,,$$

where γ_P is a certain number depending only on a permutation P of $1, \cdots, s$.

This theorem may be considered as another proof of Theorem 5.4.

COROLLARY 6.4. *If $\psi_1, \cdots, \psi_d$ are linearly independent in $V(2, s)$, then $J^{\#}\psi_1, \cdots, J^{\#}\psi_d, I^{\#}\psi_1, \cdots, I^{\#}\psi_d$ are also linearly independent.*

7. Elevation.

Let us consider an element C of $W(m, s)$ which satisfies

$$C(w, v, \cdots, v; w, \cdots, w) = 0 \tag{7.1}$$

for any $v, w \in \boldsymbol{R}^{m+1}$. Then for any natural number k there exists a sequence $a_0, a_1, \cdots, a_e$, where $e=[k/2]$ such that C_k, which is a bi-symmetric tensor of bi-degree $(s+k, s+k)$ defined by

$$C_k(v; w) = \sum_{p=0}^{e} a_p \langle v, w\rangle^{k-2p}\langle v, v\rangle^p \langle w, w\rangle^p C(v; w)\,, \tag{7.2}$$

belongs to $W(m, s+k)$. Since C_k must be harmonic, a_p are determined by

$$(k-2p+2)(k-2p+1)a_{p-1} + 2p(2s+m+2k-2p-1) = 0\,, \tag{7.3}$$

where a_0 can be chosen arbitrarily.

As it is easy to see, C_k satisfies the condition (iv) in §1 and we can state the following theorem.

THEOREM 7.1. *Let C be an element of $W(m, s)$ which satisfies* (7.1) *and a_p $(p=0, 1, \cdots, [k/2])$ be constants determined by* (7.3) *for a given natural number k. Then bi-symmetric tensor C_k of bi-degree $(s+k, s+k)$ given by* (7.2) *belongs to $W(m, s+k)$.*

DEFINITION 7.2. The tensor C_k stated in Theorem 7.1 is called an element *elevated* from C.

If $C \in W(3, s)$ belongs to $J^{*}V(2, s)$ or $I^{*}V(2, s)$, then C satisfies (7.1). Therefore we have the following.

COROLLARY 7.3. *Let C be an element of $J^{*}V(2, s)$ or $I^{*}V(2, s)$ and a_p $(p=0, 1, \cdots, [k/2])$ be constants determined by* (7.3). *Then C_k given by* (7.2) *belongs to* $W(3, s+k)$.

8. Geodesics.

We consider geodesics in minimal immersions of S^3 into spheres where the immersions $f_{3,s}$ are those associated with $J^{*}V(2, s)$ or $I^{*}V(2, s)$. Thus a geodesic is the image in $f_{3,s}(S^3(1))$ of a great circle

$$u(t)=a\cos t+b\sin t$$

of $S^3(1)$, where $\{a, b\}$ is an orthonormal set of vectors of $\boldsymbol{R}^4$. As we consider the geodesic as a curve

$$X(t)=\sum_{A=1}^{n} X^A(t)\tilde{e}_A$$

in $\boldsymbol{R}^n$, we can put

$$X^A(t)=F^A(u(t), \cdots, u(t)) .$$

Since we can follow almost the same way as in [7], we can omit some formulas stated in [7] though s is now an even number ≥ 4. Let us define $V_p(t)$ by

$$V_p(t)=\sum_{A=1}^{n} F_p^A(t)\tilde{e}_a ,$$

where

$$F_p^A(t)=F^A(u(t), \cdots, u(t), u'(t), \cdots, u'(t))$$

is of degree p in $u'(t)$. $V_p(t)$ satisfy

$$\frac{dV_p}{dt}=(s-p)V_{p+1}-pV_{p-1} ,$$

which is the result of $u''(t)=-u(t)$. Then as it is stated in [7], we have

$$X(t)=V_0(t) , \qquad \frac{dX(t)}{dt}=sV_1(t) ,$$

$$\frac{d^2X(t)}{dt^2}=-sV_0(t)+s(s-1)V_2(t) ,$$

$$\frac{d^3X(t)}{dt^3}=(-3s^2+2s)V_1(t)+s(s-1)(s-2)V_3(t)\,,$$

....

or

$$\frac{d^{2p}X(t)}{dt^{2p}}=\sum_{q=0}^{p}a_{p,q}V_{2q}(t)\,,$$

$$\frac{d^{2p+1}X(t)}{dt^{2p+1}}=\sum_{q=0}^{p}b_{p,q}V_{2q+1}(t)\,,$$

where $a_{p,q}$ and $b_{p,q}$ are some polynomials with respect to s. We define $V_{q,r}(t)$ by

$$V_{q,r}(t)=\langle V_q(t),\,V_r(t)\rangle=\sum_A F_q^A(t)F_r^A(t)\,.$$

Then we get

$$V_{q,r}(t)=C_{q,r}(u(t),\,u'(t))+c'u_{q,r}$$

from (2.3), (2.5), and (2.7) (for the details see [7]).

Because of $s=2\sigma$ we have

$$\frac{d^sX(t)}{dt^s}=\sum_{q=0}^{\sigma-1}a_{\sigma,q}V_{2q}(t)+s!\,V_s(t)\,,\qquad \frac{d^{s+1}X(t)}{dt^{s+1}}=\sum_{q=0}^{\sigma-1}b_{\sigma,q}V_{2q+1}(t)\,.$$

This shows that $d^{s+1}X/dt^{s+1}$ is a linear combination of $dX/dt,\cdots,d^{s-1}X/dt^{s-1}$. The Frenet formula of a geodesic considered as a curve in $\boldsymbol{R}^n$ is therefore written

$$\frac{dX}{dt}=i_1\,,\quad \frac{di_1}{dt}=k_1i_2\,,\quad \frac{di_2}{dt}=-k_1i_1+k_2i_3\,,$$

....

$$\frac{di_{s-1}}{dt}=-k_{s-2}i_{s-2}+k_{s-1}i_s\,,\quad \frac{di_s}{dt}=-k_{s-1}i_{s-1}\,.$$

First we get from $k_1i_2=d^2X/dt^2$

$$(k_1)^2=s^2V_{0,0}-2s^2(s-1)V_{0,2}+s^2(s-1)^2V_{2,2}$$

and hence

$$(k_1)^2=s^2\{c'u_{0,0}-2(s-1)c'u_{0.2}+(s-1)^2c'u_{2,2}+(s-1)^2C_{2,2}(u(t),\,u'(t))\}\,.$$

Now let us consider the property of $C_{p,q}(u(t),\,u'(t))$ when C belongs to $J^{\#}V(2,s)$ or $I^{\#}V(2,s)$. In view of (3.1) and $\langle J_\lambda v,\,v\rangle=0$ we have

$$C_{p,q}(v,\,w)=0\qquad \text{if}\quad p+q\neq s\,,$$

hence

$$C_{p,q}(u(t), u'(t))=0 \qquad \text{if} \quad p+q\neq s\,.$$

Then, in view of

$$\frac{d}{dt}C_{p,s-p}(u(t), u'(t))=(s-p)C_{p+1,s-p}(u(t), u'(t))+pC_{p,s-p+1}(u(t), u'(t))$$
$$-pC_{p-1,s-p}(u(t), u'(t))-(s-p)C_{p,s-p-1}(u(t), u'(t))=0\,,$$

we can see that every $C_{p,q}(u(t), u'(t))$ is independent of t.

This proves that k_1 does not depend on t. Calculation similar to that performed in [7] results in the following theorem.

THEOREM 8.1. *Let $\Gamma=f(\gamma)$ be a geodesic of $f_{3,s}(S^3(1))$, where $f_{3,s}$ are isometric minimal immersions associated with $J^{\#}V(2, s)$ or $I^{\#}V(2, s)$ and $k_1, \cdots, k_{s-1}$ be curvatures of Γ when it is considered as a curve in $\boldsymbol{R}^n$. Then the curvatures are constants which depend on the geodesic.*

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