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Isometric Deformation of Surfaces in the Hyperbolic 3-Manifold Preserving the Mean Curvature

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1. Introduction.

Let $(N^3(c), h)$ be a complete simply connected Riemannian 3-manifold of constant curvature c with metric h. Let $X : M \to N^3(c)$ be an isometric immersion of a Riemannian 2-manifold M into $N^3(c)$ and H the mean curvature of X. The isomertic immersion X is called *H-deformable* if there exists a non-trivial 1-parameter family of immersions X_t such that

(1) $$X_0 = X ,$$

(2) $$X_t^* h = X_0^* h ,$$

(3) $$H_t = H ,$$

where H_t denotes the mean curvature of X_t. An H-deformation $\{X_t\}$ is trivial if for each parameter t, there exists an isometry L of $N^3(c)$ such that $X_t = L \circ X_0$. An isometric immersion X is called *locally H-deformable* if each point of M has a neighborhood restricted to which X is H-deformable.

There are some papers on the H-deformable surfaces in Euclidean 3-space. O. Bonnet [1] proved that a surface of constant mean curvature in Euclidean space can be locally isometrically deformed preserving the mean curvature. É. Cartan [4] has studied such deformations for surfaces of nonconstant mean curvature and showed that they are W-surfaces. Chen and Peng [5] and K. Kenmotsu [8] characterized in some detail the Riemannian metrics and the mean curvature functions of the surfaces. Colares and Kenmotsu [7] and Roussos [14] proved that if a surface of constant Gaussian curvature in Euclidean 3-space is locally H-deformable, then the Gaussian curvature must be zero and such a deformation starts from a cylinder over a logarithmic spiral. Kokubu [9] studied such a deformation of hypersurfaces in Euclidean n-space ($n \geq 3$).

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In sections 2 and 3 of this paper we study locally H-deformable surfaces in $N^3(c)$. In section 4 we discuss the case of $c=-1$ in detail. The study of H-deformable surfaces in the hyperbolic space is more complicated than that in Euclidean space. The results we get are the following: If the Gaussian curvature K of the surface is constant, then $K=0$ or $K=-1$. In case of $K=0$, the mean curvature of the surface is constant. In case of $K=-1$, we can explicitly determine the mean curvature function in section 4.2. In both cases we can completely determine the first and second fundamental forms of the H-deformation $\{X_t\}$.

In this paper we deal with the local one. In fact for compact surfaces, Umehara [17] showed that the H-deformability characterizes surfaces with constant mean curvature in $N^3(c)$, which extends Tribuzy's result [16] for higher genus.

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2. Preliminaries.

We consider a piece of oriented surface M in $N^3(c)$, which does not contain any umbilic point. We apply the method by which Colares and Kenmotsu [7] studied an H-deformable surface in Euclidean 3-space. We get similar formulas for the surface M in $N^3(c)$ to those in Euclidean space.

Let $\{e_1, e_2, e_3\}$ be an orthonormal vector fields on M such that e_1 and e_2 are unit tangent vectors at $x \in M$ and e_3 is the unit normal vector at $x \in M \subset N^3(c)$. Let $\{\omega^1, \omega^2, \omega^3\}$ be the system of dual 1-forms of $\{e_1, e_2, e_3\}$, and ω_B^A the connection form given by

$$\omega^i(X)=\langle X, e_i\rangle, \qquad \omega_B^A(X)=\langle \nabla_X e_A, e_B\rangle,$$

where the indices A, B, C run through 1 to 3, and X is any vector field on M. Here ∇ denotes the Levi-Civita connection of the Riemannian metric h and $\langle\ ,\ \rangle$ the inner product induced by h on $N^3(c)$. We then have the structure equations of $N^3(c)$,

$$d\omega^A=-\sum_{B=1}^{3}\omega_B^A\wedge\omega^B,$$

$$d\omega_B^A=-\sum_{C=1}^{3}\omega_C^A\wedge\omega_B^C+c\cdot\omega^A\wedge\omega^B.$$

Restricting these equations to M, we get

$$d\omega^1=-\omega_2^1\wedge\omega^2, \tag{4}$$

$$d\omega^2=-\omega_1^2\wedge\omega^1, \tag{5}$$

$$d\omega_j^i=-\sum_{k=1}^{3}\omega_k^i\wedge\omega_j^k+c\cdot\omega^i\wedge\omega^j, \tag{6}$$

where the indices i, j run through 1 to 2. From $X^*\omega^3|_M=0$, and $d\omega^3=-\sum_{j=1}^2 \omega_j^3\wedge\omega^j$, we can write

$$\omega_j^3=\sum_{k=1}^2 h_{jk}\omega^k ,$$

where h_{jk} is the coefficient of the second fundamental form of the immersion X. The Gaussian curvature K of M is defined by

$$d\omega_2^1=K\cdot\omega^1\wedge\omega^2 .$$

Then the Gauss equation is given by

$$K=c+\det(h) .$$

We may write as $\omega_1^3=(H+x)\omega^1+y\omega^2$, $\omega_2^3=y\omega^1+(H-x)\omega^2$ for some functions x and y on M. By the Gauss equation, we have $H^2-K+c=x^2+y^2$. Thus we can write as

$$\omega_1^3=(H+\sqrt{H^2-K+c}\cos\alpha)\omega^1+\sqrt{H^2-K+c}\sin\alpha\omega^2 , \tag{7}$$

$$\omega_2^3=\sqrt{H^2-K+c}\sin\alpha\omega^1+(H-\sqrt{H^2-K+c}\cos\alpha)\omega^2 , \tag{8}$$

where α is a locally defined function on M. We define

$$D\alpha:=d\alpha+2\omega_1^2=\alpha_1\omega^1+\alpha_2\omega^2 , \tag{9}$$

where α_1 and α_2 are coefficients of the 1-form $D\alpha$. For any tensor field of (0, 1)-type α_i, we define its covariant derivatives $\alpha_{i,j}$ as follows:

$$D\alpha_i:=d\alpha_i-\sum\alpha_s\omega_i^s=\sum\alpha_{i,j}\omega^j . \tag{10}$$

Exterior differentiations of (7) and (8) gives

$$\begin{aligned}\sqrt{H^2-K+c}D\alpha&=\cos\alpha(H_1\omega^2+H_2\omega^1)-\sin\alpha(H_1\omega^1-H_2\omega^2)\\&\quad+(\sqrt{H^2-K+c})_2\omega^1-(\sqrt{H^2-K+c})_1\omega^2 ,\end{aligned} \tag{11}$$

where H_i and $(\sqrt{H^2-K+c})_i$, $i=1, 2$, are exterior derivatives of the scalar functions H and $\sqrt{H^2-K+c}$ defined as in (9), respectively.

We define the 1-forms

$$\beta_1=\frac{H_1\omega^1-H_2\omega^2}{\sqrt{H^2-K+c}} , \qquad \beta_2=\frac{H_2\omega^1+H_1\omega^2}{\sqrt{H^2-K+c}} .$$

Since the $*$-operator of Hodge is given by $*\omega^1=\omega^2$ and $*\omega^2=-\omega^1$, the formula (11) can be written as

$$D\alpha=-\sin\alpha\cdot\beta_1+\cos\alpha\cdot\beta_2-*d\log\sqrt{H^2-K+c} . \tag{12}$$

We calculate the exterior derivatives of β_i as follows:

$$(13)\begin{cases} d\beta_1=\dfrac{1}{\sqrt{H^2-K+c}}[\{(\log\sqrt{H^2-K+c})_1H_2+(\log\sqrt{H^2-K+c})_2 \\ \qquad \times H_1-2H_{1,2}\}\omega^1\wedge\omega^2-2\sqrt{H^2-K+c}\beta_2\wedge\omega_1^2]\,, \\ d\beta_2=\dfrac{1}{\sqrt{H^2-K+c}}[\{(\log\sqrt{H^2-K+c})_1H_1+(\log\sqrt{H^2-K+c})_2 \\ \qquad \times H_2+H_{1,1}-H_{2,2}\}\omega^1\wedge\omega^2+2\sqrt{H^2-K+c}\beta_1\wedge\omega_1^2]\,, \end{cases}$$

where $H_{i,j}$'s are covariant derivatives of H_i defined in (10). Using (9) and (13), the exterior differentiation of (12) gives the condition:

$$(14)\qquad -2A\sin\alpha+B\cos\alpha+P=0\,,$$

where we put

$$(15)\begin{cases} A=H_{1,2}\sqrt{H^2-K+c}-H_2(\sqrt{H^2-K+c})_1-H_1(\sqrt{H^2-K+c})_2\,, \\ B=(H_{2,2}-H_{1,1})\sqrt{H^2-K+c}+2H_1(\sqrt{H^2-K+c})_1-2H_2(\sqrt{H^2-K+c})_2\,, \\ P=(H^2-K+c)(\Delta\log\sqrt{H^2-K+c}-2K)-|\operatorname{grad}H|^2\,. \end{cases}$$

We shall give another formula obtained from (12). Applying the $*$-operator to (12), we obtain

$$(16)\qquad \alpha_1\omega^2-\alpha_2\omega^1=-\sin\alpha\cdot\beta_2-\cos\alpha\cdot\beta_1+d(\log\sqrt{H^2-K+c})\,.$$

By exterior differentiation of (16), we obtain

$$(17)\qquad (H^2-K+c)\Delta\alpha=2A\cos\alpha+B\sin\alpha\,.$$

It follows from (14) and (17) that the conditions $A=B=0$ are equivalent to the conditions $P=\Delta\alpha=0$.

3. *H*-deformable surfaces with constant Gaussian curvature in $N^3(c)$.

In this section we study a surface with constant Gaussian curvature in $N^3(c)$ which admits an isometric deformation preserving the mean curvature. We denote by ∇ the covariant differentiation of the induced metric from an isometric immersion $X: M\to N^3(c)$ and we put $Z=(e_1-ie_2)/2$. Then we get the following theorem which can be proved in a similar way to [7].

THEOREM 1. *Let M be a piece of an oriented surface in $N^3(c)$ such that it has no umbilic points. Then, M admits a non-trivial isometric deformation preserving the mean curvature if and only if one of the following conditions holds*:

(18) $$\nabla\left(\frac{\nabla H}{H^2-K+c}\right)(Z, Z)=0\,,$$

(19) $$P=0 \quad and \quad \Delta\alpha=0\,.$$

We classify surfaces with constant Gaussian curvature K in $N^3(c)$ which admit an isometric deformation preserving the mean curvature function H.

THEOREM 2. *Let M be a piece of an oriented surface in $N^3(c)$ without umbilic points such that K is constant on M. If M admits a non-trivial isometric deformation preserving the mean curvature function, then $K=c$ or $K=0$.*

PROOF. First we consider the case M is a minimal suface. By results of [2], [3], [11], and [13], we have $K=0$ or $K=1$ when $c=1$, we have $K=0$ when $c=0$, and we have $K=-1$ when $c=-1$. Next suppose that $H\neq 0$. We define a tensor field of (0, 1)-type defined by $f_i=H_i/(H^2-K+c)$ for $i=1, 2$. A computation shows us $f_{i,j}=\{(H^2-K+c)H_{i,j}-2HH_iH_j\}/(H^2-K+c)^2$. The condition $A=0$ implies

$$H_{1,2}(H^2-K+c)-2H_1HH_2=0\,.$$

The condition $B=0$ implies

$$H_{2,2}(H^2-K+c)-2HH_2^2=H_{1,1}(H^2-K+c)-2HH_1^2\,.$$

Hence there exists a scalar function λ with $f_{i,j}=\lambda\delta_{i,j}$. By taking the trace of these equations, we get

$$2\lambda=\sum f_{i,i}=\frac{(H^2-K+c)\Delta H-2H\,|\operatorname{grad} H\,|^2}{(H^2-K+c)^2}\,.$$

On the other hand the condition $P=0$ is equivalent to

$$(H^2-K+c)\Delta H-2H\,|\operatorname{grad} H\,|^2=\frac{2K(H^2-K+c)^2}{H}\,.$$

These formulas follow $\lambda=K/H$, which implies

(20) $$Hf_{i,j}=K\delta_{i,j}\,, \qquad 1\leq i, j\leq 2\,.$$

We have from (20),

(21) $$H_kf_{i,j}-H_jf_{i,k}+H(f_{i,j,k}-f_{i,k,j})=0\,.$$

We use the Ricci identities of the tensor field f_i:

(22) $$f_{1,2,1}-f_{1,1,2}=Kf_2\,,$$

(23) $$f_{2,1,2}-f_{2,2,1}=Kf_1\,.$$

By (20), (21), (22), and (23), we have $K(K-c)H_i=0$, $i=1, 2$. Thus if $H_i=0$, $i=1, 2$, then H is constant and $f_{i,j}$'s vanish identically. Therefore K must be zero. This completes the proof.

4. Deformations of surfaces in the hyperbolic space $H^3(-1)$.

In this section we study deformations of H-deformable surfaces with constant curvature in the hyperbolic 3-space $H^3(-1)$. We consider the hyperbolic 3-space as the upper half space model $\{(x_1, x_2, x_3)\in \boldsymbol{R}^3 : x_3>0\}$ with the metric $(dx_1^2+dx_2^2+dx_3^2)/x_3^2$. Recall that the theorem 2 implies the curvature of the surface is $K=0$ or $K=-1$.

4.1. The case of the Gaussian curvature $K=0$. Let M be a piece of Euclidean 2-plane with flat metric du^2+dv^2. Consider an isometric immersion $X(u, v): M\to H^3(-1)$ such that it has no umbilic points and satisfies the condition (18). Putting $\omega^1=du$ and $\omega^2=dv$, we have $\omega_2^1=0$. The condition (18) is equivalent to

$$\sqrt{H^2-1}H_{uv}-H_v(\sqrt{H^2-1})_u-H_u(\sqrt{H^2-1})_v=0\,, \tag{24}$$

$$(H^2-1)(H_{vv}-H_{uu})+2H(H_u^2-H_v^2)=0\,. \tag{25}$$

The general solutions of (24) are

$$H(u, v)=\frac{1+c_1(u)c_2(v)}{1-c_1(u)c_2(v)}\,,$$

where $c_1(u)$ and $c_2(v)$ are any functions depending only on u and v satisfying $c_1(u)c_2(u)>0$, respectively. Considering (25), we get

$$H(u, v)=\frac{1+K_1\exp(K_2(u^2+v^2))}{1-K_1\exp(K_2(u^2+v^2))}\,, \tag{26}$$

where K_1 is a positive constant and K_2 is a constant independent of u and v. Furthermore, considering (19), K_2 must be zero. As a result the surface has the constant mean curvature $H=(1+K_1)/(1-K_1)$. This is an equidistance surface from a geodesic line in $H^3(-1)$ (see [15], [16]). Then the isometric deformation preserving the mean curvature H is given by

$$X_t(u, v)=(r\cos\theta, r\sin\theta, r\tan\omega)\,, \tag{27}$$

where we set $r=\exp(\sin\omega\cdot\tilde{u})$ and $\theta=\tan\omega\cdot\tilde{v}$, also $\tilde{u}=\cos t\cdot u-\sin t\cdot v$ and $\tilde{v}=\sin t\cdot u+\cos t\cdot v$, and ω is a constant.

4.2. The case of the Gaussian curvature $K=-1$. In this section we study surfaces with the curvature $K=-1$ having the properties (18) and (19). Let M be a piece of the hyperbolic surface as the upper half-space model $\{(u, v)\in\boldsymbol{R}^2 : v>0\}$ with the metric $ds^2=(du^2+dv^2)/v^2$. We consider an isometric immersion $X(u, v): M\to H^3(-1)$ such that it has no umbilic points and satisfies the condition (18). Putting $\omega^1=du/v$

and $\omega^2 = dv/v$, we have $\omega_2^1 = -du/v$. The condition (18), which means the conditions that A and B in (15) are zero, is equivalent to

$$H\{vH_{uv}+H_u\}=2vH_uH_v\,, \tag{28}$$

$$H\{v(H_{uu}-H_{vv})-2H_v\}=2v(H_u^2-H_v^2)\,. \tag{29}$$

We can easily see that these formulas are also equivalent to

$$\left(\frac{H}{v}\right)\left(\frac{H}{v}\right)_{uv}-2\left(\frac{H}{v}\right)_u\left(\frac{H}{v}\right)_v=0\,, \tag{30}$$

$$\frac{H}{v}\left\{\left(\frac{H}{v}\right)_{vv}-\left(\frac{H}{v}\right)_{uu}\right\}+2\left\{\left(\left(\frac{H}{v}\right)_u\right)^2-\left(\left(\frac{H}{v}\right)_v\right)^2\right\}=0\,. \tag{31}$$

The general solution of (28) is

$$\frac{H(u,v)}{v}=\frac{1}{(\phi(u)+\psi(v))}\,, \tag{32}$$

where ϕ and ψ are any functions. Considering (29), we get

$$\phi''(u)-\psi''(v)=0\,. \tag{33}$$

Thus, we have

$$\phi(u)=au^2+bu+d'\,,\qquad \psi(v)=av^2+ev+d''\,.$$

Therefore we have

$$\frac{H(u,v)}{v}=\frac{1}{a(u^2+v^2)+bu+ev+d}\,, \tag{34}$$

where a, b, e and $d=d'+d''$ are some real numbers. When $K=c=-1$, the equation $P=0$ in (19) becomes

$$\Delta\log H+2-|d\log H|^2=0\,. \tag{35}$$

Substituting (34) into (35), we have $e=0$.

Case 1. We first consider the case $b^2-4ad=0$. By taking an isometric transformation of the hyperbolic surface such that $u\mapsto u+b/(2a)$, we may assume that the mean curvature H is

$$H=\frac{v}{a(u^2+v^2)}\,.$$

Furthermore, by taking isometric transformations such that

$$u\mapsto -\frac{u}{u^2+v^2}\,,\qquad v\mapsto\frac{v}{u^2+v^2}\,,$$

and

$$u \mapsto u/a\,, \qquad v \mapsto v/a\,,$$

we may assume

(36) $$H = v\,.$$

By (11), we have

(37) $$\frac{\partial\alpha}{\partial u} = \frac{\cos\alpha - 1}{v}\,, \qquad \frac{\partial\alpha}{\partial v} = \frac{\sin\alpha}{v}\,.$$

We can easily check $\Delta\alpha = 0$. This system is integrable, and general solutions of (37) are

(38) $$\tan\frac{\alpha}{2} = \frac{tv}{tu+1}\,,$$

where t is any real number. In this case we get a 1-parameter family of isometric immersions X_t preserving the mean curvature $H = v$: The second fundamental tensors h_{ij}'s of X_t are given by

(39) $$(h_{ij}) = \begin{pmatrix} v + v\cos\alpha & v\sin\alpha \\ v\sin a & v - v\cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{2v(1+tu)^2}{(1+tu)^2+(tv)^2} & \dfrac{(2tv^2)(1+tu)}{(1+tu)^2+(tv)^2} \\ \dfrac{(2tv^2)(1+tu)}{(1+tu)^2+(tv)^2} & \dfrac{2t^2v^3}{(1+tu)^2+(tv)^2} \end{pmatrix}.$$

We have the following system of the differential equations for $\{X_t, e_1, e_2, e_3\}$:

(40) $$\begin{cases} \nabla_{e_1}e_1 = \omega_1^2(e_1)e_2 + \omega_1^3(e_1)e_3 = e_2 + h_{11}\cdot e_3 \\ \nabla_{e_2}e_1 = \omega_1^2(e_2)e_2 + \omega_1^3(e_2)e_3 = h_{12}\cdot e_3 \\ \nabla_{e_1}e_2 = \omega_2^1(e_1)e_1 + \omega_2^3(e_1)e_3 = -e_1 + h_{21}\cdot e_3 \\ \nabla_{e_2}e_2 = \omega_2^1(e_2)e_1 + \omega_2^3(e_2)e_3 = h_{22}\cdot e_3 \\ \nabla_{e_1}e_3 = \omega_3^1(e_1)e_1 + \omega_3^2(e_1)e_3 = -h_{11}\cdot e_1 - h_{12}\cdot e_2 \\ \nabla_{e_2}e_3 = \omega_3^1(e_2)e_1 + \omega_3^2(e_2)e_2 = -h_{12}\cdot e_1 - h_{22}\cdot e_2\,, \end{cases}$$

where h_{ij}'s are given in (39), and $\{e_1 = v\partial X_t/\partial u, e_2 = v\partial X_t/\partial v, e_3\}$ is a system of orthonormal vector fields. When $t=0$, we have $h_{11} = 2v$, $h_{12} = h_{21} = h_{22} = 0$. Thus the integral curve of e_2 is a geodesic. The surface X_0 is realized as the following one:

(41) $$X_0(u, v) = (c\sin(2u), c\cos(2u), 2cv)\,,$$

or

(42) $$X_0(u,v)=\left(\frac{c}{1+4v^2}\cdot\cos(2u),\ \frac{c}{1+4v^2}\cdot\sin(2u),\ \frac{2cv}{1+4v^2}\right),$$

where c is a constant. We can see easily that $X_0(u,v)$ satisfies (40), and $X_0(u_0,v)$ is a geodesic for fixed u_0. When $t=\infty$, we have $h_{11}=2u^2v/(u^2+v^2)$, $h_{12}=h_{21}=2uv^2/(u^2+v^2)$, $h_{22}=2v^3/(u^2+v^2)$. The surface X_∞ is realized as the following hyperbolic cylinder [12]:

(43) $$X_\infty(u,v)=\left(\zeta(\varphi)\cdot\frac{u}{\sqrt{u^2+v^2}},\ \eta(\varphi),\ \zeta(\varphi)\cdot\frac{v}{\sqrt{u^2+v^2}}\right),$$

where $\exp(\varphi)=\sqrt{u^2+v^2}$. The functions ζ and η depending on φ are defined by the following equation:

$$\frac{d\zeta}{d\varphi}=\zeta\cos y\,,\qquad \frac{d\eta}{d\varphi}=\zeta\sin y\,,$$

where y is the function of φ, which satisfies

$$\frac{dy}{d\varphi}-\sin y=-2e^{\varphi}\,.$$

Case 2. We consider the case $b^2-4ad>0$. By taking an isometric transformation such that $u\mapsto u+(-b\pm\sqrt{b^2-4ad})/(2a)$, we may assume that H is

$$H=\frac{v}{a(u^2+v^2)+bu}\,.$$

Furthermore, by

$$u\mapsto-\frac{u}{u^2+v^2}\,,\qquad v\mapsto\frac{v}{u^2+v^2}\,,$$

and $u\mapsto u-a/b$, we may assume that

(44) $$H(u,v)=\frac{v}{bu}\,.$$

By (11) and (44), we have

(45) $$\frac{\partial\alpha}{\partial u}=\frac{\sin\alpha}{u}+\frac{(\cos\alpha-1)}{v}\,,\qquad \frac{\partial\alpha}{\partial v}=\frac{1-\cos\alpha}{u}+\frac{\sin\alpha}{v}\,.$$

We can easily check $\Delta\alpha=0$. This system is integrable, and general solutions of this system are

(46) $$\tan\frac{\alpha}{2}=\frac{2uvt}{t(u^2-v^2)+1}\,,$$

where t is any real number. Therefore we get a 1-parameter family of isometric immersion X_t preserving the mean curvature $H=v/(bu)$: The second fundamental tensors h_{ij}'s of X_t are given by

$$(47)\qquad (h_{ij})=\frac{1}{b}\begin{pmatrix} \frac{v}{u}+\frac{v}{u}\cos\alpha & \frac{v}{u}\sin\alpha \\ \frac{v}{u}\sin\alpha & \frac{v}{u}-\frac{v}{u}\cos\alpha \end{pmatrix}$$

$$=\frac{1}{b}\begin{pmatrix} \frac{2v(t(u^2-v^2)+1)^2}{u\{t^2(u^2+v^2)^2+2t(u^2-v^2)+1\}} & \frac{4uv^2t(t(u^2-v^2)+1)}{u\{t^2(u^2+v^2)^2+2t(u^2-v^2)+1\}} \\ \frac{4uv^2t(t(u^2-v^2)+1)}{u\{t^2(u^2+v^2)^2+2t(u^2-v^2)+1\}} & \frac{8t^2u^2v^3}{u\{t^2(u^2+v^2)^2+2t(u^2-v^2)+1\}} \end{pmatrix}.$$

We get again the total differential equation (40) for X_t, where h_{ij}'s are given in (47). When $t=0$, it follows that $h_{12}=h_{21}=h_{22}=0$, $h_{11}=2v/(bu)$. Thus the integral curve of e_2 is a geodesic. The surface X_0 is realized as the following:

$$(48)\qquad X_0(u, v)=2\left(\int\sin\frac{\log u}{b}du,\ \int\cos\frac{\log u}{b}du,\ v\right)$$

When $b=1$, this formula is the same as that of H-deformable surfaces in Euclidean 3-space [7]. But the first and the second fundamental forms are different from those of Euclidean case.

Case 3. We consider the case $b^2-4ad<0$. By taking isometric transformations such that $u\mapsto u+b/(2a)$, and $u\mapsto au$, $u\mapsto av$, we may assume that

$$(49)\qquad H=\frac{v}{u^2+v^2+d},$$

where d is a positive number. By (11) and (49), we have

$$(50)\qquad \begin{cases} \dfrac{\partial\alpha}{\partial u}=\dfrac{2u}{u^2+v^2+d}\sin\alpha+\dfrac{u^2-v^2+d}{v(u^2+v^2+d)}(\cos\alpha+1)-\dfrac{2}{v} \\ \dfrac{\partial\alpha}{\partial v}=\dfrac{2u}{u^2+v^2+d}(1-\cos\alpha)+\dfrac{u^2-v^2+d}{v(u^2+v^2+d)}\sin\alpha\,. \end{cases}$$

This system is integrable, and general solutions are given by

$$\psi\left(u-v\cot\frac{\alpha}{2},\ \frac{\sin\alpha/2}{v}\right)=0,$$

for an arbitrary function ψ. We get a 1-parameter family of isometric immersions X_t preserving the mean curvature $H=v/(u^2+v^2+d)$: The second fundamental tensors h_{ij}'s of X_t are given by

$$(h_{ij})=\frac{v}{(u^2+v^2+d)}\begin{pmatrix}1+\cos\alpha & \sin\alpha \\ \sin\alpha & 1-\cos\alpha\end{pmatrix}. \tag{51}$$

Therefore we have proved the following:

THEOREM 3. *Let M be a piece of surface with constant Gaussian curvature $K=0$ or $K=-1$ in the hyperbolic 3-manifold $H^3(-1)$, which does not contain any umbilic point. Suppose that M admits a non-trivial isometric deformation preserving the mean curvature function H.*

1. *If $K=0$, then H becomes constant. Consequently M is an equidistance surface from a geodesic line in $H^3(-1)$, and the H-deformation is given by* (27).
2. *If $K=-1$, then we get*

2.1. *$H=v$, and the second fundamental forms of M are determined by* (39) *for some t and the H-deformation of M starts from the surface which is given by* (41) *or* (42),
or

2.2. *$H=v/(bu)$ for any real number b, and the second fundamental forms of M are determined by* (47) *for some t and the H-deformation of M starts from the surface which is given by* (48),
or

2.3. *$H=v/(u^2+v^2+d)$ for a positive number d, and the second fundamental forms of M are determined by* (51).

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