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The Hasse Norm Principle for the Maximal Real Subfields of Cyclotomic Fields

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§1. Introduction.

Let K/k be a finite extension of number fields. Let J_K be the idele group of K and $N_{K/k}$ the norm map from K to k. The group $N_{K/k}K^{\times}$ of global norms is a subgroup of finite index in $k^{\times}\cap N_{K/k}J_K$. We say that the Hasse norm principle (abbreviated to HNP) holds for K/k if $k^{\times}\cap N_{K/k}J_K=N_{K/k}K^{\times}$. We simply say that HNP holds for K if HNP holds for $K/\boldsymbol{Q}$. The classical Hasse norm theorem asserts that if K/k is a cyclic extension, then HNP holds for K/k.

Several authors have studied the validity of HNP for abelian extensions. In [3] and [4], Gerth and Gurak independently gave necessary and sufficient conditions for HNP to hold for $\boldsymbol{Q}(\zeta_m)$, where $m\not\equiv 2 \pmod 4$ is a positive integer and ζ_m is a primitive m-th root of unity. If HNP holds for $\boldsymbol{Q}(\zeta_m)$, then it holds also for its maximal real subfield $\boldsymbol{Q}(\zeta_m)^+$ (Proposition 1 below). However, the converse is not always true. In this paper, we will give a necessary and sufficient condition for HNP to hold for $\boldsymbol{Q}(\zeta_m)^+$.

§2. Theorems.

Let $m\not\equiv 2 \pmod 4$ be a positive integer, and let p_1, p_2, p_3 and p_4 be distinct odd primes, and e, a_1, a_2, a_3, a_4 non-negative integers. We denote by $\left(\frac{*}{*}\right)$ the Legendre symbol and define ε_i and $\varepsilon_{i,j}$ ($\in\{0, 1\}$) by $(-1)^{\varepsilon_i}=\left(\frac{2}{p_i}\right)$ and $(-1)^{\varepsilon_{i,j}}=\left(\frac{p_j}{p_i}\right)$, respectively.

(A) Suppose that m has at most three distinct prime divisors and that $m\neq 2^e p_1^{a_1}p_2^{a_2}$, $e\geq 3$. In this case, we know necessary and sufficient conditions for HNP to hold for $\boldsymbol{Q}(\zeta_m)$ (cf. [3, 4]).

THEOREM 1. *HNP does not hold for $\boldsymbol{Q}(\zeta_m)$ but does hold for $\boldsymbol{Q}(\zeta_m)^+$ if and only if*

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it holds for every maximal subfield of $\boldsymbol{Q}(\zeta_m)^+$ *whose Galois group over* $\boldsymbol{Q}$ *has odd prime exponent, and moreover, one of the following five conditions is satisfied*:

(1) $m=4p_1^{a_1}$ *and* $p_1\equiv 1 \pmod 8$.

(2) $m=2^e p_1^{a_1}$, $e\geq 3$ *and* $p_1\equiv 7 \pmod 8$.

(3) $m=p_1^{a_1}p_2^{a_2}$, $p_i\equiv 1$, $p_j\equiv 3 \pmod 4$ *and* $\left(\frac{p_2}{p_1}\right)=1$, *where* $\{i,j\}=\{1,2\}$.

(4) $m=4p_1^{a_1}p_2^{a_2}$,

(a) $p_1\equiv p_2\equiv 1 \pmod 4$ *and* $\left(\frac{p_2}{p_1}\right)=-1$, *or*

(b) $p_i\equiv 1$, $p_j\equiv 3 \pmod 4$ *and* $\left(\frac{p_2}{p_1}\right)\neq\left(\frac{2}{p_i}\right)$, *where* $\{i,j\}=\{1,2\}$, *or*

(c) $p_1\equiv p_2\equiv 3 \pmod 4$ *and at least one of* $\left(\frac{2}{p_i}\right)$ *and* $\left(\frac{p_i}{p_j}\right)$ *is equal to* 1, *where* $\{i,j\}=\{1,2\}$.

(5) $m=p_1^{a_1}p_2^{a_2}p_3^{a_3}$,

(a) $p_i\equiv 1$, $p_j\equiv p_k\equiv 3 \pmod 4$ *and* $\left(\frac{p_j}{p_i}\right)\neq\left(\frac{p_k}{p_i}\right)$, *where* $\{i,j,k\}=\{1,2,3\}$, *or*

(b) $p_i\equiv p_j\equiv 1$, $p_k\equiv 3 \pmod 4$ *and* $\left(\frac{p_i}{p_j}\right)=-1$, *where* $\{i,j,k\}=\{1,2,3\}$, *or*

(c) $p_1\equiv p_2\equiv p_3\equiv 3 \pmod 4$, *and* $\left(\frac{p_2}{p_1}\right)=\left(\frac{p_3}{p_2}\right)=\left(\frac{p_1}{p_3}\right)$ *does not hold.*

(B) Suppose that $m=2^e p_1^{a_1}p_2^{a_2}$, $e\geq 3$ or that m has more than three distinct prime divisors. Then HNP does not hold for $\boldsymbol{Q}(\zeta_m)$ (cf. [3, 4]).

THEOREM 2. *HNP holds for* $\boldsymbol{Q}(\zeta_m)^+$ *if and only if one of the following four conditions is satisfied*:

(1) $m=2^e p_1^{a_1}p_2^{a_2}$, $e\geq 3$,
HNP holds for $\boldsymbol{Q}(\zeta_{p_1^{a_1}p_2^{a_2}})$, *and* $p_i\equiv 3 \pmod 4$, $p_j\equiv 3, 5 \pmod 8$, $\{i,j\}=\{1,2\}$.

(2) $m=4p_1^{a_1}p_2^{a_2}p_3^{a_3}$,
HNP holds for $\boldsymbol{Q}(\zeta_{p_1^{a_1}p_2^{a_2}p_3^{a_3}})$ *and*

(a) $p_i\equiv 3$, $p_j\equiv p_k\equiv 1 \pmod 4$, $\left(\frac{p_k}{p_j}\right)=-1$ *and* $\varepsilon_k\varepsilon_{i,j}\neq\varepsilon_j\varepsilon_{k,i}$, *where* $\{i,j,k\}=\{1,2,3\}$, *or*

(b) $p_i\equiv p_j\equiv 3$, $p_k\equiv 1 \pmod 4$ *and at most one of* $\left(\frac{2}{p_k}\right)$, $\left(\frac{p_k}{p_i}\right)$ *and* $\left(\frac{p_k}{p_j}\right)$ *is equal to* 1, *where* $\{i,j,k\}=\{1,2,3\}$, *or*

(c) $p_1\equiv p_2\equiv p_3\equiv 3 \pmod 4$,

- $p_1\equiv p_2\equiv p_3 \pmod 8$ *and* $\left(\frac{p_2}{p_1}\right)=\left(\frac{p_3}{p_2}\right)=\left(\frac{p_1}{p_3}\right)$, *or*
- $p_i\equiv p_j\not\equiv p_k \pmod 8$ *and at least one of* $\left(\frac{p_i}{p_k}\right)$ *and* $\left(\frac{p_j}{p_k}\right)$ *is equal to* 1, *where*

$\{i, j, k\}=\{1, 2, 3\}$.

(3) $m=2^e p_1^{a_1} p_2^{a_2} p_3^{a_3}$,

HNP holds for $Q(\zeta_{p_1^{a_1} p_2^{a_2} p_3^{a_3}})$, *and*

(a) $p_i \equiv p_j \equiv 3 \pmod 4$, $p_k \equiv 5 \pmod 8$ *and* $\varepsilon_i \varepsilon_{j,k} \neq \varepsilon_j \varepsilon_{k,i}$, *or*

(b) $p_i \equiv 7$, $p_j \equiv p_k \equiv 3 \pmod 8$ *and* $\left(\frac{p_j}{p_i}\right) \neq \left(\frac{p_k}{p_i}\right)$.

(4) $m=p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$,

the greatest common divisor of $\varphi(p_1^{a_1})$, $\varphi(p_2^{a_2})$, $\varphi(p_3^{a_3})$ *and* $\varphi(p_4^{a_4})$ *is a power of* 2, *where* φ *is the Euler function, HNP holds for* $Q(\zeta_{p_i^{a_i} p_j^{a_j} p_k^{a_k}})$ $(1 \le i < j < k \le 4)$, *and*

(a) $p_i \equiv p_j \equiv 3$, $p_k \equiv p_l \equiv 1 \pmod 4$, $\left(\frac{p_l}{p_k}\right) = -1$ *and* $\varepsilon_{i,k} \varepsilon_{j,l} \neq \varepsilon_{i,l} \varepsilon_{j,k}$, *where* $\{i, j, k, l\}=\{1, 2, 3, 4\}$, *or*

(b) $p_i \equiv p_j \equiv p_k \equiv 3$, $p_l \equiv 1 \pmod 4$ *and at most one of* $\left(\frac{p_l}{p_i}\right)$, $\left(\frac{p_l}{p_j}\right)$ *and* $\left(\frac{p_l}{p_k}\right)$ *is equal to* 1, *where* $\{i, j, k, l\}=\{1, 2, 3, 4\}$, *or*

(c) $p_1 \equiv p_2 \equiv p_3 \equiv p_4 \equiv 3 \pmod 4$,

- $\left(\frac{p_j}{p_i}\right)=\left(\frac{p_k}{p_j}\right)=\left(\frac{p_l}{p_k}\right)=\left(\frac{p_i}{p_l}\right)$, $\{i, j, k, l\}=\{1, 2, 3, 4\}$, *or*
- $\left(\frac{p_j}{p_i}\right)=\left(\frac{p_k}{p_j}\right)=\left(\frac{p_i}{p_k}\right)$, $\left(\frac{p_l}{p_i}\right)=\left(\frac{p_l}{p_j}\right)=\left(\frac{p_l}{p_k}\right)$, $\{i, j, k, l\}=\{1, 2, 3, 4\}$.

§3. Proof of Theorems.

We essentially use the following two facts which are well-known:

PROPOSITION 1 (Proposition 6 of Razar [8]). *Let K/k be a finite abelian extension of algebraic number fields. If HNP holds for K/k, then it holds also for all subextensions of K/k.*

PROPOSITION 2 (Theorem 1, 2 of Gerth [2], Theorem 2 of Razar [8]). *Let K/k be a finite abelian extension of algebraic number fields. Then HNP holds for K/k if and only if it holds for every maximal subextension of K/k whose Galois group has prime exponent.*

Case (A): If m is a power of a prime, then HNP holds for $Q(\zeta_m)$, therefore also for $Q(\zeta_m)^+$.

Each maximal subfield of $Q(\zeta_m)^+$ with odd prime exponent is identical with that of $Q(\zeta_m)$. Hence to deal with the other cases, we have only to consider HNP for the elementary abelian 2-extensions in $Q(\zeta_m)^+$.

Case (B): If $m=2^e d$, $e \ge 2$ and d is odd, then $Q(\zeta_m)=Q(\zeta_{2^e})Q(\zeta_d)$. So HNP holds for $Q(\zeta_m)^+$ if and only if it holds for both its maximal subfield of exponent 2 and $Q(\zeta_d)$. Hence, in the same way as in Case (A), we have only to consider the elementary abelian 2-extensions in $Q(\zeta_m)^+$.

In the case where $m=p_1^{a_1}p_2^{a_2}p_3^{a_3}p_4^{a_4}$, if the greatest common divisor d_0 of $\varphi(p_1^{a_1})$, $\varphi(p_2^{a_2})$, $\varphi(p_3^{a_3})$ and $\varphi(p_4^{a_4})$ has an odd prime divisor p, then, by Theorem 10 of Garbanati [1], HNP does not hold for the maximal subfield of exponent p of $\boldsymbol{Q}(\zeta_m)^+$. So d_0 must be a power of 2.

By Propositions 1 and 2, to show that HNP does not hold for $\boldsymbol{Q}(\zeta_m)^+$ in the case where m has five or more distinct prime divisors, it is sufficient to show that it does not hold for the maximal abelian subfield of exponent 2 of $\boldsymbol{Q}(\zeta_m)^+$ in the case where m has exactly five distinct prime divisors.

To determine whether HNP holds for elementary abelian 2-extensions of $\boldsymbol{Q}$, we use Theorem 7 of Gurak [5], which states that HNP holds for an abelian extension of $\boldsymbol{Q}$ with Galois group isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z})^n$ if and only if a certain matrix D has rank $n(n-1)/2$ over $\boldsymbol{Z}/2\boldsymbol{Z}$.

We denote by K_2 the maximal subfield of exponent 2 of $\boldsymbol{Q}(\zeta_m)^+$. We give the list of K_2 and the corresponding D in the following; we omit D if K_2 is cyclic, because HNP holds for such a field K_2. By calculating the rank of D and referring to the case of cyclotomic fields, we obtain our theorems immediately.

- $m=4p_1^{a_1}$: $K_2=\boldsymbol{Q}(\sqrt{p_1})$.
- $m=2^e p_1^{a_1}$, $e\geq 3$: $K_2=\boldsymbol{Q}(\sqrt{2}, \sqrt{p_1})$,

  $p_1\equiv 1(4)$, $D=\begin{bmatrix}\varepsilon_1\\ \varepsilon_1\end{bmatrix}$;

  $p_1\equiv 3(4)$, $D=\begin{bmatrix}1\\ 0\\ 0\\ \varepsilon_1\end{bmatrix}$.
- $m=p_1^{a_1}p_2^{a_2}$:

  $p_1\equiv p_2\equiv 1(4)$, $K_2=\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2})$, $D=\begin{bmatrix}\varepsilon_{1,2}\\ \varepsilon_{1,2}\end{bmatrix}$;

  $p_i\equiv 1$, $p_j\equiv 3(4)$, $K_2=\boldsymbol{Q}(\sqrt{p_i})$;

  $p_1\equiv p_2\equiv 3(4)$, $K_2=\boldsymbol{Q}(\sqrt{p_1 p_2})$.
- $m=4p_1^{a_1}p_2^{a_2}$: $K_2=\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2})$,

  $p_1\equiv p_2\equiv 1(4)$, $D=\begin{bmatrix}\varepsilon_{1,2}\\ \varepsilon_{1,2}\end{bmatrix}$;

  $p_i\equiv 1$, $p_j\equiv 3(4)$, $D=\begin{bmatrix}\varepsilon_i\\ \varepsilon_{i,j}\\ \varepsilon_{i,j}\end{bmatrix}$;

  $p_1\equiv p_2\equiv 3(4)$, $D=\begin{bmatrix}\varepsilon_1+\varepsilon_2\\ \varepsilon_{1,2}\\ 1+\varepsilon_{1,2}\end{bmatrix}$.
- $m=p_1^{a_1}p_2^{a_2}p_3^{a_3}$:

  $p_1\equiv p_2\equiv p_3\equiv 1(4)$, $K_2=\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$, $D=\begin{bmatrix}\varepsilon_{1,2} & \varepsilon_{1,3} & 0\\ \varepsilon_{1,2} & 0 & \varepsilon_{2,3}\\ 0 & \varepsilon_{1,3} & \varepsilon_{2,3}\end{bmatrix}$;

$$p_i \equiv 1,\ p_j \equiv p_k \equiv 3(4),\ K_2 = \boldsymbol{Q}(\sqrt{p_i}, \sqrt{p_j p_k}),\ D = \begin{bmatrix} \varepsilon_{i,j} + \varepsilon_{i,k} \\ \varepsilon_{i,j} \\ \varepsilon_{i,k} \end{bmatrix};$$

$$p_i \equiv p_j \equiv 1,\ p_k \equiv 3(4),\ K_2 = \boldsymbol{Q}(\sqrt{p_i}, \sqrt{p_j}),\ D = \begin{bmatrix} \varepsilon_{i,j} \\ \varepsilon_{i,j} \end{bmatrix};$$

$$p_1 \equiv p_2 \equiv p_3 \equiv 3(4),\ K_2 = \boldsymbol{Q}(\sqrt{p_1 p_2}, \sqrt{p_1 p_3}),\ D = \begin{bmatrix} \varepsilon_{1,2} + \varepsilon_{1,3} \\ 1 + \varepsilon_{1,2} + \varepsilon_{2,3} \\ \varepsilon_{1,3} + \varepsilon_{2,3} \end{bmatrix}.$$

- $m = 2^e p_1^{a_1} p_2^{a_2},\ e \geq 3 : K_2 = \boldsymbol{Q}(\sqrt{2}, \sqrt{p_1}, \sqrt{p_2}),$

$$p_1 \equiv 1(8),\ p_2 \equiv 1(8),\ D = \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix};$$

$$p_i \equiv 1(8),\ p_j \equiv 5(8),\ D = \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix};$$

$$p_1 \equiv 5(8),\ p_2 \equiv 5(8),\ D = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & \varepsilon_{1,2} \\ 0 & 1 & \varepsilon_{1,2} \end{bmatrix};$$

$$p_1 \equiv 3(8),\ p_2 \equiv 3(8),\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \varepsilon_{1,2} & 1 \\ 0 & 1 + \varepsilon_{1,2} & 1 \end{bmatrix};$$

$$p_1 \equiv 7(8),\ p_2 \equiv 7(8),\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \varepsilon_{1,2} & 0 \\ 0 & 1 + \varepsilon_{1,2} & 0 \end{bmatrix};$$

$$p_i \equiv 1(8),\ p_j \equiv 3(4),\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon_{i,j} & \varepsilon_{i,j} & 0 \\ \varepsilon_j & \varepsilon_{i,j} & 0 \end{bmatrix};$$

$$p_i \equiv 3(4),\ p_j \equiv \pm 3(8),\ D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \\ * & * & * \end{bmatrix}.$$

- $m = 4 p_1^{a_1} p_2^{a_2} p_3^{a_3} : K_2 = \boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}),$

$$p_1 \equiv p_2 \equiv p_3 \equiv 1(4),\ D = \begin{bmatrix} \varepsilon_{1,2} & \varepsilon_{1,3} & 0 \\ \varepsilon_{1,2} & 0 & \varepsilon_{2,3} \\ 0 & \varepsilon_{1,3} & \varepsilon_{2,3} \end{bmatrix};$$

$$p_i\equiv 3,\ p_j\equiv p_k\equiv 1(4),\ D=\begin{bmatrix} \varepsilon_j & \varepsilon_k & 0 \\ \varepsilon_{i,j} & \varepsilon_{i,k} & 0 \\ \varepsilon_{i,j} & 0 & \varepsilon_{j,k} \\ 0 & \varepsilon_{i,k} & \varepsilon_{j,k} \end{bmatrix};$$

$$p_i\equiv p_j\equiv 3,\ p_k\equiv 1(4),\ D=\begin{bmatrix} \varepsilon_i+\varepsilon_j & \varepsilon_k & 0 \\ \varepsilon_{i,j} & \varepsilon_{i,k} & \varepsilon_{i,k} \\ 1+\varepsilon_{i,j} & 0 & \varepsilon_{j,k} \\ 0 & \varepsilon_{i,k} & \varepsilon_{i,k}+\varepsilon_{j,k} \end{bmatrix};$$

$$p_1\equiv p_2\equiv p_3\equiv 3(4),\ D=\begin{bmatrix} \varepsilon_1+\varepsilon_2 & \varepsilon_1+\varepsilon_3 & 0 \\ \varepsilon_{1,2} & \varepsilon_{1,3} & \varepsilon_{1,2}+\varepsilon_{1,3} \\ 1+\varepsilon_{1,2} & 0 & 1+\varepsilon_{1,2}+\varepsilon_{2,3} \\ 0 & 1+\varepsilon_{1,3} & \varepsilon_{1,3}+\varepsilon_{2,3} \end{bmatrix}.$$

- $m=2^e p_1^{a_1} p_2^{a_2} p_3^{a_3}$, $e\geq 3$: $K_2=\boldsymbol{Q}(\sqrt{2}, \sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$,
  $p_1\equiv p_2\equiv p_3\equiv 1(4)$, D is a 4×6 matrix;

$$p_i\equiv 3,\ p_j\equiv p_k\equiv 1(4),\ D=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_j & \varepsilon_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_j & \varepsilon_k & 0 \\ \varepsilon_i & \varepsilon_{i,j} & \varepsilon_{i,k} & 0 & 0 & 0 \\ 0 & \varepsilon_{i,j} & 0 & \varepsilon_j & 0 & \varepsilon_{j,k} \\ 0 & 0 & \varepsilon_{i,k} & 0 & \varepsilon_k & \varepsilon_{j,k} \end{bmatrix};$$

$$p_i\equiv p_j\equiv 3,\ p_k\equiv 1(4),\ D=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_i+\varepsilon_j & \varepsilon_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_i+\varepsilon_j & \varepsilon_k & 0 \\ \varepsilon_i & \varepsilon_{i,j} & \varepsilon_{i,k} & \varepsilon_j & 0 & \varepsilon_{i,k} \\ 0 & 1+\varepsilon_{i,j} & 0 & \varepsilon_j & 0 & \varepsilon_{i,k} \\ 0 & 0 & \varepsilon_{i,k} & 0 & \varepsilon_k & \varepsilon_{i,k}+\varepsilon_{j,k} \end{bmatrix};$$

$$p_1\equiv p_2\equiv p_3\equiv 3(4),\ D=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_1+\varepsilon_2 & \varepsilon_1+\varepsilon_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_1+\varepsilon_2 & \varepsilon_1+\varepsilon_3 & 0 \\ \varepsilon_1 & \varepsilon_{1,2} & \varepsilon_{1,3} & \varepsilon_1 & \varepsilon_1 & \varepsilon_{1,2}+\varepsilon_{1,3} \\ 0 & 1+\varepsilon_{1,2} & 0 & \varepsilon_2 & 0 & 1+\varepsilon_{1,2}+\varepsilon_{2,3} \\ 0 & 0 & 1+\varepsilon_{1,3} & 0 & \varepsilon_3 & \varepsilon_{1,3}+\varepsilon_{2,3} \end{bmatrix}.$$

- $m=p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$:
  $p_1\equiv p_2\equiv p_3\equiv p_4\equiv 1(4)$, $K_2=\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{p_4})$, D is a 4×6 matrix;
  $p_i\equiv 3$, $p_j\equiv p_k\equiv p_l\equiv 1(4)$, $K_2=\boldsymbol{Q}(\sqrt{p_i}, \sqrt{p_j}, \sqrt{p_k})$,

$$D=\begin{bmatrix} \varepsilon_{i,j} & \varepsilon_{i,k} & 0 \\ \varepsilon_{i,j} & 0 & \varepsilon_{j,k} \\ 0 & \varepsilon_{i,k} & \varepsilon_{j,k} \end{bmatrix};$$

$p_i\equiv p_j\equiv 3$, $p_k\equiv p_l\equiv 1(4)$, $K_2=\boldsymbol{Q}(\sqrt{p_i p_j}, \sqrt{p_k}, \sqrt{p_l})$,

$$D=\begin{bmatrix} \varepsilon_{i,k} & \varepsilon_{i,l} & 0 \\ \varepsilon_{j,k} & \varepsilon_{j,l} & 0 \\ \varepsilon_{i,k}+\varepsilon_{j,k} & 0 & \varepsilon_{k,l} \\ 0 & \varepsilon_{i,l}+\varepsilon_{j,l} & \varepsilon_{k,l} \end{bmatrix};$$

$p_i \equiv p_j \equiv p_k \equiv 3,\ p_l \equiv 1(4),\ K_2 = \mathbf{Q}(\sqrt{p_i p_j}, \sqrt{p_i p_k}, \sqrt{p_l}),$

$$D = \begin{bmatrix} \varepsilon_{i,j} + \varepsilon_{i,k} & \varepsilon_{i,l} & \varepsilon_{i,l} \\ \varepsilon_{i,j} + \varepsilon_{j,k} + 1 & \varepsilon_{j,l} & 0 \\ \varepsilon_{i,k} + \varepsilon_{j,k} & 0 & \varepsilon_{k,l} \\ 0 & \varepsilon_{i,l} + \varepsilon_{j,l} & \varepsilon_{i,l} + \varepsilon_{k,l} \end{bmatrix};$$

$p_1 \equiv p_2 \equiv p_3 \equiv p_4 \equiv 3(4)$: $K_2 = \mathbf{Q}(\sqrt{p_1 p_2}, \sqrt{p_1 p_3}, \sqrt{p_1 p_4}),$

$$D = \begin{bmatrix} \varepsilon_{1,2} + \varepsilon_{1,3} & \varepsilon_{1,2} + \varepsilon_{1,4} & \varepsilon_{1,3} + \varepsilon_{1,4} \\ 1 + \varepsilon_{1,2} + \varepsilon_{2,3} & 1 + \varepsilon_{1,2} + \varepsilon_{2,4} & 0 \\ \varepsilon_{1,3} + \varepsilon_{2,3} & 0 & 1 + \varepsilon_{1,3} + \varepsilon_{3,4} \\ 0 & \varepsilon_{1,4} + \varepsilon_{2,4} & \varepsilon_{1,4} + \varepsilon_{3,4} \end{bmatrix}.$$

- $m = 4p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$: $K_2 = \mathbf{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{p_4}),$
  $p_i \equiv 1 \pmod 4$ for all i, D is a 4×6 matrix;
  $p_i \equiv 3 \pmod 4$ for some i, D is a 5×6 matrix.
- $m = 2^e p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$, $e \geq 3$: $K_2 = \mathbf{Q}(\sqrt{2}, \sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{p_4}),$
  $p_i \equiv 1 \pmod 4$ for all i, D is a 5×10 matrix;
  $p_i \equiv 3 \pmod 4$ for some i, D is a 7×10 matrix.
- $m = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4} p_5^{a_5}$:
  $p_1 \equiv p_2 \equiv p_3 \equiv p_4 \equiv p_5 \equiv 1(4)$, $K_2 = \mathbf{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}, \sqrt{p_4}, \sqrt{p_5}),$
  D is a 5×10 matrix;
  $p_i \equiv p_j \equiv p_k \equiv p_l \equiv 1,\ p_m \equiv 3(4)$, $K_2 \equiv \mathbf{Q}(\sqrt{p_i}, \sqrt{p_j}, \sqrt{p_k}, \sqrt{p_l}),$
  D is a 4×6 matrix;
  $p_i \equiv p_j \equiv p_k \equiv 1,\ p_l \equiv p_m \equiv 3(4)$, $K_2 = \mathbf{Q}(\sqrt{p_i}, \sqrt{p_j}, \sqrt{p_k}, \sqrt{p_l p_m}),$
  D is a 5×6 matrix;
  $p_i \equiv p_j \equiv 1,\ p_k \equiv p_l \equiv p_m \equiv 3(4)$, $K_2 = \mathbf{Q}(\sqrt{p_i}, \sqrt{p_j}, \sqrt{p_k p_l}, \sqrt{p_l p_m}),$
  D is a 5×6 matrix;
  $p_i \equiv 1,\ p_j \equiv p_k \equiv p_l \equiv p_m \equiv 3(4)$, $K_2 = \mathbf{Q}(\sqrt{p_i}, \sqrt{p_j p_k}, \sqrt{p_k p_l}, \sqrt{p_l p_m}),$
  D is a 5×6 matrix;
  $p_i \equiv p_j \equiv p_k \equiv p_l \equiv p_m \equiv 3(4)$, $K_2 = \mathbf{Q}(\sqrt{p_i p_j}, \sqrt{p_j p_k}, \sqrt{p_k p_l}, \sqrt{p_l p_m}),$
  D is a 5×6 matrix.

§4. Numerical results.

We give all $m \leq 1200$ such that HNP fails to hold for $\mathbf{Q}(\zeta_m)$. If HNP fails to hold for $\mathbf{Q}(\zeta_m)^+$, we put m in boldface.

(1) $m = 4p_1^{a_1}$:
68, 164, 292, 356, 388, 452, 548, 772, 932, 964, 1028, 1124, 1156.

(2) $m = 2^e p_1^{a_1}$, $e \geq 3$:
56, 112, **136**, 184, 224, 248, **272**, **328**, 368, 376, 392, 448, 496, **544**, 568, **584**, 632, **656**, **712**, 736, 752, **776**, 784, 824, 896, **904**, 992, 1016, **1088**, **1096**, 1136, **1168**.

(3) $m=p_1^{a_1}p_2^{a_2}$:

39, 55, 95, 111, 117, **145**, 155, 183, 203, **205**, 219, **221**, 259, 275, 291, 295, 299, **305**, 323, 327, 333, 351, 355, 371, **377**, 395, 407, **445**, 471, 475, **505**, 507, 543, **545**, 549, 559, 579, 583, 605, 655, 657, 667, 687, **689**, 695, 723, **725**, 731, **745**, 755, 763, 775, 791, **793**, 799, 831, 873, 895, **901**, **905**, 939, 943, 955, 959, 979, 981, 995, 999, 1003, 1011, **1025**, 1027, 1043, 1047, 1053, 1055, 1067, 1119, 1139, **1145**, 1159, 1191, 1195.

(4) $m=4p_1^{a_1}p_2^{a_2}$:

156, 220, 380, 444, 468, **580**, 620, 732, 812, **820**, 876, **884**, 1036, 1100, 1164, 1180, 1196.

(5) $m=p_1^{a_1}p_2^{a_2}p_3^{a_3}$:

165, 285, **435**, 465, 495, 609, **615**, **663**, 777, 825, 855, 885, 897, **915**, 969, 1015, 1065, **1105**, 1113, 1131, 1185.

(6) $m=2^e p_1^{a_1}p_2^{a_2}$, $e\geq 3$:

120, 168, 240, 264, 280, 312, 336, 360, 408, 440, 456, 480, 504, 520, 528, 552, 560, 600, 616, 624, 672, 680, 696, 720, 728, 744, 760, 792, 816, 880, 888, 912, 920, 936, 952, 960, 984, 1008, 1032, 1040, 1056, 1064, 1080, 1104, 1120, 1128, 1144, **1160**, 1176, 1200.

(7) $m=4p_1^{a_1}p_2^{a_2}p_3^{a_3}$:

420, 660, 780, 924, 1020, 1092, 1140.

(8) $m=2^e p_1^{a_1}p_2^{a_2}p_3^{a_3}$, $e\geq 3$:

840.

(9) $m=p_1^{a_1}p_2^{a_2}p_3^{a_3}p_4^{a_4}$:

1155.

§5. Remarks.

(1) To determine that HNP holds for a biquadratic field, we can also use Corollary 5.3 of Gurak [4], Example 1 of Razar [8] or Corollary 7 of Garbanati [1].

(2) For a triquadratic field K, it follows from Theorem 2 of Horie [6] that HNP holds for K if and only if it holds for every biquadratic subfield of K. Hence using this and biquadratic case, we can also determine whether HNP holds for K or not.

Let p_1, p_2, p_3 be distinct primes congruent to 1 mod 4. It is already shown (cf. Corollary 8 of Garbanati [1]) that HNP does not hold for $L=\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$. Hence, by Proposition 1, HNP does not hold either for any abelian field containing L, say $\boldsymbol{Q}(\sqrt{2}, \sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$.

(3) Jehne's paper [7] contains an error, in which he states that HNP holds for $K=\boldsymbol{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$, where $p_1\equiv -1 \pmod 8)$, $p_2\equiv 3 \pmod 8$, $p_3\equiv 5 \pmod 8$ are primes.

He uses the theorem of Scholz-Tate (cf. [7] P. 221 or [9] P. 198), by which HNP holds for K if there exists a prime which is not decomposed in K; he states that 2 is such a prime, hence HNP holds for K. But it is easily seen that 2 is always decomposed in K. So HNP does not always hold. Our calculation in the case $m=4p_1^{a_1}p_2^{a_2}p_3^{a_3}$ shows that HNP holds for K if and only if $\left(\frac{p_3}{p_1}\right)=-1$ or $\left(\frac{p_3}{p_2}\right)=-1$.

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