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Defining Ideals of Complete Intersection Monoid Rings

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In this note, we will extend Delorme's result about monomial curves [1] to $\boldsymbol{Z}^n$-graded rings. To do this, we will define an ideal $I(V)$ associated with a submodule V of $\boldsymbol{Z}^N$. It is generated by polynomials associated with vectors of V (see §1). And we have various examples of such ideals, e.g., defining ideals of monomial curves, that of $\boldsymbol{Z}^n$-graded ring, and an ideal generated by 2×2 minors of a matrix. In general, $\operatorname{ht} I(V)=\operatorname{rank} V$, $I(V)$ is not necessarily prime, and we will give a condition that $I(V)$ is prime (Proposition 1.3).

In section 2, we will give the condition that $I(V)$ is a complete intersection ideal when V is contained in the kernel of a map $\boldsymbol{Z}^p\to\boldsymbol{Z}^q$ consisting of positive integers (Theorem 2.4). And we give a proof of the Delorme's result that any complete intersection monomial curve in $\boldsymbol{A}^r$ is induced by a complete intersection monomial curve in $\boldsymbol{A}^{r-1}$ (Corollary 2.5). We also show that if $\operatorname{rank} V<N-1$ and if $I(V)$ is a complete intersection, it is generated by a part of a minimal generating system of a complete intersection homogeneous ideal of height $N-1$ of the form $I(V')$ (Theorem 2.10).

1. Definitions and preliminaries.

Let $A=k[X_1,\cdots,X_N]$ be a polynominal ring over a field k. For $v\in\boldsymbol{Z}^N$, we denote the i-th entry of v by $\sigma_i(v)$, and put

$$F_+(v)=\prod_{\sigma_i(v)>0} X_i^{\sigma_i(v)}$$

$$F_-(v)=\prod_{\sigma_i(v)<0} X_i^{-\sigma_i(v)}$$

$$F(v)=F_-(v)-F_+(v)\,.$$

(If $\sigma_i(v)<0$ for all i, we put $F_+(v)=1$. And if $\sigma_i(v)>0$ for all i, we put $F_-(v)=1$.) For a submodule V of rank r of $\boldsymbol{Z}^N$ with $0<r<N$, we define an ideal $I(V)$ of A generated

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by $F(v)$ for all $v \in V$. Note that V is torsion-free of rank r hence isomorphic to $\boldsymbol{Z}^r$.

Let $B=k[X_1^{\pm 1}, \cdots, X_N^{\pm 1}]$ and $V=\langle v_1, \cdots, v_r\rangle$ (this means that V is generated by $v_1, \cdots, v_r$). We claim that an ideal $I(V)B \cong I(V) \otimes B$ in B is generated by $F(v_j)$ for $1 \leq j \leq r$. For, since any vector in V is a linear combination of v_j, it is sufficient to prove

$$F(dw) \in (F(w)), \qquad F(w_1+w_2) \in (F(w_1), F(w_2)).$$

The first assertion is clear. And

$$\begin{aligned}
&1-F_-(w_1+w_2)^{-1}F_+(w_1+w_2)\\
&=1-F_-(w_1)^{-1}F_+(w_1)F_-(w_2)^{-1}F_+(w_2)\\
&=(1-F_-(w_1)^{-1}F_+(w_1))+F_-(w_1)^{-1}F_+(w_1)(1-F_-(w_2)^{-1}F_+(w_2))\\
&\in(F(w_1), F(w_2)).
\end{aligned}$$

Hence the second assertion is proved. And we notice that, if $F(v_1), \cdots, F(v_s)$ generate $I(V)$, $v_1, \cdots, v_s$ generate V.

Next, we have $\operatorname{rank}\operatorname{Coker}(V \hookrightarrow \boldsymbol{Z}^N)=r'=N-r$. Hence it is of the form $\boldsymbol{Z}^{r'} \oplus T$ where T is a torsion module. Then we have the following commutative diagram;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & \boldsymbol{Z}^N & \longrightarrow & \boldsymbol{Z}^{r'} \oplus T & \longrightarrow & 0\\
 & & \downarrow & & \| & & \downarrow{\scriptstyle pr} & & \\
0 & \longrightarrow & V' & \longrightarrow & \boldsymbol{Z}^N & \xrightarrow{\ \phi\ } & \boldsymbol{Z}^{r'} & \longrightarrow & 0,
\end{array}$$

where $V'=\operatorname{Ker}\phi$.

Let $\phi=(n_{pq})$ and ρ a homomorphism from B to $k[t_1^{\pm 1}, \cdots, t_{r'}^{\pm 1}]$ which sends X_i to $\prod t_p^{n_{pi}}$ for each i. Then $F(v)$ is contained in $\operatorname{Ker}\rho$ for any $v \in V$. For,

$$\begin{aligned}
\rho(F(v))&=\rho(F_-(v)(1-F_-(v)^{-1}F_+(v)))\\
&=\rho(F_-(v))\rho(1-\prod X_i^{qi(v)})=0.
\end{aligned}$$

We can regard B as a group algebra $k[\boldsymbol{Z}^N]$. Then $I(V)B$ is the kernel of the group algebra homomorphism $B \to k[\boldsymbol{Z}^N/V]$, which is induced from the group homomorphism $\boldsymbol{Z}^N \to \boldsymbol{Z}^N/V$. Since $\dim k[\boldsymbol{Z}^N/V]=\operatorname{rank}\boldsymbol{Z}^N/V=N-r$, we have $\operatorname{ht} I(V)B=r$. Hence

Proposition 1.1. *Let $V \subset \boldsymbol{Z}^N$ be a submodule of rank r, $0<r<N$. Then* $\operatorname{ht} I(V)=r$.

For later use, we prove a lemma;

Lemma 1.2. *Let $V \subset \boldsymbol{Z}^N$ a submodule of rank r where $0<r<N$. If $I(V)+(X_1)$ is a proper ideal in A, it is of height $r+1$.*

Proof. Since $I(V)B \cap A=I(V)$ and since X_1 is a unit in B, X_1 is not a zero divisor on $A/I(V)$. Hence the assertion is clear. Q.E.D.

By the definition of $I(V)$, we have

PROPOSITION 1.3. *Let V be a submodule of $\boldsymbol{Z}^N$ of rank r. Then $I(V)$ is prime if and only if there is a surjective homomorphism $\phi: \boldsymbol{Z}^N \to \boldsymbol{Z}^{N-r}$ with $V = \mathrm{Ker}\,\phi$.*

2. Complete intersection ideals.

For $\phi = (m_{ij}) \in \mathrm{Hom}(\boldsymbol{Z}^N, \boldsymbol{Z}^{N'})$, we say that ϕ is *positive* if $m_{ij} \geq 0$ for any i, j and $\sum_i m_{ij} > 0$ for each j. For $v \in \boldsymbol{Z}^N$, we say that v is *usual* if there are i, i' with $\sigma_i(v) > 0$ and $\sigma_{i'}(v) < 0$.

In this section, we assume that V is contained in $\mathrm{Ker}\,\phi$ where $\phi: \boldsymbol{Z}^N \to \boldsymbol{Z}^{r'}$ is a positive homomorphism. Then $I(V)$ is a homogeneous ideal in a positively multigraded ring $A = k[X_1, \cdots, X_N]$. And there is a minimal generating system of $I(V)$ consisting of polynomials of the form $F(v)$ where v is a usual vector.

We say that the signatures of z and z' $(z, z' \in \boldsymbol{Z})$ are the same if $zz' \geq 0$. For $v_1, \cdots, v_l \in V$, we consider the condition

(*) for any s $(2 \leq s \leq l)$, for any numbers $i_1, \cdots, i_s$ and $j_1, \cdots, j_s$, there exists m such that the signatures of $\sigma_{i_l}(v_{j_m})$ are the same for $l = 1, \cdots, s$.

PROPOSITION 2.1. *Let V be a submodule of $\boldsymbol{Z}^N$ of rank r and assume that there are $v_1, \cdots, v_r \in V$ such that $I(V)$ is generated by $F(v_1), \cdots, F(v_r)$. Then $v_1, \cdots, v_r$ satisfy (*).*

PROOF. We fix s. By renumbering, if necessary, we may assume $i_l = j_l = l$ for $l = 1, \cdots, s$. Assume that for any m $(1 \leq m \leq s)$, there exist i_m, i'_m such that $\sigma_{i_m}(v_m) > 0$, $\sigma_{i'_m}(v_m) < 0$. Then $F(v_m)$ is contained in the ideal $(X_{i_m}, X_{i'_m})$. Consider the ideal $J = I(V) + (X_1, \cdots, X_s)$. By Lemma 1.2, we have $\mathrm{ht}(I(V) + (X_1)) = r + 1$. Since J contains it, we have $\mathrm{ht}\,J \geq r + 1$.

On the other hand, $F(v_m)$ is contained in the ideal $(X_1, \cdots, X_s)$ for any m. Hence $\mu(J) \leq r + s - s = r$. This contradicts $\mathrm{ht}\,J \geq r + 1$. Q.E.D.

In section 1, we proved that in a Laurent polynomial ring, $F(v_1 + v_2)$ is contained in the ideal generated by $F(v_1)$ and $F(v_2)$. But in a polynomial ring, it is not always contained in $(F(v_1), F(v_2))$. In [2], the following lemma is proved.

LEMMA 2.2 ([2, Lemma 1.2]). *Let $v, v_1, v_2 \in V$.*

(1) *For any $d \in \boldsymbol{Z}$, $F(dv)$ is contained in the ideal $(F(v))$.*

(2) *$F(v_1 + v_2) \in (F(v_1), F(v_2))$, if there is no pair (i, i') such that*

$$\sigma_i(v_1) < 0\,, \ \sigma_i(v_2) > 0\,, \quad \text{and that} \quad \sigma_{i'}(v_1) > 0\,, \ \sigma_{i'}(v_2) < 0\,.$$

PROPOSITION 2.3. *Assume $v_1, \cdots, v_l \in V$ satisfy (*) and let $V' = \langle v_1, \cdots, v_l \rangle$. Then $I(V')$ is generated by $(F(v_j))_{1 \leq j \leq l}$.*

PROOF. We prove the assertion by induction on l. It is obvious if $l = 1$. Assume $l > 1$. Let $J = (F(v_j))_{1 \leq j \leq l}$. For $w = \sum d_j v_j \in V$, we claim that $F(w)$ is contained in J. By induction hypothesis, if some $d_j = 0$, $F(w)$ is contained in J. So, assume $d_j \neq 0$ for all j.

If necessary, replace v_j by $-v_j$, then we may assume $d_j>0$ for all j.

If $F(w)$ is contained in the ideal $(F(v_j), F(w-d_jv_j))$ for some j, it is contained in J by induction hypothesis. Hence we also assume $F(w)$ is not contained in $(F(v_j), F(w-d_jv_j))$ for any j.

Since $F(w)\notin(F(v_1), F(w-d_1v_1))$, there are i_1, i_2 such that

$$\sigma_{i_1}(v_1)>0\,, \qquad \sigma_{i_2}(v_1)<0\,,$$
$$\sigma_{i_1}(w-d_1v_1)<0\,, \qquad \sigma_{i_2}(w-d_1v_1)>0\,.$$

Say $i_1=1$, $i_2=2$. Since $\sigma_2(w-d_1v_1)>0$, there is j such that $\sigma_2(v_j)>0$. For, $w-d_1v_1=d_2v_2+\cdots+d_lv_l$ and $d_j>0$. Say $j=2$.

Since $F(w)\notin(F(v_2), F(w-d_2v_2))$, there is i such that

$$\sigma_i(v_2)<0\,, \qquad \sigma_i(w-d_2v_2)>0\,.$$

If $i=1$, v_1, v_2 do not satisfy (*), a contradiction. Hence $i>2$. Say $i=3$. As the same argument as before, there is $j\neq 2$ such that $\sigma_3(v_j)>0$. If $j=1$, v_2, v_3 do not satisfy (*), a contradiction. Hence $j>2$. Say $j=3$.

Repeating this process l times, we have

$$\sigma_i(v_i)>0\,, \quad \sigma_{i+1}(v_i)<0\,, \qquad \text{for} \quad i=1,\cdots,l-1\,.$$

Then, since $F(w)\notin(F(v_l), F(w-d_lv_l))$, there is $j\neq l$ such that $\sigma_l(v_j)>0$. Then $v_1,\cdots,v_l$ do not satisfy (*), a contradiction. Q.E.D.

From Proposition 2.1 and Proposition 2.3, we have

THEOREM 2.4. *$I(V)$ is a complete intersection if and only if there exist $v_1,\cdots,v_r$ satisfying (*) which generate V.*

In the case of rank $V=N-1$, we have

COROLLARY 2.5 (Delorme [1, Lemma 6]). *A complete intersection monomial curve is obtained from unimodular vectors of less length than N, which define complete intersection monomial curves, respectively.*

We will give a proof: Assume $V=\mathrm{Ker}\,u$ where $u=(n_1,\cdots,n_N)$ is a unimodular vector of length N whose entries are positive integers. Then $I(V)$ is the defining ideal of a monomial curve. Since rank $V=N-1$, for each i, there is v with $\sigma_i(v)<0$ and $\sigma_{i'}(v)\geq 0$ if $i\neq i'$. Hence $I(V)$ contains polynomials of the form $X_i^{-\sigma_i(v)}-\prod_{i'\neq i}X_{i'}^{\sigma_{i'}(v)}$. Then, if $I(V)$ is a complete intersection, its generating system must contain a polynomial of the form $X_i^{\alpha_i}-X_{i'}^{\alpha_{i'}}$. Hence we may assume $\sigma_1(v_1)=-\alpha_1$, $\sigma_2(v_1)=\alpha_2$ and $\sigma_{i''}(v_1)=0$ otherwise.

Now let d be the g.c.d. of n_1, n_2. Then $\alpha_1=d^{-1}n_2$ and $\alpha_2=d^{-1}n_1$. Put $u'=(d, n_3,\cdots,n_N)$ be a positive unimodular vector of length $N-1$ and $V'=\mathrm{Ker}\,u'$. And consider a map $\phi\colon Z^N\to Z^{N-1}$ which sends e_1 to α_2e_1, e_2 to α_1e_1 and e_i to e_{i-1} for

$i \geq 3$. Then $\phi(V)=V'$ and $\phi(v_2), \cdots, \phi(v_{N-1})$ satisfy (*). Hence $I(V')$ is a complete intersection by Theorem 2.4. Therefore an ideal $I(V)$ is obtained from unimodular vectors u' and $(d^{-1}n_1, d^{-1}n_2)$, which define complete intersection monomial curves, respectively. Q.E.D.

Now we investigate the case $r<N-1$.

LEMMA 2.6. *Let $v_1, \cdots, v_l \in \boldsymbol{Z}^N$ be usual vectors satisfying (*) and assume $N \geq 3$. If $l<N-1$, there are i, i' with $\sigma_i(v_j)\sigma_{i'}(v_j) \geq 0$ for any j.*

PROOF. We prove the lemma by induction on N. If $N=3$, we have $l=1$ and the assertion is clear. In general, we assume that $\sigma_1(v_1)>0$ and $\sigma_N(v_1)<0$. For each j, let v'_j be the image of v_j by the map $\boldsymbol{Z}^N \to \boldsymbol{Z}^{N-1}$ which sends e_i to e_i $(i=1, \cdots, N-1)$ and e_N to e_1. Then $v'_2, \cdots, v'_l$ satisfy (*). Hence, by the induction hypothesis, there are i, i' with $\sigma_i(v'_j)\sigma_{i'}(v'_j) \geq 0$ for $j \geq 2$. And $\sigma_i(v_j)\sigma_{i'}(v_j) \geq 0$ for $j \geq 2$.

If $i=1$ and $\sigma_{i'}(v_1) \geq 0$, then $\sigma_1(v_1)\sigma_{i'}(v_1) \geq 0$ and the assertion is proved. If $i=1$ and $\sigma_{i'}(v_1)<0$, then $\sigma_N(v_1)\sigma_{i'}(v_1)>0$. And $\sigma_1(v_j)\sigma_N(v_j) \geq 0$ for $j \geq 2$, since $v_1, \cdots, v_l$ satisfy (*). Thus $\sigma_N(v_j)\sigma_{i'}(v_j) \geq 0$ for $j \geq 2$, and we obtain the result.

If $i, i'>1$ and $\sigma_i(v_1)\sigma_{i'}(v_1) \geq 0$, the assertion is clear. Assume $\sigma_i(v_1)>0$ and $\sigma_{i'}(v_1)<0$. Then $\sigma_1(v_j)\sigma_{i'}(v_j) \geq 0$ for $j \geq 2$ since v_1, v_l satisfy (*) and $\sigma_1(v_1)>0$, $\sigma_{i'}(v_1)<0$. Thus $\sigma_1(v_j)\sigma_{i'}(v_j) \geq 0$ for any j, since $\sigma_i(v_j)\sigma_{i'}(v_j) \geq 0$ for $j \geq 2$. This completes the proof. Q.E.D.

PROPOSITION 2.7. *Let $v_1, \cdots, v_r \in \boldsymbol{Z}^N$ be usual vectors satisfying (*). If $r<N-1$, there are usual vectors $v_{r+1}, \cdots, v_{N-1} \in \boldsymbol{Z}^N$ such that $v_1, \cdots, v_{N-1}$ satisfy (*).*

PROOF. By Lemma 2.6, there are i, i' with $\sigma_i(v_j)\sigma_{i'}(v_j) \geq 0$ for any j. We choose a vector v_{r+1} with $\sigma_i(v_{r+1})\sigma_{i'}(v_{r+1})<0$ and $\sigma_{i''}(v_{r+1})=0$ if $i'' \neq i, i'$. Then $v_1, \cdots, v_{r+1}$ satisfy (*). We can repeat this process $N-r-1$ times. Q.E.D.

LEMMA 2.8. *Let $v \in \boldsymbol{Z}^N$ be a usual vector with $\sigma_i(v)=0$ if $i>s$. Then there are a positive surjective homomorphism $\psi: \boldsymbol{Z}^N \to \boldsymbol{Z}^{N-1}$ with $\psi(v)=0$ and $\psi(e_i)=e_{i-1}$ if $i>s$.*

PROOF. Let d be the g.c.d. of $\sigma_1(v), \cdots, \sigma_s(v)$. Since v is usual, there is a positive matrix $M \in \mathrm{GL}_s(\boldsymbol{Z})$ with $M(d^{-1}v)=e_1$. Then M induces a positive surjective homomorphism $\beta: \boldsymbol{Z}^s \to \boldsymbol{Z}^{s-1}$ with $\beta(v)=0$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{Z} & \xrightarrow{d^{-1}v} & \boldsymbol{Z}^s & \xrightarrow{\beta} & \boldsymbol{Z}^{s-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow M & & \| & & \\
0 & \longrightarrow & \boldsymbol{Z} & \xrightarrow{e_1} & \boldsymbol{Z}^s & \longrightarrow & \boldsymbol{Z}^{s-1} & \longrightarrow & 0 .
\end{array}$$

Now $\psi=\begin{pmatrix} \beta & 0 \\ 0 & E_{N-s} \end{pmatrix}$ satisfies the condition of the lemma. Q.E.D.

PROPOSITION 2.9. *Let* $v_1, \cdots, v_r \in Z^N$ *be usual vectors satisfying* $(*)$ *and* $V = \langle v_1, \cdots, v_r \rangle$. *Then* rank $V = r$ *and* V *is contained in the kernel of a positive surjective homomorphism.*

PROOF. We will prove the assertion by induction on r. It is clear, if $r=1$. Assume $r>1$. Since $v_1, \cdots, v_r$ satisfy $(*)$, there is some j such that for each $j' \neq j$, we have $\sigma_i(v_{j'})\sigma_{i'}(v_{j'}) \geq 0$, for i, i' with $\sigma_i(v_j)\sigma_{i'}(v_j) \neq 0$. Say $j=1$ and assume $\sigma_i(v_1) \neq 0$ if $i \leq s$ and $\sigma_i(v_1)=0$ if $i>s$. Note $s<N$. Applying Lemma 2.8 to v_1, there is $\psi: Z^N \to Z^{N-1}$ a positive surjective homomorphism with $\psi(v_1)=0$ and $\psi(e_i)=e_{i-1}$ if $i>s$. Then $\psi(v_2), \cdots, \psi(v_r)$ satisfy $(*)$, hence by induction hypothesis, they form a space of rank $r-1$ and contained in the kernel of a positive surjective homomorphism $\gamma: Z^{N-1} \to Z^{r'}$. If $d_1v_1 + \cdots + d_rv_r = 0$, then $d_2\psi(v_2) + \cdots + d_r\psi(v_r) = 0$ and $d_2 = \cdots = d_r = 0$, hence $d_1 = 0$. Thus rank $V = r$. And V is contained in the kernel of a positive surjective homomorphism $\gamma\psi$. Q.E.D.

From Proposition 2.7 and Proposition 2.9, we obtain

THEOREM 2.10. *Let* V *be a submodule of* Z^N *of rank* r *with* $r<N-1$. *Assume that* V *is contained in the kernel of a positive surjective homomorphism. If* $I(V)$ *is a complete intersection and generated by* $F(v_1), \cdots, F(v_r)$, *there are* $F(v_{r+1}), \cdots, F(v_{N-1})$ *such that* $F(v_j)$*'s generate a complete intersection ideal of the form* $I(V')$ *of height* $N-1$, *which is homogeneous in the positive graded ring* A.

Hence if $I(V)$ *is a complete intersection, it is generated by a part of a minimal generating system of a complete intersection homogeneous ideal of height* $N-1$.

Finally, we remark that we cannot take V' so that $I(V')$ is prime even if $I(V)$ is prime.

For example, let $V = \operatorname{Ker}\begin{pmatrix} 0 & 2 & 1 & 1 \\ 8 & 0 & 2 & 3 \end{pmatrix}$. Then V is generated by ${}^t(-1, -2, 4, 0)$ and ${}^t(-1, -1, -2, 4)$, hence $I(V)$ is prime and is a complete intersection.

To extend $I(V)$ to a complete intersection of height 3, we must choose a vector of the form ${}^t(-a, b, 0, 0)$ with $a>0$, $b>0$ by Theorem 2.4. But it is never prime for any a, b, since the cokernel of the injection $V + \langle w \rangle$ to Z^4 has a torsion.

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