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Collisions of Markov Processes

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Abstract. Let X_1 and X_2 be two independent Hunt processes which take values in a metric space and have the same transition density functions with respect to a reference measure. We describe explicit conditions on the transition density functions so that X_1 and X_2 have collisions with positive probability or with probability one or do not have any collision. The applications to Lévy processes, diffusions driven by s.d.e.'s and Brownian motions on fractals are exhibited.

§1. Introductions and notations.

Let X_1 and X_2 be two independent Brownian motions in $\boldsymbol{R}^d$. It is well-known that X_1 and X_2 have intersections, i.e. $X_1(t_1)=X_2(t_2)$ for some t_2, $t_2>0$, when $d\leq 3$. However, X_1 and X_2 have collisions, i.e. $X_1(t)=X_2(t)$ for some $t>0$, only when $d=1$. Jain-Pruitt [8], Hawkes [9] and Pruitt [13] studied the collisions of two stable processes on the line. Shieh [14] considered the collisions of two Lévy processes on the line. In this paper, we present explicit conditions on the transition density functions of two independent Hunt processes, taking values in a metric space, so that they have collisions with positive probability or with probability one or do not have any collision. Our general results are directly applicable to some specific processes; we concern with Lévy processes, diffusions driven by s.d.e.'s, and Brownian motions on fractals. As one consequence, we prove that two independent Brownian motions on Sierpinski Gasket or Carpet have collisions with probability one (see (§5, iii), while it is interesting to note that the intersections of these processes are trivially true). The basic idea of deriving our results follows somewhat "standard" one in the various problems on intersections and collisions. To prove the collisions, we construct explicitly a non-trivial measure supported on $\{t: X_1(t)=X_2(t)\}$; to prove the non-collision, we show that the diagonal $\{(x,x)\}$ is polar for the product process $Y(t)=(X_1(t), X_2(t))$. However, our methods and calculations in this paper are quite different from those in [8, 9, 13]; we are mainly motivated by a device of Marcus [12] on level sets of real-valued stochastic processes.

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Moreover, even for Lévy processes and diffusions on the line, our results about the time when and the interval where the collisions happen appear to be novel (§5, i and ii).

We prove our result on the collisions with positive probability in §2 and that with probability one in §3. We prove a result on the non-collision in §4. In the final §5 we give specific applications. Now, we commence with some basic notations and facts used throughout this paper. We assume that the state space is a locally compact separable complete metric space (E, ρ), and assume that there is an everywhere dense Radon measure $m(dx)$ defined on E which plays the role of reference measure. Let $B(x, \varepsilon)$, $x \in E$ and $\varepsilon > 0$, denote the closed ball in E with the center x and the radius ε. We assume that there exists some ε_0 such that $B(x, \varepsilon_0)$ is compact for each x; moreover, for $0 < \varepsilon \leq \varepsilon_0$

$$m(B(x, \varepsilon)) = m(B(y, \varepsilon)) \qquad \text{for all } x, y \in E.$$

Note that the above quantity is finitely positive. We always let X_1 and X_2 be two independent E-valued Hunt processes (no explosions) which are defined on a certain underlying probability space $w \in \Omega$ and have the same transition density functions $p(t, x, y)$ with respect to $m(dx)$, i.e. for $s, t > 0$ and $x \in E$, Borel $A \subset E$

$$P[X_j(t+s) \in A \mid X_j(s) = x] = \int_A p(t, x, y) m(dy).$$

We assume that

(1.1) $p(t, x, y)$ is jointly measurable in (t, x, y);

(1.2) $p(t, x, y)$ is continuous in (x, y) for each $t > 0$, and

(1.3) the Chapman-Kolmogorov equation

$$\int_{z \in E} p(t, x, z) p(s, z, y) m(dz) = p(t+s, x, y)$$

holds for all $s, t > 0$ and $x, y \in E$. We shall only need (1.1) and (1.2) in §§2, 3 and (1.3) is needed only in §4. We also use the notations $P^{x,y}$ and $E^{x,y}$ to indicate the probability and the expectation corresponding to the initial conditions $X_1(0) = x$ and $X_2(0) = y$. We need frequently a simple identity that

$$\int_E p(t, x, y) m(dy) = 1 \qquad \text{for all} \quad t > 0 \quad \text{and} \quad x \in E. \tag{1.4}$$

§2. The collisions with positive probability.

Let $X_1(t)$ and $X_2(t)$ be two independent E-valued Hunt processes with $p(t, x, y)$ as transition density functions, as those mentioned in §1. We assume further in this section that

(2.1) $$p(t, x, y) > 0 \quad \text{for all} \quad t > 0 \quad \text{and} \quad x, y \in E.$$

For compact $K \subset E$, we set

$$\alpha_K(t) = \sup_{x,y \in K} p(t, x, y); \qquad \beta_K(t) = \inf_{x,y \in K} p(t, x, y).$$

Note that $\alpha_K(t) < \infty$ and $\beta_K(t) > 0$ by our conditions (1.2) and (2.1).

THEOREM 2.1. *Assume* (1.1), (1.2) *and* (2.1). *Let compact $K \subset E$ be with positive m-measure and let $x, y \in K$. If*

(2.2) $$\int_0^1 \alpha_K(t)dt < \infty,$$

then

$$P^{x,y}[w : \exists \text{ uncountably many } t \in (0, 1] \text{ s.t. } X_1(t, w) = X_2(t, w) \in K] > 0.$$

PROOF. We aim to construct a non-trivial measure supported on $\{t : X_1(t) = X_2(t) \in K\}$; the arguments are motivated from Marcus' [12, Theorem 1] work on the capacity of level sets of real-valued processes. Define the approximation kernels by

$$\psi_\varepsilon(x, y) = \frac{1_{B(x,\varepsilon)}(y)}{m(B(x, \varepsilon))}.$$

By our assumptions on the measure $m(dx)$ mentioned in §1, $\psi_\varepsilon(x, y)$ is well-defined when ε is small enough and $\psi_\varepsilon(x, y) = \psi_\varepsilon(y, x)$. We define

(2.3) $$Z_\varepsilon(t) = \int_0^t 1_K(X_1(s))\psi_\varepsilon(X_1(s), X_2(s))ds, \qquad 0 \le t \le 1.$$

We firstly prove that

(2.4) $$\lim_{\varepsilon,\varepsilon' \downarrow 0} E^{x,y}Z_\varepsilon(t)Z_{\varepsilon'}(t)$$

$$= 2\int_0^t \int_0^{s'} \left\{ \int_{z_2 \in K} \int_{z_1 \in K} p(s, x, z_2)p(s, y, z_1)p^2(s'-s, z_1, z_2)m(dz_1)m(dz_2) \right\} ds\,ds' < \infty,$$

where the finiteness of the multiple integral is a part of the assertion. By (1.1) the right-handed side of (2.4) is well-defined, and we observe that

(2.5) $$E^{x,y}Z_\varepsilon(t)Z_{\varepsilon'}(t) = 2\int_0^t \int_0^{s'} \left\{ \int_{y_2 \in K_{\varepsilon'}} \int_{x_2 \in K} \int_{y_1 \in K_\varepsilon} \int_{x_1 \in K} \right.$$

$$\psi_\varepsilon(x_1, y_1)\psi_{\varepsilon'}(x_2, y_2)p(s, x, x_1)p(s'-s, x_1, x_2)$$

$$\left. p(s, y, y_1)p(s'-s, y_1, y_2)m(dx_1)m(dy_1)m(dx_2)m(dy_2) \right\} ds\,ds',$$

where $K_\varepsilon = \{y \mid \rho(y, K) \le \varepsilon\}$. For fixed s and s', we claim that the integral in the brace of (2.5) converges to the corresponding integral of (2.4) as $\varepsilon, \varepsilon' \downarrow 0$. We use the notations $\{\cdot\}_{\varepsilon,\varepsilon'}$ and $\{\cdot\}$ respectively for these two integrals. Since for all fixed x_1 and y_2

$$(2.6) \qquad \int_{y_1} \psi_\varepsilon(x_1, y_1) m(dy_1) = 1 \quad \text{and} \quad \int_{x_2} \psi_{\varepsilon'}(x_2, y_2) m(dx_2) = 1 \,,$$

changing z_1, z_2 into x_1, y_2 and inserting x_2, y_1 we can write $\{\cdot\}$ as

$$\{\cdot\} = \int_{y_2 \in K} \int_{x_2 \in K} \int_{y_1 \in K} \int_{x_1 \in K} p(s, x, x_1) p(s, y, x_1) \psi_\varepsilon(x_1, y_1)$$
$$p(s'-s, x_1, y_2) p(s'-s, x_1, y_2) \psi_{\varepsilon'}(x_2, y_2) m(dx_1) m(dy_1) m(dx_2) m(dy_2) \,.$$

Thus, the convergence of $\{\cdot\}_{\varepsilon,\varepsilon'}$ to $\{\cdot\}$ as $\varepsilon, \varepsilon' \downarrow 0$ follows from the uniform continuity of $p(s, x, y)$ in $(x, y) \in K_\varepsilon \times K_\varepsilon$, which is a consequence of (1.2). Now, we show that we may apply the dominated convergence theorem to assert (2.4). We have

$$\{\cdot\}_{\varepsilon,\varepsilon'} \le \alpha_K(s) \alpha_K(s'-s)$$
$$\times \int_{y_2} \int_{x_2} \int_{y_1} \int_{x_1} \psi_\varepsilon(x_1, y_1) \psi_{\varepsilon'}(x_2, y_2)$$
$$p(s, y, y_1) p(s'-s, y_1, y_2) m(dx_1) m(dy_1) m(dx_2) m(dy_2) \,;$$

the above multiple integral is ≤ 1 by repeatedly using (1.4) and (2.6). By (2.2) $g(s, s') := \alpha_K(s) \alpha_K(s'-s)$ is integrable over (s, s'), and thus the dominated convergence theorem is indeed applicable. Now, (2.4) implies that $Z_\varepsilon(t)$ is Cauchy in $L^2(dP^{x,y})$ as $\varepsilon \downarrow 0$. Therefore, we can find a sequence $\varepsilon_n \downarrow 0$ such that

$$(2.7) \qquad P^{x,y}\left[w : \lim_{n \to \infty} Z_{\varepsilon_n}(t, w) := Z(t, w) \text{ exists for all rational } t \in [0, 1] \right] = 1 \,.$$

Obviously, $Z(t, w)$ is nondecreasing in t, and thus it determines a Borel measure on $[0, 1]$ which we denote by $Z(dt, w)$. From the integrability of $\alpha_K(s)$, we can construct a decreasing continuous $\phi : r \in (0, \infty) \to \phi(r) \in [1, \infty)$ with $\phi(0+) = \infty$ such that

$$\int_0^1 \alpha_K(t) \phi(t) dt < \infty \,,$$

see Shieh [14, p. 504]. By the same arguments as those in Marcus [12, (2.5)–(2.8)], we have

$$(2.8) \qquad P^{x,y}\left[\int_0^1 \int_0^1 \phi(|s'-s|) Z(ds) Z(ds') < \infty \right] = 1 \,.$$

From the construction of $Z(dt)$, the support of $Z(dt)$ is contained in $\{t : X_1(t) = X_2(t)$ and both X_j are continuous at $t\}$. By (2.8) and that $\phi(0+) = \infty$, $Z(dt)$ has no atoms;

moreover both X_j have at most countably many discontinuities (being Hunt processes). Therefore, we may use a simple Fubini argument to assert that

$$P^{x,y}[\operatorname{supp} Z(dt) \subset \{t : X_1(t) = X_2(t)\}] = 1 . \tag{2.9}$$

By (2.8), (2.9) and an elementary Schwartz-type inequality, *cf.* Marcus [12, (2.10)], we have

$$\begin{aligned} & P^{x,y}[\text{the set } \{t \in (0, 1] : X_1(t) = X_2(t)\} \text{ has positive } \phi\text{-capacity}] \\ & \geq P^{x,y}[Z[0, 1] > 0] \\ & \geq \frac{(E^{x,y} Z[0, 1])^2}{E^{x,y} Z^2[0, 1]} . \end{aligned} \tag{2.10}$$

We can easily see from (2.3)–(2.5) and the definitions of α_K, β_K that

$$\begin{aligned} E^{x,y} Z[0, 1] &= \int_0^1 \int_{z \in K} p(t, x, z) p(t, y, z) m(dz) dt \\ &\geq m(K) \int_0^1 \beta_K^2(t) dt > 0 , \qquad \text{and} \end{aligned} \tag{2.11}$$

$$\begin{aligned} E^{x,y} Z^2[0, 1] &= 2 \int_s^1 \int_0^1 \int_{z_2 \in K} \int_{z_1 \in K} \\ & p(s, x, z_1) p(s, y, z_1) p^2(s' - s, z_1, z_2) m(dz_1) m(dz_2) ds\, ds' \\ &\leq 2 \int_s^1 \int_0^1 \alpha_K(s) \alpha_K(s' - s) ds\, ds' < \infty . \end{aligned} \tag{2.12}$$

In (2.12), we have used again the identity (1.4), and the finiteness of the last integral above follows from (2.2). Since a set with positive ϕ-capacity is certainly uncountable, (2.10)–(2.12) complete the proof of the theorem.

§3. The a.s. collisions and complementary remarks.

Theorem 2.1 concerns merely with positive probability. Under certain "non-transient" condition (see Remark 3 below the proof of Theorem 3.1), we can prove an almost sure result by some "patch-up" technique. In the following context, "a.s." means "$P^{x,y}$-a.s." etc.

THEOREM 3.1. *Assume* (1.1), (1.2) *and* (2.1); *moreover, assume that for all compact* $K \subset E$ *and* $x, y \in E$

$$\int_0^1 \alpha_K(t) dt < \infty , \qquad \textit{and} \tag{3.1}$$

$$\varliminf_{\substack{n\in N\\ n\to\infty}} \max\{\rho(x, X_1(n)), \rho(y, X_2(n))\} < \infty \qquad a.s. \tag{3.2}$$

Then, for a.s. w, there exists uncountably many t such that $X_1(t, w) = X_2(t, w)$.

PROOF. Note that (3.2) holds trivially when the space E is compact. Thus, we assume that E is non-compact and regard the compact case as a particular one. Since E is separable and locally compact, given $x, y \in E$ there exists an increasing sequence K_l of compacts such that $x, y \in K_1$ and $K_l \uparrow E$. We may also assume that each K_l is of positive m-measure and that the diameter of K_l increases to ∞. For $n, l \in N$ we set

$$A_n^l = [w : \exists \text{ uncountably many } t \in (n, n+1] \text{ s.t. } X_1(t, w) = X_2(t, w) \in K_l],$$

$$A_n = [w : \exists \text{ uncountably many } t \in (n, n+1] \text{ s.t. } X_1(t, w) = X_2(t, w)].$$

By the Markov property, for a.s. w

$$\begin{aligned} &P^{x,y}[A_n \mid X_1(t), X_2(t), t \le n-1](w) \\ &\quad = P^{X_1(n-1,w), X_2(n-1,w)}[A_1] \ge P^{X_1(n-1,w), X_2(n-1,w)}[A_1^l]. \end{aligned}$$

By the proof of Theorem 2.1, the above display

$$\ge \frac{\displaystyle\int_0^1 \int_{z\in K_l} p(t, X_1(n-1, w), z) p(t, X_2(n-1, w), z) m(dz) dt}{\displaystyle 2\int_s^1 \int_0^1 \alpha_{K_l}(s) \alpha_{K_l}(s'-s) ds\, ds'}.$$

The numerator of the above fraction

$$\ge m(K_l) \int_0^1 \beta_{K_l}^2(t) dt,$$

whenever $X_1(n-1, w), X_2(n-1, w) \in K_l$. Thus, for each K_l and for a.s. w

$$\sum_{n=1}^{\infty} P^{x,y}[A_n \mid X_1(t), X_2(t), t \le n-1](w) \tag{3.3}$$

$$\ge \frac{N_l(w) \cdot m(K_l) \cdot \displaystyle\int_0^1 \beta_{K_l}^2(t) dt}{\displaystyle 2\int_s^1 \int_0^1 \alpha_{K_l}(s) \alpha_{K_l}(s'-s) ds\, ds'},$$

where $N_l(w) = \#\{n = n(w) \mid X_1(n-1, w), X_2(n-1, w) \in K_l\}$. Under (3.2), there exists an a.s. finite positive random variable $r = r(w)$ such that almostly surely $\rho(x, X_1(n)) < r$ and $\rho(y, X_2(n)) < r$ hold for infinitely many n. Thus, by our choice of compacts K_l, for a.s. w we can find one K_l (which may depend on w) such that $N_l(w) = \infty$. By (3.3), we then have

$$\sum_{n=1}^{\infty} P^{x,y}[A_n \mid X_1(t), X_2(t), t \leq n-1] = \infty \qquad \text{a.s.} \tag{3.4}$$

By the conditional Borel-Cantelli lemma mentioned, for example, in Breiman [6, p. 96], it follows from (3.4) that $P^{x,y}[A_n\, i.o.] = 1$. This proves the theorem.

We mention several remarks concerning with the possible extensions of Theorems 2.1 and 3.1.

REMARKS. 1. We may allow X_1, X_2 to have different transition densities p_1, p_2. Theorems 2.1 and 3.1 hold when both of p_i satisfy (2.1) and one of p_i satisfies (2.2) and (3.1).

2. In (2.2) the upper bound 1 for the integral of α_K can be replaced by any positive T. Hence, we can assert in Theorem 2.1 that the collisions happen with positive probability in any small initial interval of time and in any compact neighborhood of the initial points x, y.

3. We regard (3.2) as a "non-transient" condition for the $E \times E$-valued Markov chain $Y(n) = (X_1(n), X_2(n))$. A ready sufficient condition for (3.2) is that $Y(n)$ is neighborhood-recurrent in $E \times E$; our arguments in Theorem 3.1 actually show that the collisions then happen with probability one in any compact neighborhood of the initial points x, y. Moreover, as we may see from the proof of Theorem 3.1 that (3.2) is unnecessary in case that the state space E is compact or in case that X_i are Lévy processes (the simple 0-1 law is sufficient to derive the a.s. result from the result of positive probability).

4. We may proceed most of our arguments under the (weaker) assumptions on the resolvent densities (Green functions) rather than on transition densities, as Rogers [14] did for multiple points of Markov processes.

5. We observe that, in the proofs of Theorems 2.1 and 3.1 we actually have just used the simple Markov property and the consequence that the paths have at most countably many discontinuities. Therefore, we may state our Theorems 2.1 and 3.1 for (simple) Markov processes with transition density functions $p(t, x, y)$ for which

$$\int_{y \in E - B(x,\varepsilon)} p(t, x, y) m(dy) = o(1) \qquad \text{as} \quad t \to 0,$$

uniformly on x-compacts, for each $\varepsilon > 0$. This classical condition asserts the "cadlaguity" of sample paths.

6. The translation-invariant condition $m(B(x, \varepsilon)) = m(B(y, \varepsilon))$ for the measure m can be weakened to that the two quantities are mutually bounded by certain absolute constants. We may see this from the arguments leading to the construction of the measure $Z(\cdot)$ in the proof of Theorem 2.1. I thank to the referee for this indication.

7. Professor T. Shiga indicates me the existence of a paper by S. Karlin and J.

McGregar "Coincidence Probabilities" (Pacific J. Math. **9** (1959) 1141–1161), in which the probability of non-collisions up to a given time instant was evaluated in terms of a certain determinant (Theorem 1 at p. 1149 with $n=2$). Their method is different from ours and the evaluated probability seems not comparable with our lower bound estimate in §2. Moreover, we cannot derive the a.s. collisions by their estimate and the "patch-up" technique in §3. Furthermore, the various general formulations for non-polar sets stated in Blumenthal and Getoor [1, e.g. p. 89] seem also not applicable to our specific conditions in §§2,3.

§4. The non-collision.

In this section, we give explicit conditions so that X_1, X_2 cannot have any collision. Since we shall use the capacity theory, we have to assume some duality condition. Here, we confine ourselves to the simplest symmetric case. Let X_1 and X_2 be two independent E-valued Hunt processes with the same transition density functions $p(t, x, y)$. We assume that

(4.1) $$p(t, x, y)=p(t, y, x) \qquad \text{for all } t>0 \text{ and all } x, y\in E\,.$$

We assume also that

(4.2) $\exists$ positive δ, δ' s.t. $p(t, x, y)\leq\delta$ for all $(t, x, y)\in(0, 1]\times\{\rho(x, y)>\delta'\}$.

THEOREM 4.1. *Assume* (1.1)–(1.3) *and* (4.1), (4.2). *If*

(4.3) $$\int_0^1\left\{\inf_{y\in K} p(t, y, y)\right\}dt=\infty$$

for all compact $K\subset E$ with positive m-measure, then X_1 and X_2 a.s. do not have any collision.

PROOF. We prove that the diagonal $\{(x, x)\}$ is polar for the $E\times E$-valued Hunt process $Y(t)=(X_1(t), X_2(t))$. Since $Y(t)$ is m-symmetric, the capacity theory mentioned, for example, in Blumenthal-Getoor [1, Chapter VI pp. 285–6] is applicable. Hence, it suffices to prove that, for each fixed compact $K\subset E$ with positive m-measure, the set $D_K=\{(x, x)\mid x\in K\}$ is of the natural (with respect to Y) 1-capacity zero. Note that the 1-potential kernel of $Y(t)$ is

(4.4) $$u^{(1)}((x, y), (x', y'))=\int_0^\infty e^{-t}p(t, x, x')p(t, y, y')dt\,.$$

Let $\mu(dxdy)$ be a finite positive Borel measure on $E\times E$ and let $v(dx)$ be its marginal distribution on E. Suppose that $\operatorname{supp}\mu\subset D_K$ and that

(4.5) $$U^{(1)}\mu(x, y)=\int u^{(1)}((x, y), (x', y'))\mu(dx'dy')\leq 1\,,$$

for all (x, y). By (4.4), (4.5) and the definition of $v(dx)$ we have, for all (x, y)

$$\int_0^\infty \int_K e^{-t} p(t, x, z) p(t, y, z) v(dz) dt \leq 1 . \tag{4.6}$$

Choose a compact K' such that $K \subset K'$ and that $\sigma(x, y) \geq \delta'$ whenever $x \in K$ and $y \notin K'$. Integrating (4.6) for $x = y$ over K' with respect to $m(dx)$, we have

$$\int_0^\infty \int_{x \in K'} \int_{z \in K} e^{-t} p(t, x, z) p(t, x, z) v(dz) m(dx) dt \leq m(K') .$$

By (1.3) and (4.1) we have

$$\begin{aligned} \int_{x \in K'} p(t, x, z) p(t, x, z) m(dx) &= \int_{x \in K'} p(t, z, x) p(t, x, z) m(dx) \\ &= \left[\int_{x \in E} - \int_{x \notin K'} \right] p(t, z, x) p(t, x, z) m(dx) \\ &= p(2t, z, z) - \int_{x \notin K'} p^2(t, z, x) m(dx) . \end{aligned}$$

By (4.2) and our choice of K' we have

$$\int_{x \notin K'} \int_{z \in K} p^2(t, z, x) v(dz) m(dx) \leq \delta \cdot v(K) ,$$

since, by (1.4),

$$\int_{x \notin K'} p(t, z, x) m(dx) \leq 1 \qquad \text{for all } t \text{ and } z .$$

Thus,

$$\int_0^\infty e^{-t} \int_{z \in K} p(2t, z, z) v(dz) dt \leq m(K') + \text{Const.}\, v(K) .$$

Consequently,

$$\int_0^\infty e^{-t} \left\{ \inf_{z \in K} p(2t, z, z) \right\} dt \leq \frac{m(K')}{v(K)} + \text{Const.} < \infty ,$$

which contradicts (4.3). Hence, there cannot exist a finite positive Borel measure supported on D_K for which the 1-potential function is bounded. This proves that D_K is of 1-capacity zero and hence proves the theorem.

REMARK. We have used the symmetric condition (4.1) in Theorem 4.1 twice. One is to enforce the capacity theory and the other is to relate the integral of $p^2(t, x, z)$ in

x to $p(2t, z, z)$. Is the symmetric assumption essentially needed? Consider two independent Cauchy processes on the line. They satisfy (4.3) and they do not have collisions [8, §1] no matter they are symmetric or asymmetric. However, we are unable to prove a more general non-collision result just under certain duality assumption (I thank S. Watanabe for indicating such a consideration).

§5. Applications.

(i) *Lévy processes.* Let X_1 and X_2 be two independent Lévy processes in $\boldsymbol{R}^d$ with the same characteristic exponent $\psi(\lambda)$, i.e. $Ee^{i\lambda\cdot X_1(t)}=Ee^{i\lambda\cdot X_2(t)}=e^{-t\psi(\lambda)}$ for all $t>0$ and $\lambda\in\boldsymbol{R}^d$. When the Blumenthal-Getoor lower index $\beta'':=\sup\{\zeta\geq 0: |\lambda|^{-\zeta}\operatorname{Re}\psi(\lambda)\to\infty$ as $|\lambda|\to\infty\}$ is positive, then both $X_j(t)$ have a common bound continuous density $p(t, x)$ with respect to the Lebesgue measure. We assume that $p(t, x)$ is everywhere positive. Applying §§2, 3 we have: if $d=1$ and $\beta''>1$ then X_1 and X_2 have collisions with probability one and the collisions happen in any small initial time-interval and in any compact interval containing the initial points. Applying §4 we have: if $\psi(\lambda)$ is real (so that each $X(t)$ is symmetric), then X_1 and X_2 do not have any collision in case either $d=1$, $\beta''\leq 1$ or $d\geq 2$. The above results extend greatly Shieh [15, Theorem 1] and some are new even for stable processes. In the Lévy case, the collisions with probability one follows from that of positive probability as we have remarked in Remark 3.3.

(ii) *Diffusions driven by s.d.e.'s.* Let X_1 and X_2 be two independent diffusions in $\boldsymbol{R}^d$ driven by the same stochastic differential equation $dX_j(t)=\sigma(X_j(t))dB(t)+b(X_j(t))dt$, where σ^2 is uniformly elliptic and σ, b are bounded and Hölder continuous. The transition density functions have well-known Gaussian estimates. Applying §§2, 3 we have: if $d=1$ then X_1 and X_2 have collisions with probability one and the collisions happen in any small initial time-interval and in any compact interval containing the initial points. Note that the chain $Y(n)=(X_1(n), X_2(n))$ is neighborhood-recurrent, as we can apply the criterion in Friedman [7, §2]. Applying §4 we have: if $d\geq 2$ then X_1 and X_2 do not have any collision (to ensure (4.1), we have to assume that $b\equiv 0$; then we use the Cameron-Martin theorem for $b\not\equiv 0$.). We remark that Lindvall [10] and Lindvall-Rogers [11] have studied relevant results from the viewpoint of coupling theory.

(iii) *Brownian motions on fractals.* Brownian motions on (planar compact) Sierpinski Gasket and Carpet have been constructed by Barlow-Perkins [5] and Barlow-Bass [2]. The transition density functions with respect to the natural Hausdorff measure associated with the fractal have been proven existent and been precisely estimated as follows [4, 5]

$$p(t, x, y)\sim t^{-d_s/2}e^{(\mathrm{const.}|x-y|t^{-1/d_w})^{d_w/d_w-1}},$$

where d_s is "the spectral dimension" and d_w is "the dimension of the walk". The numerical values and estimates for d_s and d_w given in [4, 5]; in both cases, $d_s \in (1, 2)$ and $d_w > 2$. By §§2, 3 we have: any two independent Brownian motions on either S.G. or S.C. have collisions with probability one. We remark that the intersections of these processes are trivially true, since they are point-recurrent [3, 5].

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