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Kähler Magnetic Flows for a Manifold of Constant Holomorphic Sectional Curvature

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Introduction.

In his paper [7], being inspired by a classical treatment of static magnetic fields in the three dimensional Euclidean space, T. Sunada studied the flow associated with a magnetic field on a Riemann surface. A closed 2-form $\boldsymbol{B}$ on a complete Riemannian manifold M is called a *magnetic field*. Let $\Omega=\Omega_{\boldsymbol{B}}$ denote the skew symmetric operator on the tangent bundle TM of M satisfying $\boldsymbol{B}(u, v)=\langle u, \Omega(v)\rangle$ with the Riemannian metric $\langle\ ,\ \rangle$ for every tangent vectors u and v. The Newton equation on this setting is of the form $\nabla_{\dot\gamma}\dot\gamma=\Omega(\dot\gamma)$ for a smooth curve γ on M. We call such a curve satisfying this equation a *trajectory* for $\boldsymbol{B}$. In terms of physics it is a trajectory of a charged particle moving on this manifold under the action of the magnetic field. The aim of this paper is to give a light in terms of magnetic fields on dynamical systems for a manifold of complex space form. The most important dynamical object associated to a Riemannian manifold is the geodesic flow. Consider the case without an action of magnetic field, $\boldsymbol{B}=0$. The Newton equation turns out to $\nabla_{\dot\gamma}\dot\gamma=0$, hence trajectories are nothing but geodesics. In the same way as the geodesic flow corresponds to geodesics, we can define a flow associated with a magnetic field in the following manner. One can easily check that every trajectory $\gamma(t)$ for $\boldsymbol{B}$ has constant speed, hence is defined for $-\infty<t<\infty$. We call a trajectory *normal* if it is parametrized by its arc length. The *magnetic flow* $\boldsymbol{B}\varphi_t$: $UM\to UM$ on the unit tangent bundle UM is defined by

$$\boldsymbol{B}\varphi_t(v)=\gamma_v(t)\,,\qquad v\in UM\,,\quad -\infty<t<\infty\,,$$

where γ_v denotes the normal trajectory for $\boldsymbol{B}$ with $\dot\gamma_v(0)=v$. If γ is a trajectory for $\boldsymbol{B}$, then the curve $\sigma(t)=\gamma(\lambda t)$ with a constant λ is a trajectory for $\lambda\boldsymbol{B}$. This represents a dynamical property of trajectories for $\boldsymbol{B}$.

On a Riemann surface, magnetic fields are of the form $f\cdot\mathrm{Vol}$ with a smooth

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function f and the volume form Vol. When f is a constant function, it is called *uniform*. The feature of trajectories are well-known for a uniform magnetic field $k\cdot\mathrm{Vol}$ on a surface of constant sectional curvature. On a sphere $S^2(c)$ of sectional curvature c, normal trajectories are small circles, which can be seen circles (in usual sense of Euclidean geometry) of radius $(k^2+c)^{-1/2}$ if we canonically embed $S^2(c)$ in $\boldsymbol{R}^3$. On a Euclidean plane $\boldsymbol{R}^2$ they are circles of radius $1/|k|$. In these two cases, every trajectory is closed. The prime period of a normal trajectory is $2\pi/\sqrt{k^2+c}$ on $S^2(c)$, and $2\pi/|k|$ on $\boldsymbol{R}^2$. On a hyperbolic plane $H^2(-c)$ of sectional curvature $-c$, the feature is quite different. When the strength $|k|$ is greater than $\sqrt{c}$, normal trajectories are still closed with prime period $2\pi/\sqrt{k^2-c}$, but when $|k|\le\sqrt{c}$, normal trajectories are open curves and go to some points on the ideal boundary as $t\to\pm\infty$. Moreover, when $|k|=\sqrt{c}$ normal trajectories are horocycles (see [4] and [7]).

We shall now consider magnetic fields on manifolds of higher dimension. We call a magnetic field uniform if the associated skew symmetric operator is parallel $\nabla\Omega=0$. This means that the magnetic field has constant strength. On real space forms of dimension greater than 2 we find there exist no nontrivial uniform magnetic fields. Another typical example of uniform magnetic fields is a Kähler magnetic field on a Kähler manifold. Let (M, J) be a Kähler manifold with a complex structure J. We call a scalar multiple of the Kähler form $\boldsymbol{B}_J$ a *Kähler magnetic field*, here the Kähler form is given by $\boldsymbol{B}_J(u, v)=\langle u, Jv\rangle$. In contrast with the feature of trajectories for uniform magnetic fields on surfaces of constant sectional curvature, it is quite natural to study Kähler magnetic fields on manifolds of constant holomorphic sectional curvature. For a complex space $\boldsymbol{C}^n$, the flat case, the situation is trivial. For a complex projective space, we announced in [1] that every trajectory for a Kähler magnetic field is a small circle on a totally geodesic embedded 2-sphere, hence is simply closed. We study in section 1 the case of a complex hyperbolic space $\boldsymbol{C}H^n(-c)$ of constant holomorphic sectional curvature $-c$. We give explicit expressions of trajectories and point out that the feature of trajectories changes according to the strength $|k|$ of a Kähler magnetic field $k\cdot\boldsymbol{B}_J$ is greater or smaller than $\sqrt{c}$. Under these consideration we shall be concerned in section 2 with dynamical theoretic relationship between the geodesic flow and Kähler magnetic flows. Two flows φ_t and ψ_t on a manifold N are said to be *smoothly conjugate* in the strong sense if there exist a diffeomorphism $f\colon N\to N$ and a nonzero constant θ such that $f^{-1}\circ\varphi_t\circ f=\psi_{\theta t}$. We show that, in the same way as the sense $n=1$, Kähler magnetic flows for $\boldsymbol{C}H^n(-c)$ are classified into three classes according to the strength of Kähler magnetic field. We also point out that Kähler magnetic flows for a complex projective space are strong smoothly conjugate each other.

Our proof is based on the fact that a complex hyperbolic space is a base manifold of a principal S^1-bundle $\pi\colon H_1^{2n+1}\to\boldsymbol{C}H^n$, which corresponds to the Hopf fibration $S^{2n+1}\to\boldsymbol{C}P^n$. Once we take horizontal lifts of trajectories, we find that they are helices of order 3, which satisfy linear ordinary differential equations in $\boldsymbol{C}^{n+1}$. Solving these equations we can give their explicit expressions, which are very useful in our argument.

Our results are natural generalization of the result of [7] on uniform magnetic flows for a hyperbolic plane.

§1. Trajectories for Kähler magnetic fields.

We shall start with giving some fundamental notations on a complex hyperbolic space. Consider a Hermitian form on $\boldsymbol{C}^{n+1}$ defined by

$$\langle\!\langle z, w\rangle\!\rangle = -z_0\overline{w_0} + \sum_{j=1}^{n} z_j\overline{w_j}\,,$$

for $z=(z_0, \cdots, z_n)$, $w=(w_0, \cdots, w_n)\in \boldsymbol{C}^{n+1}$. On the real hypersurface $H_1^{2n+1}=\{z\in \boldsymbol{C}^{n+1} \mid \langle\!\langle z, z\rangle\!\rangle = -1\}$, the group $S^1=\{e^{i\theta}\}$ acts freely; $z\to e^{i\theta}z$. We denote by $\pi\colon H_1^{2n+1}\to \boldsymbol{C}H^n$ the principal S^1-fiber bundle. For $z\in H_1^{2n+1}$, the tangent space is represented by

$$T_zH_1^{2n+1}=\{(z, u) \mid u\in \boldsymbol{C}^{n+1},\ \mathrm{Re}\langle\!\langle z, u\rangle\!\rangle=0\}\,,$$

and is decomposed into horizontal and vertical subspaces $\mathscr{H}_zH_1^{2n+1}\oplus\mathscr{V}_zH_1^{2n+1}$, where

$$\mathscr{H}_zH_1^{2n+1}=\{(z, u) \mid u\in \boldsymbol{C}^{n+1},\ \langle\!\langle z,u\rangle\!\rangle=0\}\,,\quad \text{and}\quad \mathscr{V}_zH_1^{2n+1}=\{(z, i\lambda z) \mid \lambda\in \boldsymbol{R}\}\,.$$

The tangent space $T_{\pi(z)}\boldsymbol{C}H^n$ is identified with the horizontal subspace $\mathscr{H}_zH_1^{2n+1}$, and represented by

$$T_{\pi(z)}\boldsymbol{C}H^n=\{d\pi((z, u)) \mid z\in H_1^{2n+1},\ (z, u)\in\mathscr{H}_zH_1^{2n+1}\}\,.$$

Since the restriction of the Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $T_z\boldsymbol{C}^{n+1}\simeq\boldsymbol{C}^{n+1}$ to $\mathscr{H}_zH_1^{2n+1}$ is positive-definite, we can define a metric on $\boldsymbol{C}H^n$ by

$$\langle u, v\rangle=\frac{4}{c}\mathrm{Re}\langle\!\langle u, v\rangle\!\rangle\,,\qquad u, v\in T_{\pi(z)}\boldsymbol{C}H^n\simeq\mathscr{H}_zH_1^{2n+1}\,,$$

with a positive constant c. With the complex structure J induced by the canonical complex structure on $\boldsymbol{C}^{n+1}$, the Kähler manifold $\boldsymbol{C}H^n=\boldsymbol{C}H^n(-c)$ is called a complex hyperbolic space of holomorphic sectional curvature $-c$.

We use the same $\langle\ ,\ \rangle$ to denote the pseudo-Riemannian metric $\mathrm{Re}\langle\!\langle\ ,\ \rangle\!\rangle$ on $\boldsymbol{C}^{n+1}$ and its restriction on H_1^{2n+1}. Let $\bar\nabla$ and $\tilde\nabla$ denote the Riemannian connections associated to $\langle\ ,\ \rangle$ on $\boldsymbol{C}^{n+1}$ and H_1^{2n+1} respectively. We put $N=N(z)=(z, z)\in T_zH_1^{2n+1}$, which is the "unit" normal vector at $z\in H_1^{2n+1}$. The relations between $\bar\nabla$, $\tilde\nabla$ and the Riemannian connection ∇ on $\boldsymbol{C}H^n(-4)$ are as follows:

LEMMA 1. (1) *For any vector fields X, Y on $H_1^{2n+1}\subset\boldsymbol{C}^{n+1}$ we have $\tilde\nabla_XY=\bar\nabla_XY-\langle X, Y\rangle N$.*

(2) *For horizontal vector fields X, Y on H_1^{2n+1} we have in regarding them as vector fields on $\boldsymbol{C}H^n(-4)$ that $\nabla_XY=\tilde\nabla_XY+\langle X, JY\rangle JN$.*

PROOF. (1) Since $\langle Y, N\rangle=0$ and $\bar{\nabla}_X N=X$, we have

$$0=\bar{\nabla}_X\langle Y, N\rangle=\langle\bar{\nabla}_X Y, N\rangle+\langle Y, \bar{\nabla}_X N\rangle=\langle\bar{\nabla}_X Y, N\rangle+\langle Y, X\rangle .$$

Therefore we get

$$\tilde{\nabla}_X Y=\bar{\nabla}_X Y-\frac{\langle\bar{\nabla}_X Y, N\rangle}{\langle N, N\rangle} N=\bar{\nabla}_X Y+\langle\bar{\nabla}_X Y, N\rangle N=\bar{\nabla}_X Y-\langle X, Y\rangle N .$$

(2) Similarly we have

$$\nabla_X Y=\tilde{\nabla}_X Y-\frac{\langle\tilde{\nabla}_X Y, JN\rangle}{\langle JN, JN\rangle} JN=\tilde{\nabla}_X Y-\frac{\langle\tilde{\nabla}_X Y, JN\rangle}{\langle N, N\rangle} JN=\tilde{\nabla}_X Y+\langle\tilde{\nabla}_X Y, JN\rangle JN .$$

Since $\langle Y, JN\rangle=0$ we find

$$\begin{aligned} 0=\tilde{\nabla}_X\langle Y, JN\rangle &=\langle\tilde{\nabla}_X Y, JN\rangle+\langle Y, \tilde{\nabla}_X JN\rangle \\ &=\langle\tilde{\nabla}_X Y, JN\rangle+\langle Y, \bar{\nabla}_X JN-\langle X, JN\rangle N\rangle \\ &=\langle\tilde{\nabla}_X Y, JN\rangle+\langle Y, J\bar{\nabla}_X N\rangle \\ &=\langle\tilde{\nabla}_X Y, JN\rangle+\langle Y, JX\rangle=\langle\tilde{\nabla}_X Y, JN\rangle-\langle X, JY\rangle , \end{aligned}$$

and get the relation.

By using this fundamental relation, we shall give explicit expressions for trajectories for Kähler magnetic fields on a complex hyperbolic space. We here mention to some fundamental properties of trajectories for Kähler magnetic fields. Let γ be a trajectory for a Kähler magnetic field $k\cdot \boldsymbol{B}_J$ on a Kähler manifold $(M, J, \langle\ ,\ \rangle)$. If f is a $\pm$holomorphic isometry of M, then the curve $f\circ\gamma$ is a trajectory for $\pm k\cdot\boldsymbol{B}_J$. Two trajectories γ_1 and γ_2 are called *congruent* if there exists a holomorphic isometry f with $\gamma_2=f\circ\gamma_1$. On a manifold of complex space form $\boldsymbol{C}P^n(c)$ or $\boldsymbol{C}H^n(-c)$, it is clear that trajectories for $k\cdot\boldsymbol{B}_J$ are congruent. When we change the Riemannian metric homothetically; $\langle\ ,\ \rangle\to a^2\cdot\langle\ ,\ \rangle$ for some positive constant a, the curve $\sigma(t)=\gamma(t/a)$ is a trajectory for the Kähler magnetic field $(k/a)\cdot\boldsymbol{B}_J$. In our study for Kähler magnetic fields on a complex hyperbolic space, we may therefore consider only the case of $\boldsymbol{C}H^n(-4)$.

As a direct consequence of Lemma 1, we can conclude that every horizontal lift $\tilde{\gamma}$ of a trajectory γ for a Kähler magnetic field $k\cdot\boldsymbol{B}_J$ on $\boldsymbol{C}H^n(-4)$ into H_1^{2n+1} is a helix of order 3:

$$\begin{cases} \bar{\nabla}_{\dot{\tilde{\gamma}}}\dot{\tilde{\gamma}}= & k\cdot J\dot{\tilde{\gamma}} \\ \bar{\nabla}_{\dot{\tilde{\gamma}}}J\dot{\tilde{\gamma}}= -k\cdot\dot{\tilde{\gamma}} & +JN \\ \bar{\nabla}_{\dot{\tilde{\gamma}}}JN= & J\dot{\tilde{\gamma}} . \end{cases}$$

Here we should note that $\langle JN, JN\rangle=-1$ and the equation of helix seems a bit different from usual one. If we regard $\tilde{\gamma}$ as a curve in $\boldsymbol{C}^{n+1}$, we get by Lemma 1 that $\tilde{\gamma}$ satisfies the following linear ordinary differential equation:

$$\ddot{\tilde{\gamma}}-\tilde{\gamma}=ki\dot{\tilde{\gamma}}\ .$$

Solving this equation under the initial condition that $\gamma(0)=\pi(z)$ and $\dot{\gamma}(0)=d\pi((z,u))$, we have

$$\tilde{\gamma}(t)=\begin{cases}(4-k^2)^{-1/2}\{(\alpha e^{\beta t}-\beta e^{\alpha t})z+(e^{\alpha t}-e^{\beta t})u\}\ , & \text{if}\quad k\neq 2\ ,\\ e^{\pm it}\{(1\mp it)z+tu\}\ , & \text{if}\quad k=\pm 2\ ,\end{cases}$$

where $\alpha=(ki+\sqrt{4-k^2})/2$ and $\beta=(ki-\sqrt{4-k^2})/2$. Rewriting these we have

PROPOSITION 1. *The trajectory γ for the Kähler magnetic field $k\cdot \boldsymbol{B}_J$ on $\boldsymbol{CH}^n(-4)$ with $\gamma(0)=\pi(z)$ and $\dot{\gamma}(0)=d\pi((z,u))$ is written as*

$$\begin{aligned}&\gamma(t)=\pi(\cos\sqrt{k^2-4}\,t/2\cdot z+(k^2-4)^{-1/2}\sin\sqrt{k^2-4}\,t/2\cdot(-kiz+2u))\ , && \textit{if}\quad |k|>2\ ,\\ &\gamma(t)=\pi(\{(1\mp it)z+tu\})\ , && \textit{if}\quad k=\pm 2\ ,\\ &\gamma(t)=\pi(\cosh\sqrt{4-k^2}\,t/2\cdot z+(4-k^2)^{-1/2}\sinh\sqrt{4-k^2}\,t/2\cdot(-kiz+2u))\ , && \textit{if}\quad |k|<2\ .\end{aligned}$$

Therefore it lies on a totally geodesic embedded hyperbolic plane $\pi((\boldsymbol{C}z\oplus \boldsymbol{C}u)\cap H_1^{2n+1})$ of sectional curvature -4.

A trajectory γ for a Kähler magnetic field is said to be *closed* if there exist t_0 such that $\gamma(t_0)=\gamma(0)$ and $\dot{\gamma}(t_0)=\dot{\gamma}(0)$. We call a positive number the *prime period* of γ if it is the minimum positive number satisfying these equalities. The expressions (or the Comtet's result) lead us to the following.

COROLLARY. *The feature of trajectories for the Kähler magnetic field $k\cdot \boldsymbol{B}_J$ on $\boldsymbol{CH}^n(-c)$ are as follows.*

(1) *When $|k|>\sqrt{c}$, they are simply closed with prime period $2\pi/\sqrt{k^2-c}$.*

(2) *When $|k|\leq\sqrt{c}$, they are two-sides unbounded simple open curves. Here a trajectory γ is called two-sides unbounded if the sets $\{\gamma(t)\mid t>0\}$ and $\{\gamma(t)\mid t<0\}$ are unbounded.*

Since $|\langle\!\langle z,w\rangle\!\rangle|>1$ if $\pi(z)\neq\pi(w)$, by use of Proposition 1 one can easily get the following property on trajectories for a Kähler magnetic field of weak strength.

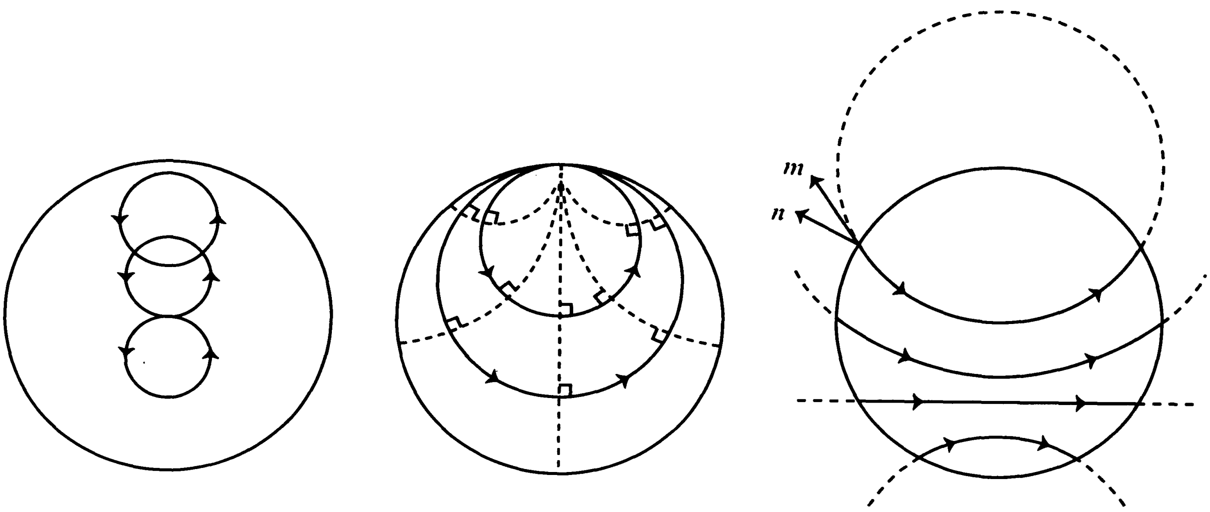
PROPOSITION 2. *Consider a Kähler magnetic field $k\cdot \boldsymbol{B}_J$ with $|k|<\sqrt{c}$ on a complex hyperbolic space $\boldsymbol{CH}^n(-c)$. Given two distinct points $p=\pi(z)$, $q=\pi(w)$ on $\boldsymbol{CH}^n$ we can find two and only two trajectories for $k\cdot \boldsymbol{B}_J$ joining them. One is from p to q and the other is from q to p. Their length l between these points are the same; it satisfies*

$$\sinh^2\sqrt{c-k^2}\,l/2=\frac{1}{c}(c-k^2)(|\langle\!\langle z,w\rangle\!\rangle|^2-1)\ .$$

A complex hyperbolic space $\boldsymbol{CH}^n$ is identified with the open unit ball $D_n(\boldsymbol{C})=\{x=(x_1,\cdots,x_n)\in \boldsymbol{C}^n \mid (\!(x,x)\!)=\sum_{j=1}^n x_j\overline{x_j}<1\}$ in $\boldsymbol{C}^n$ by the map

$$\Phi:\ CH^n \ni \pi(z_0, \cdots, z_n) \to (z_1/z_0, \cdots, z_n/z_0) \in D_n(\boldsymbol{C})\,.$$

The topological boundary $\partial D_n(\boldsymbol{C}) = \{x \in \boldsymbol{C}^n \mid \langle\!\langle x, x\rangle\!\rangle = 1\}$ is the ideal boundary as a Hadamard manifold. Under this identification the image of trajectories lies on a complex plane and can be seen as circles on $\boldsymbol{C}^n$ like the following figures.



$|k|>\sqrt{c}$ $\qquad$ $k=\pm\sqrt{c}$ $\qquad$ $|k|<\sqrt{c}$

We here mention the asymptotic behavior of two-sides unbounded trajectories. Let γ be a trajectory with $\dot\gamma(0) = d\pi((z, u))$ for a Kähler magnetic field $k\cdot \boldsymbol{B}_J$ on $CH^n(-c)$. When the strength $|k|$ is not greater than $\sqrt{c}$, the limits $\gamma(\infty) = \lim_{t\to\infty}\gamma(t)$ and $\gamma(-\infty) = \lim_{t\to-\infty}\gamma(t)$ exist in $\overline{CH^n}$, the compactification with the ideal boundary. We shall call these points the points of infinity of γ. If $|k| < \sqrt{c}$ they are distinct, and they coincide if $k = \pm\sqrt{c}$. For example, on $CH^n(-4)$ we have

$$\lim_{t\to\infty}\Phi\circ\gamma(t) = \left(\frac{2z_1 + (ki + \sqrt{4-k^2})u_1}{2z_0 + (ki+\sqrt{4-k^2})u_0}, \cdots, \frac{2z_n + (ki+\sqrt{4-k^2})u_n}{2z_0 + (ki+\sqrt{4-k^2})u_0}\right),$$

$$\lim_{t\to-\infty}\Phi\circ\gamma(t) = \left(\frac{2z_1 + (ki - \sqrt{4-k^2})u_1}{2z_0 + (ki-\sqrt{4-k^2})u_0}, \cdots, \frac{2z_n + (ki-\sqrt{4-k^2})u_n}{2z_0 + (ki-\sqrt{4-k^2})u_0}\right)$$

if $|k| < 2$, and

$$\lim_{t\to\infty}\Phi\circ\gamma(t) = \lim_{t\to-\infty}\Phi\circ\gamma(t) = \left(\frac{z_1 \pm iu_1}{z_0 \pm iu_0}, \cdots, \frac{z_n \pm iu_n}{z_0 \pm iu_0}\right)$$

if $k = \pm 2$. When $k = \pm\sqrt{c}$ we consider the geodesic ρ with $\dot\rho(0) = d\pi((z, \pm Ju))$. On $CH^n(-4)$ it satisfies the equation $\rho(t) = \pi(\cosh t\cdot z \pm \sinh t\cdot Ju)$, hence $\lim_{t\to\infty}\Phi\circ\rho(t) = \lim_{t\to\infty}\Phi\circ\gamma(t)$. This means that if γ crosses to a geodesic going to the single point of infinity then they cross orthogonally. With these properties we may call trajectories for

$\pm\sqrt{c}\cdot \boldsymbol{B}_J$ are horocyclic.

We denote by $\boldsymbol{n}$ the outer normal vector of $\overline{D_n(\boldsymbol{C})}$ at a point p of infinity for γ, and by $\boldsymbol{m}$ the outer tangent vector at p of the circle which is the image $\Phi\circ\gamma$. We shall call the angle between $\boldsymbol{n}$ and $\boldsymbol{m}$ the *magnetic angle* of γ at p. The magnetic angle of trajectories for $k\cdot\boldsymbol{B}_J$ do not depend on the choice of trajectories and on their points at infinity: the angle is $\cos^{-1}\sqrt{(c-k^2)/c}$. Summarizing up we have

PROPOSITION 3. *On $\boldsymbol{C}H^n(-c)$ trajectories for the Kähler magnetic field $\pm\sqrt{c}\cdot\boldsymbol{B}_J$ are horocyclic, and trajectories for $k\cdot\boldsymbol{B}_J$ with $|k|<\sqrt{c}$ have distinct points at infinity.*

For given two distinct points on $\partial D_n(\boldsymbol{C})$ we have a unique complex plane containing them. We therefore find out that the property in Proposition 2 is inherited to the ideal boundary.

PROPOSITION 4. *Consider the Kähler magnetic field $k\cdot\boldsymbol{B}_J$ with $|k|<\sqrt{c}$ on a complex hyperbolic space $\boldsymbol{C}H^n(-c)$. Given two distinct points at infinity we have two and only two trajectories for $k\cdot\boldsymbol{B}_J$ joining these points.*

§2. Kähler magnetic flows.

In the previous section we found that trajectories have similar properties as those of geodesics. By use of the explicit expression of trajectories we shall study Kähler magnetic flows, magnetic flows associated with Kähler magnetic fields. Our main result is the following.

THEOREM 1. *Let $\boldsymbol{B}=k\cdot\boldsymbol{B}_J$ be a Kähler magnetic field on a complex hyperbolic space $\boldsymbol{C}H^n(-c)$ of holomorphic sectional curvature $-c$. The feature of the Kähler magnetic flow $\boldsymbol{B}\varphi_t$ depends on the strength $|k|$.*

(1) *If $|k|<\sqrt{c}$, then the Kähler magnetic flow $\boldsymbol{B}\varphi_t$ is smoothly conjugate to the geodesic flow φ_t in the strong sense;*

$$f_k^{-1}\circ\boldsymbol{B}\varphi_t\circ f_k=\varphi_{\sqrt{c-k^2}\,t/\sqrt{c}}$$

for some diffeomorphism f_k on $UCH^n(-c)$.

(2) *If $|k|>\sqrt{c}$, then $\boldsymbol{B}\varphi_t$ is smoothly conjugate to the rotation flow $\boldsymbol{B}_0\varphi_t$, where $\boldsymbol{B}_0=\sqrt{2c}\cdot\boldsymbol{B}_J$ in the strong sense;*

$$f_k^{-1}\circ\boldsymbol{B}\varphi_t\circ f_k=\boldsymbol{B}_0\varphi_{\sqrt{k^2-c}\,t/\sqrt{c}}\qquad \text{if}\quad k>\sqrt{c}\,,$$
$$f_k^{-1}\circ\boldsymbol{B}\varphi_t\circ f_k=\boldsymbol{B}_0\varphi_{-\sqrt{k^2-c}\,t/\sqrt{c}}\qquad \text{if}\quad k<-\sqrt{c}$$

for some diffeomorphism f_k on $UCH^n(-c)$.

(3) *When $k=\pm\sqrt{c}$, the Kähler magnetic flow $\boldsymbol{B}\varphi_t$ is so called the horocyclic flow, and is not smoothly conjugate in the strong sense to other magnetic flows for Kähler magnetic fields of strength not equal to $\sqrt{c}$.*

PROOF. As usual we only treat the case $c=4$. We first note that the geodesic flow φ_t on $UCH^n(-4)$ is represented by $\varphi_t(d\pi(\begin{pmatrix} z \\ u \end{pmatrix}))=d\pi(A_0(t)\begin{pmatrix} z \\ u \end{pmatrix})$ with the matrix

$$A_0(t)=\begin{pmatrix} \cosh t\cdot I & \sinh t\cdot I \\ \sinh t\cdot I & \cosh t\cdot I \end{pmatrix}\in \mathrm{Mat}(2(n+1); \boldsymbol{C}),$$

where $\begin{pmatrix} z \\ u \end{pmatrix}$ denotes the transposed vector of (z, u) and $I\in \mathrm{Mat}(n+1, \boldsymbol{C})$ denotes the identity matrix. We define $A_k(t)\in \mathrm{Mat}(2(n+1); \boldsymbol{C})$ by

$$A_k(t)=$$
$$\begin{cases} \cosh\sqrt{4-k^2}\,t/2\cdot\begin{pmatrix} I & O \\ O & I \end{pmatrix}+\dfrac{\sinh\sqrt{4-k^2}\,t/2}{\sqrt{4-k^2}}\cdot\begin{pmatrix} -kiI & 2I \\ 2I & kiI \end{pmatrix}, & \text{if } |k|<2, \\ \begin{pmatrix} (1\mp it)I & tI \\ tI & (1\pm it)I \end{pmatrix} & \text{if } k=\pm 2, \\ \cos\sqrt{k^2-4}\,t/2\cdot\begin{pmatrix} I & O \\ O & I \end{pmatrix}+\dfrac{\sin\sqrt{k^2-4}\,t/2}{\sqrt{k^2-4}}\cdot\begin{pmatrix} -kiI & 2I \\ 2I & kiI \end{pmatrix}, & \text{if } |k|>2, \end{cases}$$

where $O\in \mathrm{Mat}(n+1, \boldsymbol{C})$ denotes the zero matrix. By Proposition 1 we find that the Kähler magnetic flow $\boldsymbol{B}\varphi_t$ for $\boldsymbol{B}=k\cdot \boldsymbol{B}_J$ is represented by

$$\boldsymbol{B}\varphi_t\left(d\pi\left(\begin{pmatrix} z \\ u \end{pmatrix}\right)\right)=d\pi\left(e^{kit/2}\cdot A_k(t)\begin{pmatrix} z \\ u \end{pmatrix}\right)=d\pi\left(A_k(t)\begin{pmatrix} z \\ u \end{pmatrix}\right).$$

We first treat the case that $k>2$. Put $\varepsilon_k=(\sqrt{k^2-4}+k)^{1/2}/\sqrt{2}\in \boldsymbol{R}$ and set

$$P_k=\frac{1}{(k^2-4)^{1/4}}\cdot\begin{pmatrix} -i/\varepsilon_k\cdot I & \varepsilon_k\cdot I \\ \varepsilon_k\cdot I & i/\varepsilon_k\cdot I \end{pmatrix},$$

then we have

$$P_k^{-1}\cdot A_k(t)\cdot P_k$$
$$=\begin{pmatrix} (\cos\sqrt{k^2-4}\,t/2+i\sin\sqrt{k^2-4}\,t/2)\cdot I & O \\ O & (\cos\sqrt{k^2-4}\,t/2-i\sin\sqrt{k^2-4}\,t/2)\cdot I \end{pmatrix}.$$

Hence we get $Q_k^{-1}\cdot A_k(t)\cdot Q_k=A_{2\sqrt{2}}(\sqrt{k^2-4}\,t/2)$ with $Q_k=P_k\cdot P_{2\sqrt{2}}^{-1}$. Since Q_k acts on the horizontal subbundle $\mathscr{H}H_1^{2n+1}$ and is commutative with the S^1-fiber action, it induces a diffeomorphism f_k on $UCH^n(-4)$ such that

$$f_k^{-1}\circ \boldsymbol{B}\varphi_t\circ f_k=\boldsymbol{B}_0\varphi_{\sqrt{k^2-4}\,t/2},$$

where $\boldsymbol{B}_0 = 2\sqrt{2} \cdot \boldsymbol{B}_J$. As $(-\boldsymbol{B})\varphi_t(v) = -\boldsymbol{B}\varphi_{-t}(-v)$, we have $A_{-k}(t) = \begin{pmatrix} I & O \\ O & -I \end{pmatrix} A_k(-t) \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$, hence we can find for $k < -2$ a diffeomorphism f_k on $UCH^n(-4)$ such that

$$f_k^{-1} \circ \boldsymbol{B}\varphi_t \circ f_k = \boldsymbol{B}_0 \varphi_{-\sqrt{k^2-4}\,t/2} .$$

Next we treat the case that $|k| < 2$. Put $\varepsilon_k = (2 - \sqrt{4-k^2})^{1/2}$ and define $Q_k \in \mathrm{Mat}(2(n+1); \boldsymbol{C})$ by

$$Q_k = \frac{1}{\sqrt{2}\,(4-k^2)^{1/4}} \cdot \begin{pmatrix} k/\varepsilon_k \cdot I & -i\varepsilon_k \cdot I \\ i\varepsilon_k \cdot I & k/\varepsilon_k \cdot I \end{pmatrix} .$$

We then have

$$Q_k^{-1} \cdot A_k(t) \cdot Q_k = A_0(\sqrt{4-k^2}\,t/2) .$$

Since we can easily check that Q_k acts on $\mathscr{H}H_1^{2n+1}$ and is commutative with the S^1-fiber action, we can conclude that there exists a diffeomorphism f_k on $UCH^n(-4)$ satisfying

$$f_k^{-1} \circ \boldsymbol{B}\varphi_t \circ f_k = \varphi_{\sqrt{4-k^2}\,t/2} ,$$

hence we obtain Theorem 2.

We now make mention of the hyperbolicity of magnetic flows. We shall call a flow ψ_t on a manifold N satisfies the *hyperbolic* condition if the tangent bundle TN has a continuous splitting $TN = E^t \oplus E^s \oplus E^u$ into three $d\psi_t$-invariant subbundles with the following properties. The line bundle E^t is tangent to the flow, and for the stable and unstable subbundles E^s and E^u, there exist positive constants C, λ such that for $t \geq 0$

$$\|d\psi_t(\xi)\| \leq Ce^{-\lambda t}\|\xi\| \quad \text{if } \xi \in E^s , \qquad \|d\psi_{-t}(\xi)\| \leq Ce^{-\lambda t}\|\xi\| \quad \text{if } \xi \in E^u .$$

It is well-known that the geodesic flow on UM of a complete Riemannian manifold M satisfies the hyperbolic condition if M is of bounded negative curvature. Since the action of Q_k on $\mathscr{H}H_1^{2n+1}$ (in the proof of Theorem 1) is commutative with the induced actions of isometries on $\boldsymbol{C}H^n$, we find that the classification of Kähler magnetic flows holds for a complete manifold of constant negative holomorphic sectional curvature. Therefore we have

COROLLARY. *Let M be a complete Kähler manifold of holomorphic sectional curvature $-c$. If the strength of the Kähler magnetic field $\boldsymbol{B} = k \cdot \boldsymbol{B}_J$ is smaller than $\sqrt{c}$, then the magnetic flow $\boldsymbol{B}\varphi_t$ satisfies the hyperbolic condition. When M is compact, its topological entropy is $\sqrt{c-k^2}/2$.*

Finally we concern with Kähler magnetic flows on a complex projective space $\boldsymbol{C}P^n(c)$ of holomorphic sectional curvature c. Let $\tilde{\pi}$: $S^{2n+1} \to \boldsymbol{C}P^n$ denote the Hopf fibration. The tangent bundle $T\boldsymbol{C}P^n$ can be denoted as $\{d\tilde{\pi}((z, u)) \mid z \in S^{2n+1} \subset \boldsymbol{C}^{n+1}, u \in$

$C^{n+1}, ((z, u))=0\}$. By the similar argument as in section 1, we obtain an explicit expression of trajectories for Kähler magnetic fields on $CP^n(c)$ (see [1]). We find that Kähler magnetic flow $\boldsymbol{B}\varphi_t$ for $\boldsymbol{B}=k\cdot\boldsymbol{B}_J$ on $UCP^n(4)$ is represented as

$$\boldsymbol{B}\varphi_t\left(d\pi\left(\begin{pmatrix} z \\ u \end{pmatrix}\right)\right)=d\pi\left(e^{-kit/2}\cdot\tilde{A}_k(t)\begin{pmatrix} z \\ u \end{pmatrix}\right)=d\pi\left(\tilde{A}_k(t)\begin{pmatrix} z \\ u \end{pmatrix}\right)$$

with the matrix $\tilde{A}_k\in\mathrm{Mat}(2(n+1), \boldsymbol{C})$ defined by

$$\tilde{A}_k(t)=\cos\sqrt{k^2+4}\,t/2\cdot\begin{pmatrix} I & O \\ O & I \end{pmatrix}+\frac{\sin\sqrt{k^2+4}\,t/2}{\sqrt{k^2+4}}\cdot\begin{pmatrix} kiI & 2I \\ -2I & -kiI \end{pmatrix}.$$

This flow is a rotation flow with prime period $2\pi/\sqrt{k^2+4}$. Putting $\tilde{\varepsilon}_k=(\sqrt{k^2+4}+k)^{1/2}/\sqrt{2}$ and

$$\tilde{P}_k=\frac{1}{(k^2+4)^{1/4}}\cdot\begin{pmatrix} i\tilde{\varepsilon}_k\cdot I & i/\tilde{\varepsilon}_k\cdot I \\ -1/\tilde{\varepsilon}_k\cdot I & \tilde{\varepsilon}_k\cdot I \end{pmatrix},$$

we have

$$\begin{aligned}&\tilde{P}_k^{-1}\cdot\tilde{A}_k(t)\cdot\tilde{P}_k\\ &=\begin{pmatrix} (\cos\sqrt{k^2+4}\,t/2+i\sin\sqrt{k^2+4}\,t/2)\cdot I & O \\ O & (\cos\sqrt{k^2+4}\,t/2-i\sin\sqrt{k^2+4}\,t/2)\cdot I \end{pmatrix}.\end{aligned}$$

Hence we get $\tilde{Q}_k^{-1}\cdot\tilde{A}_k(t)\cdot\tilde{Q}_k=\tilde{A}_0(\sqrt{k^2+4}\,t/2)$ with $\tilde{Q}_k=\tilde{P}_k\cdot\tilde{P}_0^{-1}$. One can easily check that $\tilde{Q}_k$ acts on the horizontal subbundle $\mathscr{H}S^{2n+1}$ which is identified with TCP^n. Since $\tilde{A}_0(t)$ corresponds to the geodesic flow φ_t and $\tilde{Q}_k$ is commutative with the S^1-fiber action, it induces a diffeomorphism f_k on $UCP(4)$ such that $f_k^{-1}\circ\boldsymbol{B}\varphi_t\circ f_k=\varphi_{\sqrt{k^2+4}\,t/2}$. We therefore get the following.

THEOREM 2. *Let $\boldsymbol{B}=k\cdot\boldsymbol{B}_J$ be a Kähler magnetic field on a complex projective space $CP^n(c)$ of constant holomorphic sectional curvature c. The Kähler magnetic flow $\boldsymbol{B}\varphi_t$ is smoothly conjugate to the geodesic flow φ_t, which is a rotation flow in the strong sense. More precisely, we can find a diffeomorphism f_k on the unit sphere bundle $UCP^n(c)$ such that $f_k^{-1}\circ\boldsymbol{B}\varphi_t\circ f_k=\varphi_{\sqrt{k^2+c}\,t/\sqrt{c}}$.*

Since the action of $\tilde{Q}_k$ on $\mathscr{H}S^{2n+1}$ is commutative with the induced actions of isometries on CP^n, we have that for a complete manifold of positive holomorphic sectional curvature every two Kähler magnetic flows are smoothly conjugate in the strong sense.

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