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Braid Type of the Fixed Point Set for Orientation-Preserving Embeddings on the Disk

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Abstract. We consider orientation-preserving embeddings of the disk having seven or less fixed points all of which are transversal. We classify all the braid types which can be realized as the braid type of the fixed point set for such an embedding.

1. Introduction.

In this paper, we study the topological structure of the fixed point set for orientation-preserving embeddings of the two-dimensional disk.

The braid is a useful invariant for the topological characterization of a collection of fixed points, or more generally of a collection of periodic points, for such an embedding f. Many authors have investigated the role of the braid invariant in the study of dynamical systems on the disk, and obtained various results concerned with the relationship between the braid type of a given set of periodic points and the dynamical complexity of f (e.g. [3, 7, 8, 9, 11, 13, 14]).

In this paper, we study braid types from a different point of view: We limit ourselves to embeddings having only transversal fixed points. Then the whole set $\mathrm{Fix}(f)$ of its fixed points is a finite set, and its braid type can be defined. We consider the question of what kind of braid type occurs as the braid type of the fixed point set $\mathrm{Fix}(f)$ for such an embedding f. In other words, our objective is to ask for conditions on the braid type of a collection of fixed points which ensure the existence of some other fixed point.

Here, we give an answer to this problem in the case where f has seven or less fixed points. The main result (Theorem 1) shows that such braid types are exactly the braid types obtained from the braid type with only one string by applying the operation which twines a pair of parallel "cables" around one string repeatedly. In particular, it shows that all of them do not contain pseudo-Anosov components.

The main tools used in the proof are Nielsen fixed point theory and a result proved

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in [17] concerning the relationship between three topological invariants for fixed points—the fixed point index, the torsion number, and the braid type.

2. Braid types.

In this section, we consider the necessary preliminaries on the braid and the braid type. For general references of the braid theory, see [1], [10], or [18].

Let n be a positive integer. Let V_n (resp. W_n) denote the set of ordered (resp. unordered) n-tuples of distinct points in the plane $\boldsymbol{R}^2$, i.e.,

$$V_n=\{(x_1, \cdots, x_n) \mid x_i \in \boldsymbol{R}^2 \text{ for } i=1, \cdots, n, x_i \neq x_j \text{ if } i \neq j\} ,$$

$$W_n=\{S \subset \boldsymbol{R}^2 \mid \# S=n\} ,$$

where # denotes the cardinality. We equip V_n with the topology induced from the product topology on $\boldsymbol{R}^{2n}$. Define a surjective map $\pi: V_n \to W_n$ by

$$\pi((x_1, \cdots, x_n))=\{x_1, \cdots, x_n\} .$$

Then since π is surjective, we can give W_n the identification topology determined by this map π. It is easy to see that the map π becomes an $n!$-fold regular covering map.

The fundamental group $\pi_1(W_n)$ of W_n is called the n-braid group and denoted by B_n. An element of B_n is called an *n-braid*, or simply a braid. Thus a braid is defined to be a homotopy class of loops in W_n with base point being fixed.

On the other hand, if we allow the base point to vary in W_n during the homotopy, we obtain the notion of a "braid type". We say two loops l, l': $[0, 1] \to W_n$ are *freely homotopic*, if there is a family l_μ: $[0, 1] \to W_n$ of loops in W_n with $l_0=l$, $l_1=l'$ which depends continuously on the parameter $0 \leq \mu \leq 1$. We call a free homotopy class of loops in W_n an *n-braid type*, and denote by BT_n the set of all n-braid types. This set BT_n can be identified with the set of all conjugacy classes of the group B_n. In fact, the surjective map from B_n to BT_n which is defined by sending the homotopy class of a loop in W_n into its free homotopy class induces a bijective correspondence between the set of conjugacy classes of B_n and the set BT_n, since a change of base point in W_n corresponds to a conjugation in B_n. For $b \in B_n$, we denote by $[b]$ the braid type corresponding to the conjugacy class of b.

Let P_n denote the image of the homomorphism $\pi_*: \pi_1(V_n) \to B_n$ induced by the covering map $\pi: V_n \to W_n$. We call it the pure n-braid group and its element a *pure n-braid.* It is easy to see that an n-braid represented by a loop l is a pure braid if and only if any lifting of l to V_n is also a loop. A braid type which is represented by a pure n-braid is called a *pure n-braid type*. Let PT_n denote the set of all pure n-braid types.

Let l be a loop in W_n. Then since l can be lifted to the covering space V_n, there are continuous paths $x_1(t), \cdots, x_n(t)$ in the plane such that $\{x_1(t), \cdots, x_n(t)\}=l(t)$. Denote by $A(i)$ the embedded arc $\{(x_i(t), t) \mid 0 \leq t \leq 1\}$ in $\boldsymbol{R}^2 \times [0, 1]$, and call it the *string* corresponding to the path $x_i(t)$. Also, we call the collection $\mathscr{A}=\{A(1), \cdots, A(n)\}$ of the strings a *geometric n-braid* corresponding to the loop l. If $\mathscr{A}$ is a geometric braid, we denote by $\mathrm{bt}(\mathscr{A})$ the braid type represented by a loop l which corresponds to $\mathscr{A}$.

Define the full twist braid $\theta_n \in B_n$ as the braid represented by the loop $R_t(S)$ $(0 \leq t \leq 1)$, where R_t is the rigid rotation of the disk with angle $2\pi t$ and $S \in W_n$ is the base point of the fundamental group B_n. It is known that if $n \geq 3$, the center $Z(B_n)$ of B_n coincides with the infinite cyclic group generated by θ_n (see [1, Corollary 1.8.4]). Let $Z(B_n)$ act on BT_n by $\theta_n \cdot [b]=[\theta_n b]$, where $b \in B_n$.

The group structure of B_n is given as follows ([1, Theorem 1.8]): We choose $E_n=\{e_1, \cdots, e_n\}$ as the base point of B_n, where $e_i=(i, 0)$. For $1 \leq i \leq n-1$, let σ_i be the elementary braid represented by the geometric n-braid in which the i-th string just overcrosses the $(i+1)$-th string once and all other strings go straight from the top to the bottom. Then B_n has a presentation with generators $\sigma_1, \cdots, \sigma_{n-1}$ and relations $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j=\sigma_j\sigma_i$ where $|i-j|>1$. It is clear that $\theta_n=(\sigma_1 \cdots \sigma_{n-1})^n$. Also, the group P_n of pure n-braids coincides with the kernel of the homomorphism from B_n to the group of all permutations on the set $\{1, \cdots, n\}$ which sends σ_i into the transposition of i and $i+1$.

3. Main result.

Let $f\colon D \to D$ be an orientation-preserving C^1-embedding of the two-dimensional disk D. We say a fixed point x of f is *transversal* if x is in the interior of D and the differential $Df(x)$ does not have an eigenvalue 1, i.e., the graph of f in $D \times D$ is transversal to the diagonal.

Throughout the remainder of this paper, we assume that every fixed point of f is transversal. Under this assumption, f has a finite number of fixed points, since each fixed point is isolated. Moreover, the number of fixed points is odd, because the fixed point index $\mathrm{ind}(x, f)$ of each fixed point x is known to be $(-1)^u$, where u is the number (counted with multiplicity) of real eigenvalues of $Df(x)$ greater than 1 (see e.g. [12, p. 12]), and the sum of the fixed point indices for all x must be equal to the euler characteristic of the disk, that is 1.

We define the notion of the braid type of a set of fixed points for f. Let $f_t\colon D \to D$ $(0 \leq t \leq 1)$ be an isotopy with $f_0=\mathrm{id}$, $f_1=f$, where id denotes the identity map. Such an isotopy always exists, since we can isotope f to a homeomorphism of D which fixes the boundary ∂D of D pointwise and therefore obtain the isotopy $\{f_t\}$ by the Alexander trick (see [1, Lemma 4.4.1]). Let S be a set of fixed points of f. Then S has a finite number, say n, of elements, and the path $f_t(S)$, $0 \leq t \leq 1$, in W_n becomes a loop with base point S.

Definition 1. We denote by $\mathrm{bt}(S; \{f_t\})$ the pure braid type represented by the loop $f_t(S)$ and call it the *braid type* of S with respect to the isotopy $\{f_t\}$. In the case where the isotopy $\{f_t\}$ is fixed, the braid type of S will be abbreviated to bt(S, f) or bt(S).

The definition of the braid type of S depends on the choice of the isotopy $\{f_t\}$ as follows: Let $\{f_t\}$ and $\{f'_t\}$ be isotopies from id to f. Then it is known that there is an integer k such that two isotopies $\{R_{kt} \circ f_t\}$ and $\{f'_t\}$ are homotopic through isotopies from id to f. This easily implies that

$$\mathrm{bt}(S; \{f'_t\}) = \theta_n^k \cdot \mathrm{bt}(S; \{f_t\}) .$$

In particular, the braid type of S is unique up to multiples of the full twist braid.

In this paper, we consider the question of what kind of braid types appear as the braid type of the fixed point set Fix(f) of f. To state the main theorem, we define an operation (C) which gives a new pure braid type $\beta' \in PT_{n+2}$ from a pure braid type $\beta \in PT_n$. Choose a geometric n-braid $\mathscr{A}$ which represents β. Let $\mathscr{A}'$ be the geometric $(n+2)$-braid which is obtained by first splitting one string of $\mathscr{A}$ into three parallel strings and then applying a number of full twists to the three parallel strings. (The number of full twists may be zero.) Define β' to be the braid type which is represented by $\mathscr{A}'$. Any β' obtained in this way is called a braid type obtained from β by applying the operation (C). Roughly speaking, to apply this operation (C) means to twine a pair of parallel cables around a string of a given braid type. Figure 1 gives an example of the braid types β' obtained from $\beta = [\sigma_1^2] \in PT_3$ by applying the operation (C).

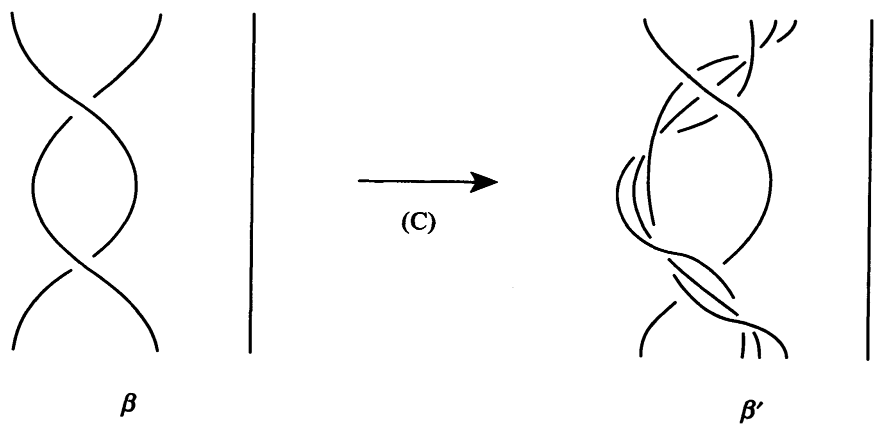

Figure 1

It is trivial that the resulting braid type β' depends on the number of full twists applied to the three parallel strings. β' also depends on the choice of the string. For instance, let $\beta(b, i, r)$ be the braid type obtained from a braid type $[b]$, where $b \in P_n$, by applying the operation (C) where we choose the string containing e_i as the string to be splitted and choose an integer r as the number of applied full twists. Let $b = \sigma_1^2 \in P_3$. Then, we see $\beta(b, 3, 0) = [\sigma_1^2] \in PT_5$. This is clearly different from

$$\beta(b, 1, 0) = \beta(b, 2, 0) = [\sigma_3 \sigma_2 (\sigma_1)^2 \sigma_2 \sigma_3] \in PT_5 .$$

The following theorem characterizes the braid types which can be realized as the braid type of Fix(f) for f having seven or less fixed points.

THEOREM 1. *Let β be a pure n-braid type, where $n=1, 3, 5$, or 7. Then the following conditions are equivalent.*

(1) *β is realized as the braid type of the fixed point set* Fix(f) *for some orientation-preserving C^1-embedding f of the disk having exactly n fixed points all of which are transversal.*

(2) *β is obtained from the* 1*-braid type by applying the operation* (C) *repeatedly.*

This theorem is an immediate consequence of Theorem 2 in the next section.

We close this section by giving the braid types satisfying the condition (2) of the above theorem explicitely. Let Cable(n) be the set of such pure n-braid types. Here, we choose E_n as the base point of B_n, and consider B_n as a subgroup of B_{n+2} by regarding $b\in B_n$ as an $(n+1)$-braid by adding $(n+1)$-th and $(n+2)$-th straight strings to b. Let e be the unit element of B_n, and let $\overline{Z(B_3)}$ be the subgroup of B_7 generated by $\bar{\theta}_3=(\sigma_5\sigma_6)^3$. For a subgroup H of B_n, let $[H]$ denote the set of braid types $[b]$ for all $b\in H$.

PROPOSITION 1. (1) $\mathrm{Cable}(1)=\{[e]\}$,

(2) $\mathrm{Cable}(3)=[Z(B_3)]=\{[\theta_3^k] \mid k\in \boldsymbol{Z}\}$,

(3) $\mathrm{Cable}(5)=[Z(B_3)\cdot Z(B_5)]=\{[\theta_3^k\theta_5^l] \mid k, l\in \boldsymbol{Z}\}$,

(4) $\mathrm{Cable}(7)=[Z(B_3)\cdot Z(B_5)\cdot Z(B_7)]\cup[Z(B_3)\cdot \overline{Z(B_3)}\cdot Z(B_7)]$.

PROOF. (1) and (2) are trivial. Note that β belongs to Cable(n) if and only if β is obtained from some element of Cable($n-2$) by applying the operation (C). Therefore we have

$$\mathrm{Cable}(5)=\{\beta(\theta_3^l, i, r) \mid i=1, 2, 3, \text{ and } l, r\in \boldsymbol{Z}\}\ .$$

Since $\beta(\theta_3^l, i, r)=[\theta_3^{r-l}\theta_5^l]$ for any i, we have (3).

Similarly since

$$\mathrm{Cable}(7)=\{\beta(\theta_3^k\theta_5^l, i, r) \mid 1\le i\le 5,\ k, l, r\in \boldsymbol{Z}\}\ ,$$

and $\beta(\theta_3^k\theta_5^l, i, r)$ is equal to $\theta_3^{r-k-l}\theta_5^k\theta_7^l$ if $i\le 3$, and $\theta_3^k\bar{\theta}_3^{r-l}\theta_7^l$ if $i\ge 4$, we have (4). □

REMARK. There is an alternative but equivalent way of defining a braid type: An n-braid type is defined as a conjugacy class of the group of isotopy classes of orientation-preserving homeomorphisms on the n-punctured disk which fix the outer boundary pointwise (see e.g. Boyland [5]). Thus the braid types can be classified according to the Nielsen-Thurston theory [19] of classification of surface homeomorphisms up to isotopy. It is clear that any element of Cable(n) has no pseudo-Anosov component. (In fact, any element of Cable(n) is equal to a braid type which corresponds to a "disk tree" consisting of $(n-1)/2$ copies of the diffeomorphism $a_{1,3,0}$ introduced by Llibre and MacKay [13].) Therefore, Theorem 1 implies that if f has at most seven fixed points and all of them are transversal, then the braid type of the fixed point set has no pseudo-Anosov component. This result fails for $n=9$. In fact, let g_1, g_2 be the generalized horseshoe diffeomorphisms introduced in [15, p. 63]. They have $E_3=\{e_1, e_2, e_3\}$ as an attractive invariant set and the braids of E_3 determined by g_1 and g_2 are equal to σ_1 and σ_2^{-1} respectively. Let $f=g_1^2\circ g_2^2$. Then f has nine fixed points all of which are hyperbolic. However, bt(Fix(f)) has a pseudo-Anosov component, since the braid type bt(E_3, f) is clearly equal to the pseudo-Anosov braid type $[\sigma_1^2\sigma_2^{-2}]$.

4. Indexed braid types.

We first define the notion of an "indexed" braid type, which is necessary to state Theorem 2. Let $\boldsymbol{Z}^*$ denote the integers $\boldsymbol{Z}$ with a symbol $\{*\}$ added, i.e.,

$$\boldsymbol{Z}^*=\boldsymbol{Z}\cup\{*\}.$$

We call a pair (l, τ) of a loop l in W_n and a map $\tau: l(0)\to\boldsymbol{Z}^*$ an *indexed loop* in W_n. In other words, an indexed loop is a loop in W_n where an index having a value in the set $\boldsymbol{Z}^*$ is assigned to each point of the subset $l(0)$ of the plane. Two indexed loops (l, τ) and (l', τ') in W_n are said to be *homotopic* if there exists a family $\{(l_\mu, \tau_\mu)\}$, $0\le\mu\le 1$, of indexed loops in W_n with $(l_0, \tau_0)=(l, \tau)$, $(l_1, \tau_1)=(l', \tau')$ such that

(i) $\{l_\mu\}$ is a free homotopy of loops in W_n, and

(ii) $\tau_\mu(x_i(\mu))$ is constant with respect to μ for each $i=1, \cdots, n$, where $x_1(\mu), \cdots, x_n(\mu)$ are continuous paths in $\boldsymbol{R}^2$ with $\{x_1(\mu), \cdots, x_n(\mu)\}=S_\mu$. (Note that such paths always exist, since S_μ is a continuous path in W_n with respect to μ.)

We denote by $[l, \tau]$ the homotopy class of (l, τ), and call it an *indexed n-braid type*. Thus an indexed braid type is considered as a braid type with index assigned to each string.

In the case where both l and l' represent pure braid types, a necessary and sufficient condition for two indexed loops (l, τ) and (l', τ') to be homotopic is that there is a homotopy $\{\tilde{l}_\mu\}$ of loops in V_n such that

$$\pi\circ\tilde{l}_0=l, \quad \pi\circ\tilde{l}_1=l', \quad \tau(x_i(0, 0))=\tau'(x_i(0, 1)),$$

where $x_i(t, \mu)$ is the i-th coordinate of $\tilde{l}_\mu(t)\in V_n$.

We introduce another expression of an indexed braid type which is more convenient in some cases. Choose a base point S of B_n, and arrange the points in S as $x_1, \cdots, x_n$. For $b\in B_n$ and an n-tuple $J=(j_1, \cdots, j_n)$ of elements of $\boldsymbol{Z}^*$, let $[b, J]$ denote the indexed braid type represented by (l, τ_J), where l is a loop based at S representing b and τ_J is a map from S to $\boldsymbol{Z}^*$ defined by $\tau_J(x_i)=j_i$. It is easy to see that any indexed braid type can be expressed in this form $[b, J]$. Note that J is not unique, as the following example shows: Let $b=\sigma_1^2\sigma_3^2\in B_4$. Then

$$[b, J] = [b, (j_{\nu(1)}, \cdots, j_{\nu(4)})]\,,$$

for any permutation ν on the set $\{1, 2, 3, 4\}$ with $\{\nu(1), \nu(2)\} = \{1, 2\}$ or $\{3, 4\}$.

If $n \geq 3$, we define an action of the center $Z(B_n) = \{\theta_n^k \mid k \in Z\}$ on the set of indexed braid types as follows:

$$\theta_n^k \cdot [b, J] = [\theta_n^k b, (j'_1, \cdots, j'_n)]\,,$$

where $j'_i = j_i + k$ if $j_i \in \boldsymbol{Z}$ and $*$ if $j_i = *$.

Recall that every fixed point of the embedding f is assumed to be transversal and hence its fixed point index $\mathrm{ind}(x, f)$ is always equal to 1 or -1. We call a fixed point x of f a *positive* (resp. *negative*) fixed point if $\mathrm{ind}(x, f) = 1$ (resp. -1). Let $\mathrm{Fix}_+(f)$ (resp. $\mathrm{Fix}_-(f)$) be the set of all positive (resp. negative) fixed points of f.

We give some examples of positive and negative fixed points. Let x be a fixed point of f, and λ_1, λ_2 denote the eigenvalues of the differential $Df(x)$ with $|\lambda_2| \leq |\lambda_1|$. The fixed point x is called a sink, a source, a twisted saddle, or an untwisted saddle if $|\lambda_1| < 1$, $|\lambda_2| > 1$, $\lambda_1 < -1 < \lambda_2 < 0$, or $0 < \lambda_2 < 1 < \lambda_1$ respectively. Then a sink, a source, and a twisted saddle are examples of positive fixed points, since $u = 0$ or 2 for such x. On the other hand, it is easily shown that a fixed point is negative if and only if it is an untwisted saddle.

We next define a topological invariant for negative fixed points which we call the "torsion number". Let x be a negative fixed point. Choose an arbitrary nonzero eigenvector v in the stable (or equivalently unstable) eigenspace of $Df(x)$. This is possible since x must be a saddle. Then the path $[0, 1] \ni t \to Df_t(x)v / |Df_t(x)v|$ becomes a loop in the unit circle. We call the topological degree of this loop the *torsion number* of x and denote it by $\mathrm{tor}(x, \{f_t\})$ or by $\mathrm{tor}(x, f)$. Thus the torsion number counts the number of rotations of eigenvectors of $Df(x)$ around x while t varies from 0 to 1. It is trivial this definition does not depend on the choice of v.

We generalize the definition of the torsion number to positive fixed points by putting $\mathrm{tor}(x, f) = *$ for positive x. By this generalization, the torsion number is improved to contain the information on the fixed point indices as well. Denote by

$$\mathrm{tor}(f) : \mathrm{Fix}(f) \to \boldsymbol{Z}^*$$

the map defined by $\mathrm{tor}(f)(x) = \mathrm{tor}(x, f)$.

Now we introduce the notion of an indexed braid type of a fixed point set which combines all the information on the fixed point indices, the torsion numbers, and the braid type.

DEFINITION 2. Let S be a subset of $\mathrm{Fix}(f)$. Define the indexed braid type of S denoted by $\mathrm{bt}(S; \{f_t\})^*$ as the indexed braid type represented by the loop $f_t(S)$ together with the map $\mathrm{tor}(f)|_S : S \to \boldsymbol{Z}^*$. When the isotopy $\{f_t\}$ is fixed, we write it simply by $\mathrm{bt}(S, f)^*$ or by $\mathrm{bt}(S)^*$.

Likewise the braid type of S, the indexed braid type of S is also unique up to multiplication by full twist braids. More precisely, if $\mathrm{bt}(S; \{f'_t\}) = \theta_n^k \mathrm{bt}(S; \{f_t\})$, then $\mathrm{bt}(S; \{f'_t\})^* = \theta_n^k \mathrm{bt}(S; \{f_t\})^*$.

We define an operation (C)* which gives a new indexed pure $(n+2)$-braid type β'^* from an indexed pure n-braid type β^* which has at least one string with index $*$. Let $\mathscr{A}$ be a geometric braid which represents β^*. Choose a string of $\mathscr{A}$ with index $*$ assigned. Split this string into three parallel strings and then apply a number, say r, of full twists to these three parallel strings. We assign the index $*$ to two of these three parallel strings, and the index r to the other one of them. We do not change the indices of the other strings. Let β'^* be the indexed braid type obtained through this operation.

We have the following theorem concerning the indexed braid type of the fixed point set.

THEOREM 2. *Let β^* be an indexed pure n-braid type, where $n = 1, 3, 5,$ or 7. Then the following two conditions are equivalent.*

(1) *β^* is realized as the indexed braid type of the fixed point set for some orientation-preserving C^1-embedding f of the disk having exactly n fixed points all of which are transversal.*

(2) *β^* is obtained from the 1-braid type with index $*$ by applying the operation (C)* repeatedly.*

This theorem will be proved in Section 6. We show here that Theorem 1 follows from Theorem 2. Suppose a braid type β satisfies the condition (1) of Theorem 1, i.e., $\beta = \mathrm{bt}(\mathrm{Fix}(f))$ for some f. By Theorem 2, $\mathrm{bt}(\mathrm{Fix}(f))^*$ must satisfy the condition (2) of Theorem 2. Therefore, by comparing the two operations (C), (C)*, we see that $\mathrm{bt}(\mathrm{Fix}(f))$ satisfies the condition (2) of Theorem 1. Conversely, suppose $\beta = [b]$ satisfies the condition (2) of Theorem 1. Then it is easy to see that there is a (unique) J such that $[b, J]$ satisfies the condition (2) of Theorem 2, and hence by Theorem 2, $[b, J] = \mathrm{bt}(\mathrm{Fix}(f))^*$ for some f. Therefore by ignoring the indices, we have $\beta = \mathrm{bt}(\mathrm{Fix}(f))$.

5. Some properties of braid types.

Here, we will describe some results on the braid type of a fixed point set, which will be used in the proof of Theorem 2.

Suppose an orientation-preserving C^1-embedding $h: D \to D$ is given. We assume h has only finitely many fixed points, and all of them are contained in the interior of the disk. We choose and fix an isotopy $\{h_t\}$ of the disk with $h_0 = \mathrm{id}$ and $h_1 = h$. For distinct fixed points x and y of h, define their *linking number* $lk(x, y; h)$ as the topological degree of the loop $h_t(x) - h_t(y)$ in $\boldsymbol{R}^2 - \{0\}$.

If x is a negative fixed point of f, define the linking number $lk(x, x; f)$ of x with itself to be the torsion number $\mathrm{tor}(x, f)$.

Let S be a subset of $\mathrm{Fix}(h)$. Let $D(S)$ denote the n-punctured disk obtained by removing S from D and recompactifying by adding a circle to each end of $D-S$, where n is the number of points in S. ($D(S)$ is considered as a subspace of D.) Then h_t can be extended to a continuous embedding from $D(S)$ to $D(h_t(S))$ [2, p. 24], which will be denoted by $h_{S,t}$. Let C_x denote the boundary circle of $D(S)$ which corresponds to $x \in S$. We extend $h_{S,t}$ further to an isotopy $\bar{h}_{S,t}: D \to D$ so that the center d_x of the circle C_x is a fixed point. Let $h_S = h_{S,1}$ and $\bar{h}_S = \bar{h}_{S,1}$. For a fixed point x of h_S and an inner boundary C_y of $D(S)$, define the linking number $lk(x, C_y; h_S)$ of x with C_y as $lk(x, d_y; \bar{h}_S)$. Note that if x is a fixed point of h which is not contained in S and $y \in S$, then $lk(x, y; h) = lk(x, C_y; h_S)$.

Now we consider the embedding f. Let S be a subset of $\mathrm{Fix}(f)$. Then $f_S: D(S) \to D(S)$ can be defined. If $x \in S$ is positive, then $\mathrm{Fix}(f_S) \cap C_x$ is an isolated fixed point set for f_S. This fixed point set has fixed point index zero. In fact, by a local perturbation of f around x we get a map f' such that x is fixed by f' and the differential $Df'(x)$ has no real eigenvalues. Then f'_S has no fixed points on C_x and therefore by the homotopy invariance of fixed point index (see e.g. [6], [12]), $\mathrm{Fix}(f_S) \cap C_x$ has fixed point index zero. Also, if $x \in S$ is negative (i.e., if x is an untwisted saddle), it is easy to see that f_S has four fixed points on C_x, two of which have fixed point index -1 and the other two have fixed point index zero.

Let Λ denote the ring $\boldsymbol{Z}[t, t^{-1}]$ of integer polynomials in the variable t and its inverse, and $\mathrm{GL}(n-1, \Lambda)$ the group of all invertible matrices of size $n-1$ with entries in Λ. Let $R: B_n \to \mathrm{GL}(n-1, \Lambda)$ denote the reduced Burau representation ([1, Lemma 3.11.1], [18, (16.4)]). For a braid type β, define a polynomial $\Gamma(\beta) \in \Lambda$ by

$$\Gamma(\beta) = \mathrm{tr}\, R(b)\,,$$

where $b \in B_n$ with $[b] = \beta$. This is well defined since if two elements b and b' of B_n represent the same braid type, then they are conjugate and hence the matrices $R(b)$ and $R(b')$ are also conjugate. We have:

PROPOSITION 2.

$$-\Gamma(\mathrm{bt}(S, f)) = \sum_{x \in S^c} \mathrm{ind}(x, f) t^{lk(x,S)} - 2 \sum_{x \in S_-} t^{lk(x,S)}\,,$$

where $S^c = \mathrm{Fix}(f) - S$, $S_- = S \cap \mathrm{Fix}_-(f)$, *and* $lk(x, S) = \sum_{y \in S} lk(x, y; f)$.

PROOF. For $x \in S_-$, let u_x^1, u_x^2 be the fixed points in C_x which have fixed point index -1.

For fixed points x of f_S, let $lk(x; f_S) = \sum_{y \in S} lk(x, C_y; f_S)$. We can assume that f_S has no fixed points on C_x for any positive $x \in S$. Then f_S has finitely many fixed points. The following formula is known (see e.g. [11, Sect. 2 (D)], [4, p. 29]):

$$-\Gamma(\mathrm{bt}(S, f)) = \sum_{x \in \mathrm{Fix}(f_S)} \mathrm{ind}(x, f_S) t^{lk(x; f_S)}\,.$$

It is easy to see that for any $y\in S$, $x\in S_-$, and $\varepsilon=1, 2$, we have

$$lk(u_x^\varepsilon, C_y; f_S)=lk(x, y; f).$$

This implies that $lk(u_x^\varepsilon; f_S)=lk(x, S)$. Also, $lk(x; f_S)=lk(x, S)$ for any $x\in S^c$. Therefore, if we let $U=\{u_x^\varepsilon \mid x\in S_-, \varepsilon=1, 2\}$, then we have

$$\begin{aligned}-\Gamma(\mathrm{bt}(S, f)) &= \sum_{x\in S^c} \mathrm{ind}(x, f_S)t^{lk(x;f_S)} + \sum_{u\in U} \mathrm{ind}(u, f_S)t^{lk(u;f_S)} \\ &= \sum_{x\in S^c} \mathrm{ind}(x, f)t^{lk(x,S)} - 2\sum_{x\in S_-} t^{lk(x,S)}.\end{aligned}$$

□

If we take S to be $\mathrm{Fix}(f)$ in the above proposition, we obtain the following result which gives a necessary condition, in terms of the Burau matrices, for a braid type to be realized as the braid type of the fixed point set for some f. (This result will not be used to prove the theorems in this paper.)

COROLLARY. *The polynomial* $\Gamma(\mathrm{bt}(\mathrm{Fix}(f)))$ *has the following form*:

$$\Gamma(\mathrm{bt}(\mathrm{Fix}(f)))=2(t^{p_1}+\cdots+t^{p_l}),$$

where $l=(\#\mathrm{Fix}(f)-1)/2$ *and* $p_1, \cdots, p_l$ *are integers.*

The following result plays an essential role in the proof of Theorem 2.

PROPOSITION 3. *Suppose* $\#\mathrm{Fix}(f)\leq 7$. *Then for any subset* S *of* $\mathrm{Fix}(f)$ *consisting of three points, we have* $\mathrm{bt}(S, f)=[\sigma_1^{2i}]$ *for some integer* i *up to multiplication with full twist* 3*-braids.*

PROOF. Since $\mathrm{bt}(S, f)$ is a 3-braid type, by an argument in the proof of Proposition in [16], $\mathrm{bt}(S, f)$ is equal to one of the following braid types up to full twists:

(i) $[\sigma_1^{2i}]$, $i\in \boldsymbol{Z}$,

(ii) $[(\sigma_1\sigma_2)^{\pm 1}]$, $[\sigma_1\sigma_2\sigma_1]$,

(iii) $[\alpha(i_1)\cdots\alpha(i_d)]$, $d\geq 1$, $i_1\geq 5$, $i_2, \cdots, i_d\geq 4$, where $\alpha(i)=\sigma_1^i\sigma_2$.

Case (ii) does not occur, because the braid types in case (ii) are not pure braid types.

We will show that the case (iii) is also impossible. Suppose the case (iii) holds. We first claim that there is a set S' of three positive fixed points of f such that $\mathrm{bt}(S', f)=\mathrm{bt}(S, f)$. To prove this, it is enough to show that we can replace each negative fixed point in S with a positive one without altering the braid type. Assume S contains a negative fixed point x_1. Let x_2, x_3 be the other fixed points in S. Replacing the isotopy $\{f_t\}$ if necessary, we can assume $lk(x_2, x_3; f)=0$. Moreover we can assume the curves $f_t(x_2)$, $f_t(x_3)$ $(0\leq t\leq 1)$ are constant in t, i.e., the strings corresponding to these curves are straight lines. Let $Q=\{x_2, x_3\}$. Then $Q=f_t(Q)$ for any t and $f_{Q,t}$ maps $D(Q)$ to itself. For a loop c in the 2-punctured disk $D(Q)$, let $[c]$ denote the free homotopy class of c in $D(Q)$. Then it is easy to see that fixed points x, y of f_Q are in the same fixed point class if and only if the loops $f_{Q,t}(x)$ and $f_{Q,t}(y)$ in $D(Q)$ are freely homotopic. For a free homotopy class γ of loops in $D(Q)$, let $\text{Fix}_\gamma(f)$ be the fixed point class of f_Q corresponding to γ. ($\text{Fix}_\gamma(f)$ may be empty.) Since any orientation-preserving embedding of $D(Q)$ into itself is homotopic to the identity, by the homotopy invariance of the fixed point index, we have $\text{ind}(\text{Fix}_\gamma(f))=0$ for any γ which is nontrivial, i.e., which does not contain any constant loop. Therefore, since the free homotopy class $[f_t(x_1)]$ of the loop $f_t(x_1)$ is nontrivial by the assumption that the case (iii) holds, the sum of the fixed point indices for the fixed points x of f_Q with $x\neq x_1$ and $[f_{Q,t}(x)]=[f_t(x_1)]$ is equal to one. It follows from this that there must exist a fixed point x'_1 of f_Q having positive fixed point index such that $f_{Q,t}(x'_1)$ is freely homotopic to $f_t(x_1)$ in $D(Q)$. This implies that x'_1 lies in $D-Q$ and $\text{bt}(S, f)=\text{bt}(\{x'_1, x_2, x_3\}, f)$. Hence x'_1 gives the desired positive fixed point. This proves the claim.

By [16, p. 200, (5)] the coefficient of t^k in the polynomial $\Gamma(\text{bt}(S', f))$ is nonzero if $2d\leq k\leq\sum_{s=1}^{d}(i_s-1)$. Hence, $\Gamma(\text{bt}(S', f))$ has at least $p+1$ nonzero terms, where $p=\sum_{s=1}^{d}(i_s-3)$. Therefore by applying Proposition 2 to S', we have

$$\#(\text{Fix}(f)-S')\geq p+1\ .$$

From this and the hypothesis that $\#\text{Fix}(f)\leq 7$, it follows that $p\leq 3$ and hence $(i_1, \cdots, i_d)$ must be one of the sequences (6), (5), or (5, 4). This implies $\text{bt}(S, f)$ is not a pure braid type, which is a contradiction. Therefore Case (iii) does not occur. □

The following result has been proved in [17], which reduces the problem of classifying indexed braid types for $\text{Fix}(f)$ to that of classifying indexed braid types for the set of negative fixed points.

THEOREM 3 ([17]). *Let $f, g: D\to D$ be orientation-preserving C^1-embeddings having only transversal fixed points. Suppose f and g have the same number of fixed points. Let $f_t, g_t: D\to D$ be isotopies with $f_0=g_0=\text{id}$, $f_1=f$, $g_1=g$. Then the following conditions are equivalent*:

(1) $\text{bt}(\text{Fix}(f); \{f_t\})^*=\text{bt}(\text{Fix}(g); \{g_t\})^*$,
(2) $\text{bt}(\text{Fix}_+(f); \{f_t\})=\text{bt}(\text{Fix}_+(g); \{g_t\})$,
(3) $\text{bt}(\text{Fix}_-(f); \{f_t\})^*=\text{bt}(\text{Fix}_-(g); \{g_t\})^*$.

6. Proof of Theorem 2.

The theorem is trivial for $n=1$. We will prove the theorem for $n=3, 5, 7$. We first show that the condition (2) implies the condition (1). Note that every indexed braid type satisfying the condition (2) for $n=3, 5, 7$ is equal to one of the following indexed braid types up to full twists (cf. Proposition 1):

$$\gamma^*=[e, (*, 0, *)]\ , \qquad \gamma^*_{(k)}=[\theta_3^k, (*, k, *, 0, *)]\ ,$$

$$\gamma^*_{(k,l)}=[\theta_3^k\theta_5^l, (*, k+l, *, l, *, 0, *)]\ , \qquad \bar{\gamma}^*_{(k,l)}=[\theta_3^k\bar{\theta}_3^l, (*, k, *, 0, *, l, *)]\ ,$$

where k, l are integers.

It is sufficient to verify that each of these indexed braid types can be realized as the indexed braid type of the fixed point set for some embedding, since the indexed braid type of a fixed point set is unique up to full twists.

Let X be a vector field on the unit disk D with two sinks $(1/2, 0)$, $(-1/2, 0)$ and one saddle $(0, 0)$ as in Fig. 2. Let $\phi_t: D \to D$ be the 1-parameter group of transformations associated to X.

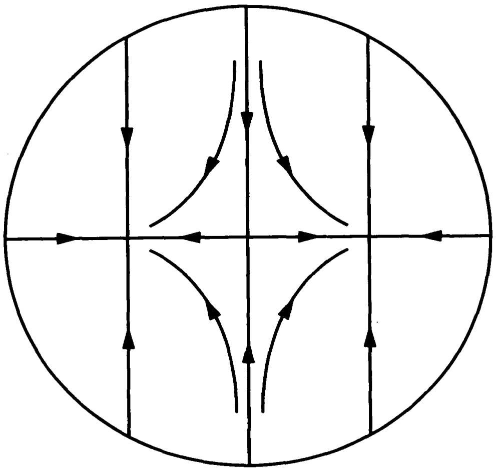

Figure 2

Let D', D'' be concentric disks centered at $(-1/2, 0)$ with radius $1/6$, $1/3$ respectively. Also, let $\bar{D}'$, $\bar{D}''$ be the concentric disks centered at $(1/2, 0)$ with radius $1/6$, $1/3$ respectively. Let k, l be integers.

Let $\{\phi_{(k),t}\}$ be an isotopy which coincides with ϕ_t outside of D'', whose restriction to D' is conjugate to $\phi_t \circ R_{kt}: D \to D$ via a rigid translation from D' to D, and which has no fixed points on $D'' - D'$.

Let $\{\phi_{(k,l),t}\}$ be an isotopy which coincides with ϕ_t outside of D'', and is conjugate to $\phi_{(k),t} \circ R_{lt}$ on D', and has no fixed points on $D'' - D'$. Also, let $\{\bar{\phi}_{(k,l),t}\}$ be an isotopy which coincides with $\phi_{(k),t}$ outside of $\bar{D}''$, and is conjugate to $\phi_t \circ R_{lt}$ on $\bar{D}'$ and has no fixed points on $\bar{D}'' - \bar{D}'$. Let

$$\phi = \phi_1 , \quad \phi_{(k)} = \phi_{(k),1} , \quad \phi_{(k,l)} = \phi_{(k,l),1} , \quad \bar{\phi}_{(k,l)} = \bar{\phi}_{(k,l),1} .$$

Then, it is easy to see that the indexed braid type of the fixed point set for each of these maps ϕ, $\phi_{(k)}$, $\phi_{(k,l)}$, $\bar{\phi}_{(k,l)}$ is equal to γ^*, $\gamma^*_{(k)}$, $\gamma^*_{(k,l)}$, $\bar{\gamma}^*_{(k,l)}$ respectively. Thus we have proved that the condition (2) implies (1).

We next show the condition (1) implies (2).

If $h: D \to D$ is an embedding as in Section 5, $y_1, \cdots, y_l$ are distinct fixed points of h, and $K = (k_1, \cdots, k_l)$ is a sequence of integers, we define a set $\mathrm{Fix}_K(h_S)$ of fixed points of $h_S: D(S) \to D(S)$, where $S = \{y_1, \cdots, y_l\}$, by

$$\mathrm{Fix}_K(h_S)=\{x\in \mathrm{Fix}(h_S) \mid lk(x, C_{y_i}; h_S)=k_i \text{ for any } i\}.$$

(This set is called a homological Nielsen class in [11, Sect. 1 (B)].)

Let f be an orientation-preserving embedding having 3, 5 or 7 fixed points all of which are transversal. We only need to show that, for a particularly chosen isotopy $\{f_t\}$, the indexed braid type of $\mathrm{Fix}(f)$ coincides with the indexed braid type of the fixed point set for one of the embeddings ϕ, $\phi_{(k)}$, $\phi_{(k,l)}$, $\bar{\phi}_{(k,l)}$.

Let $l=(n-1)/2$ $(=\#\mathrm{Fix}_-(f))$ and $y_1, \cdots, y_l$ be the elements of $\mathrm{Fix}_-(f)$. Let $\tau_i=\mathrm{tor}(y_i)$. The proof is divided into three cases.

Case 1. $n=3$. We choose an isotopy $\{f_t\}$ for which $\tau_1=0$. This is possible by rotating the disk appropriately if necessary. Then

$$\mathrm{bt}(\mathrm{Fix}_-(f))^*=[e, (\tau_1)]=[e, (0)]=\mathrm{bt}(\mathrm{Fix}_-(\phi))^*.$$

Hence by Theorem 3, we have $\mathrm{bt}(\mathrm{Fix}(f))^*=\mathrm{bt}(\mathrm{Fix}(\phi))^*$.

Case 2. $n=5$. We choose $\{f_t\}$ such that $lk(y_1, y_2; f)=0$. Then,

$$\mathrm{bt}(\mathrm{Fix}_-(f))^*=[e, (\tau_1, \tau_2)].$$

We can assume $y_1=(-1/2, 0)$, $y_2=(1/2, 0)$. Then $\mathrm{Fix}_-(f)=\mathrm{Fix}_+(\phi)$. Let $S=\mathrm{Fix}_-(f)$. Since $f_S, \phi_S\colon D(S)\to D(S)$ are isotopic, by the invariance of the fixed point index, we have

$$\mathrm{ind}(\mathrm{Fix}_{(0,0)}(f_S), f_S)=\mathrm{ind}(\mathrm{Fix}_{(0,0)}(\phi_S), \phi_S)=\mathrm{ind}(0, \phi_S)=-1.$$

This implies there is a fixed point of f_S which has negative fixed point index and whose linking numbers with C_{y_1} and C_{y_2} are both zero. Since any fixed point of f in $D-S$ has positive fixed point index, this fixed point must be in C_{y_1} or C_{y_2}. Consequently, we have that either τ_1 or τ_2 must be zero. We may assume $\tau_2=0$. Then,

$$\mathrm{bt}(\mathrm{Fix}_-(f))^*=[e, (\tau_1, 0)]=\mathrm{bt}(\mathrm{Fix}_-(\phi_{(\tau_1)}))^*.$$

Hence, by Theorem 3, $\mathrm{bt}(\mathrm{Fix}(f))^*=\mathrm{bt}(\mathrm{Fix}(\phi_{(\tau_1)}))^*$.

Case 3. $n=7$. Since $\#\mathrm{Fix}_-(f)=3$, by Proposition 3 we can choose an isotopy $\{f_t\}$ such that $\mathrm{bt}(\mathrm{Fix}_-(f))=[\sigma_1^{2i}]$ for some integer i. Therefore $\mathrm{bt}(\mathrm{Fix}_-(f))^*=[\sigma_1^{2i}, (\tau_1, \tau_2, \tau_3)]$. Since $\mathrm{bt}(\mathrm{Fix}_-(f))=\mathrm{bt}(\mathrm{Fix}_+(\phi_{(i)}))$, we may assume $\mathrm{Fix}_-(f)=\mathrm{Fix}_+(\phi_{(i)})$. Let $S=\mathrm{Fix}_-(f)$. Then f_S and $(\phi_{(i)})_S$ are isotopic. Therefore, if i is nonzero, then $\mathrm{ind}(\mathrm{Fix}_K(f_S))=-1$ for $K=(0, 0, 0), (i, i, 0)$. Also, if i is zero, then $\mathrm{ind}(\mathrm{Fix}_{(0,0,0)}(f_S))=-2$. Similarly as in Case 2, from these we have the following:

(i) If i is nonzero, then $\tau_3=0$ and either τ_1 or τ_2 is i.

(ii) If i is zero, then at least one of τ_1, τ_2, τ_3 is zero.

Consider the case $i\neq 0$. We may assume $\tau_2=i$. Let $k=\tau_1-i$. Then we have

$$\mathrm{bt}(\mathrm{Fix}_-(f))^*=[\sigma_1^{2i}, (\tau_1, i, 0)]=\mathrm{bt}(\mathrm{Fix}_-(\phi_{(k,i)}))^*.$$

Hence by Theorem 3, the indexed braid type of $\mathrm{Fix}(f)$ is equal to that of $\mathrm{Fix}(\phi_{(k,i)})$.

Now, consider the case $i=0$. We can assume $\tau_2=0$ without loss of generality. Let

$k=\tau_1$, $l=\tau_3$. Then since the indexed braid types of $\mathrm{Fix}_-(\bar{\Phi}_{(k,l)})$ and $\mathrm{Fix}_-(f)$ are equal, the indexed braid type of $\mathrm{Fix}(f)$ is equal to $\mathrm{Fix}(\bar{\Phi}_{(k,l)})$ by Theorem 3.

Thus, we have proved that the condition (1) in Theorem 2 implies the condition (2).

7. Relation between braid types and indexed braid types.

In this section, we will prove a result which shows that in classifying indexed braid types of the fixed point sets, we can neglect the information on indices. This result shows, in particular, that Theorem 2, which at a first glance seems to contain more information than Theorem 1, is actually equivalent to Theorem 1. The result is:

PROPOSITION 4. $\mathrm{bt}(\mathrm{Fix}(f); \{f_t\})^* = \mathrm{bt}(\mathrm{Fix}(g); \{g_t\})^*$ *if and only if* $\mathrm{bt}(\mathrm{Fix}(f); \{f_t\}) = \mathrm{bt}(\mathrm{Fix}(g); \{g_t\})$.

To prove the proposition, we introduce some equivalence relation on the set of strings of an indexed geometric braid. Suppose $\mathscr{A} = \{A(1), \cdots, A(n)\}$ is a geometric braid with index p_i assigned to each string $A(i)$. We say two indexed strings $A(i)$ and $A(j)$ are *equivalent* if $p_i = p_j$ and there is a homotopy of strings $\{C_\mu\}_{0 \le \mu \le 1}$ such that $C_0 = A(i)$, $C_1 = A(j)$, and $C_\mu \cap A(k)$ is empty for any $k = 1, \cdots, n$, $0 < \mu < 1$.

For a subset I of $\{1, \cdots, n\}$, define $\mathscr{A}(I) = \{A(i) \mid i \in I\}$. We denote by $\mathrm{bt}(\mathscr{A}(I))^*$ the indexed braid type represented by a geometric braid $\mathscr{A}(I)$ with index p_i assigned to $A(i)$, $i \in I$.

LEMMA. *Let I, J, K be mutually disjoint subsets of $\{1, \cdots, n\}$ such that J and K have the same cardinality d. Assume we can arrange the elements of J, K as $J = \{j_1, \cdots, j_d\}$, $K = \{k_1, \cdots, k_d\}$ so that two indexed strings $A(j_s)$ and $A(k_s)$ are equivalent for each $1 \le s \le d$. Then*

$$\mathrm{bt}(\mathscr{A}(I \cup J))^* = \mathrm{bt}(\mathscr{A}(I \cup K))^* .$$

PROOF. Let $J_s = \{j_s, \cdots, j_d\}$, $K_s = \{k_s, \cdots, k_d\}$ for $s \le d$. Let $J_{d+1} = K_{d+1} = \varnothing$ and $L_s = J \cup I_s \cup (K - K_s)$. Since $L_1 = I \cup J$ and $L_{d+1} = I \cup K$, it is sufficient to show that

$$\mathrm{bt}(\mathscr{A}(L_s))^* = \mathrm{bt}(\mathscr{A}(L_{s+1}))^*$$

for each $1 \le s \le d$ by induction on s. Let $s = 1$. Since $A(j_1)$ and $A(k_1)$ are equivalent, there is a homotopy of strings C_μ from $A(j_1)$ to $A(k_1)$ such that C_μ and $A(k)$, $k = 1, \cdots, n$, are disjoint for $0 < \mu < 1$. Therefore, $\mathrm{bt}(\mathscr{A}(I \cup J))^* = \mathrm{bt}(\mathscr{A}(I \cup J_2 \cup \{k_1\}))^*$ via the homotopy of geometric braids $\mathscr{A}(I \cup J_2) \cup C_\mu$. Thus the equality holds for $s = 1$. In the same way, the equality is proved for any s. □

PROOF OF PROPOSITION 4. It is trivial that $\mathrm{bt}(\mathrm{Fix}(f))^* = \mathrm{bt}(\mathrm{Fix}(g))^*$ implies $\mathrm{bt}(\mathrm{Fix}(f)) = \mathrm{bt}(\mathrm{Fix}(g))$. We will prove the converse. Assume $\mathrm{bt}(\mathrm{Fix}(f)) = \mathrm{bt}(\mathrm{Fix}(g))$. Let $y_1, \cdots, y_l$ and $y'_1, \cdots, y'_l$ be the elements of $\mathrm{Fix}_-(f)$ and $\mathrm{Fix}_-(g)$ respectively. We alter

f_t to f'_t in a small neighborhood of $\mathrm{Fix}_-(f)$ so that each y_i is a sink of f'_t and there are newly added two untwisted saddles u_{2i-1}, u_{2i} close to y_i. Similarly, alter g_t to g'_t so that y'_i is a sink of g'_1 and there are two untwisted saddles u'_{2i-1}, u'_{2i} added near y'_i. Let $f'=f'_1$ and $g'=g'_1$. Then

$$\mathrm{Fix}_+(f')=\mathrm{Fix}(f)\,, \qquad \mathrm{Fix}_+(g')=\mathrm{Fix}(g)\,,$$
$$\mathrm{Fix}_-(f')=\{u_1, \cdots, u_{2l}\}\,, \qquad \mathrm{Fix}_-(g')=\{u'_1, \cdots, u'_{2l}\}\,.$$

Let $A(i)$ (resp. $A'(i)$) be the string associated to $f'_t(u_i)$ (resp. $g'_t(u'_i)$) and let

$$\mathscr{A}=\{A(1), \cdots, A(2l)\}\,, \qquad \mathscr{A}'=\{A'(1), \cdots, A'(2l)\}\,.$$

Let $\tau_i=\mathrm{tor}(u_i, f')$, $\tau'_i=\mathrm{tor}(u'_i, g')$ and we assign τ_i (resp. τ'_i) to the string $A(i)$ (resp. $A(i)'$). Let $\mathscr{A}(I_1), \cdots, \mathscr{A}(I_k)$ be the equivalence classes in $\mathscr{A}$, where $I_1 \cup \cdots \cup I_k=\{1, \cdots, 2l\}$. Since the braid types of $\mathrm{Fix}(f)$ and $\mathrm{Fix}(g)$ are assumed to be equal, we have $\mathrm{bt}(\mathrm{Fix}_+(f'))=\mathrm{bt}(\mathrm{Fix}_+(g'))$. Therefore, we have by Theorem 3 that $\mathrm{bt}(\mathrm{Fix}_-(f'))^*=\mathrm{bt}(\mathrm{Fix}_-(g'))^*$. Hence, there is a homotopy of loops $\tilde{l}_\mu$: $[0, 1]\to V_n$ such that $x_i(t, 0)=f'_t(u_i)$, $x_i(t, 1)=g'_t(u'_{\nu(i)})$, and $\tau_i=\tau'_{\nu(i)}$ for some permutation ν of $\{1, \cdots, 2d\}$, where $x_i(t, \mu)$ is the i-th coordinate of $\tilde{l}_\mu(t)$. Thus if $A(i)$ and $A(j)$ are equivalent, then so are $A'(\nu(i))$ and $A'(\nu(j))$, and consequently the collection $\mathscr{A}'(\nu(I_1)), \cdots, \mathscr{A}'(\nu(I_k))$ form the equivalence classes of $\mathscr{A}'$.

Denote by O (resp. E) the set $\{i \mid 1\le i\le 2l,\ i \text{ is odd (resp. even)}\}$. Let

$$I=O\cap \nu(O)\,, \quad J_s=\nu(I_s)\cap O\cap \nu(E)\,, \quad K_s=\nu(I_s)\cap E\cap \nu(O)\,,$$
$$J=\bigcup_{s=1}^{k} J_s, \qquad K=\bigcup_{s=1}^{k} K_s\,.$$

Then $O=I\cup J$, $\nu(O)=I\cup K$. Since $\mathscr{A}'(J_s)$ and $\mathscr{A}'(K_s)$ are contained in the same equivalence class $\mathscr{A}'(\nu(I_s))$, the subsets I, J, and K satisfy the hypothesis of Lemma with respect to $\mathscr{A}'$. Hence,

$$\mathrm{bt}(\mathscr{A}'(\nu(O)))^*=\mathrm{bt}(\mathscr{A}'(I\cup K))^*=\mathrm{bt}(\mathscr{A}'(I\cup J))^*=\mathrm{bt}(\mathscr{A}'(O))^*\,.$$

Therefore, since y_i is sufficiently close to u_{2i-1} and $\mathrm{tor}(y_i, f)=\mathrm{tor}(u_{2i-1}, f')$, we have

$$\mathrm{bt}(\mathrm{Fix}_-(f))^*=\mathrm{bt}(\mathscr{A}(O))^*=\mathrm{bt}(\mathscr{A}'(\nu(O)))^*=\mathrm{bt}(\mathscr{A}'(O))^*\,.$$

On the other hand, clearly we have:

$$\mathrm{bt}(\mathrm{Fix}_-(g))^*=\mathrm{bt}(\mathscr{A}'(O))^*\,.$$

Hence we have $\mathrm{bt}(\mathrm{Fix}_-(f))^*=\mathrm{bt}(\mathrm{Fix}_-(g))^*$, and by Theorem 3 we have $\mathrm{bt}(\mathrm{Fix}(f))^*=\mathrm{bt}(\mathrm{Fix}(g))^*$. Thus the proof is completed.

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