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Defining Equations of Modular Curves $X_0(N)$

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1. Introduction.

We have as a defining equation of the modular curve $X_0(N)$ the modular equation of level N, which has many good properties; e.g. it reflects the properties of $X_0(N)$ that is the coarse moduli space of the isomorphism classes of the generalized elliptic curves with a cyclic subgroup of order N. But its degree and coefficients are too large for its application to practical calculation on $X_0(N)$. While it is an important problem to determine the algebraic points on $X_0(N)$, we need a handier form of defining equation, which will also serve to solve other related problems.

In the present paper, we give a general method to explicitly calculate a system of defining equations of an arbitrary modular curve $X_0(N)$. In case of a hyperelliptic modular curve of genus two, a kind of normal form of defining equations is given by Murabayashi (cf. [M]). We generalize his method. We list defining equations of all modular curves $X_0(N)$ of genus two to six. One should note that our algorithm works for $X_0(N)$ of genus greater than six.

The form of our defining equations is as follows. If $X_0(N)$ is hyperelliptic or of genus three, then the number of our defining equations is one. If $X_0(N)$ is non-hyperelliptic and of genus greater than three, then our system of defining equations is given as the intersection of some quadratic and cubic hypersurfaces on $\boldsymbol{P}^{g-1}$. In particular, we list the defining equations of $X_0(N)$ ($N=34$, 43, 45, 64) which are non-hyperelliptic of genus three. We note that $X_0(64)$ is the Fermat curve of degree four.

In our method, we cannot have defining equations of modular curves of genus zero or one. But defining equations of some modular curves of genus one (i.e. $X_0(N)$, $N=14$, 20, 24) can be obtained by means of a covering from some $X_0(N)$. We can explicitly give covering maps between modular curves.

To get our equations, we use the Fourier expansions of certain cusp forms of weight 2 on $\Gamma_0(N)$. Their Fourier coefficients can be given by the Brandt matrix (cf.

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[P]) and the trace formula.

We will use the following notation throughout this paper.

- N: a positive integer (= the level of a modular curve)
- $\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \middle| c \equiv 0 \bmod N \right\}$
- $\mathfrak{H} := \{z \in \boldsymbol{C} \mid \mathrm{Im}(z) > 0\}$
- $\mathfrak{H}^* := \mathfrak{H} \cup \boldsymbol{Q} \cup \{i\infty\}$
- $X_0(N) := \Gamma_0(N) \backslash \mathfrak{H}^*$, a modular curve of level N
- $g = \mathrm{genus}(X_0(N))$
- $S_2(\Gamma_0(N))$: the $\boldsymbol{Q}$-vector space of cusp forms of weight 2 on $\Gamma_0(N)$
  Let $f_1, \cdots, f_g$ be a basis of $S_2(\Gamma_0(N))$, z the natural coordinate on $\mathfrak{H}$, $q = e^{2\pi i z}$ and let the Fourier expansion of f_i be

$$f_i = a_{i,1} q + a_{i,2} q^2 + \cdots, \ (a_{i,j} \in \boldsymbol{Z}, 1 \le i \le g, j = 1, 2, \cdots).$$

(These Fourier coefficients can be taken in $\boldsymbol{Z}$ (cf. [S]).)

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2. Computation of defining equations.

If $X_0(N)$ is a hyperelliptic curve, its normal form can be obtained by Murabayashi's method. If $X_0(N)$ is a non-hyperelliptic curve, a system of its defining equations can be obtained by means of canonical map. The hyperelliptic modular curves of type $X_0(N)$ have been classified by Ogg (cf. [O]).

THEOREM 1 (Ogg). *There are exactly nineteen values of level N which makes $X_0(N)$ hyperelliptic. They are given as follows ($g = genus(X_0(N))$):*

$g=2$; $N=22, 23, 26, 28, 29, 31, 37, 50$,

$g=3$; $N=30, 33, 35, 39, 40, 41, 48$,

$g=4$; $N=47$,

$g=5$; $N=46, 59$,

$g=6$; $N=71$.

The next lemma is well known.

LEMMA 1. *Let Ω^1 be the sheaf of holomorphic 1-forms on $X_0(N)$. Then the following map Ψ is an isomorphism of $S_2(\Gamma_0(N)) \otimes \boldsymbol{C}$ to $H^0(X_0(N), \Omega^1)$:*

$$\Psi : S_2(\Gamma_0(N)) \otimes \boldsymbol{C} \xrightarrow{\sim} H^0(X_0(N), \Omega^1)$$
$$f \longmapsto 2\pi i f dz .$$

By this lemma, the relations of the holomorphic 1-forms on $X_0(N)$ can be reduced to those of $S_2(\Gamma_0(N))$.

2.1. Hyperelliptic case. Let $\overline{i\infty}$ denote the point of $X_0(N)$ which is represented by $i\infty$. If $\overline{i\infty}$ is not a Weierstrass point of $X_0(N)$, then $X_0(N)$ can be described by the following normal form:

$$y^2 = f(x) , \qquad f(x) \in \boldsymbol{Q}[x] , \quad \deg f = 2g+2 ,$$

where x is a covering map of degree two from $X_0(N)$ to $\boldsymbol{P}^1$. (Here, the normal form means a defining equation of type $y^2 = f(x)$, $f(x) \in \boldsymbol{C}[x]$ and $\boldsymbol{C}(x, y)$ is isomorphic to the function field of the curve.)

First of all, we calculate some divisors with respect to the normal form

$$y^2 = f(x) = \prod_{j=1}^{2g+2} (x - \alpha_j) \qquad (\alpha_j \in \boldsymbol{C}) ,$$

$$\frac{(x-\alpha_j)^k dx}{y} \in H^0(X_0(N), \Omega^1) , \qquad k = 0, \cdots, g-1 .$$

Let P_j be the Weierstrass point corresponding to α_j (i.e. $P_j = (\alpha_j, 0)$), $Q_1 = \overline{i\infty}$ and let Q_2 be the image of Q_1 by the hyperelliptic involution.

$$\begin{cases} (x-\alpha_j) = 2P_j - Q_1 - Q_2 \\ (y) = P_1 + P_2 + \cdots + P_{2g+2} - (g+1)Q_1 - (g+1)Q_2 \\ (dx) = P_1 + P_2 + \cdots + P_{2g+2} - 2Q_1 - 2Q_2 \\ \left(\dfrac{(x-\alpha_j)^k dx}{y} \right) = 2kP_j + (g-1-k)Q_1 + (g-1-k)Q_2 \quad (0 \le k \le g-1) . \end{cases}$$

If $\overline{i\infty}$ is not a Weierstrass point, the gap sequence at $\overline{i\infty}$ is

$$\{1, 2, \cdots, g\}$$

(cf. [ACGH]). Then there exists $h_j \in H^0(X_0(N), \Omega^1)$ such that the degree of h_j at $\overline{i\infty}$ is $j-1$ $(j = 1, \cdots, g)$. Therefore, using a linear combination of $f_1, \cdots, f_g$, we can take $g_1, \cdots, g_g$ with the following Fourier expansions:

$$\begin{cases} g_1(z) = q^g + s_{1,g+1} q^{g+1} + \cdots + s_{1,g+i} q^{g+i} + \cdots \\ g_2(z) = q^{g-1} + s_{2,g} q^g + \cdots + s_{2,g+i} q^{g+i} + \cdots \\ \cdots\cdots \\ g_g(z) = q + s_{g,2} q^2 + \cdots + s_{g,g+i} q^{g+i} + \cdots . \end{cases}$$

LEMMA 2. *Let x be $g_2(z)/g_1(z)$, then $x: X_0(N) \to \boldsymbol{P}^1$ is of degree two.*

PROOF. Let $Y=F(X)$ be a normal form of $X_0(N)$ ($\deg F(X)=2g+2$). By Lemma 1 and the fact that

$$X^j \frac{dX}{Y} \qquad (j=0, \cdots, g-1)$$

span $H^0(X_0(N), \Omega^1)$, we have

$$g_i(z)dz = \sum_{j=0}^{g-1} p_{i,j} X^j \frac{dX}{Y} \qquad (1 \le i \le g,\ p_{i,j} \in \boldsymbol{C}).$$

Comparing the order at $\overline{i\infty}$ of the divisors of both sides, $g_i(z)dz$ must be the following form:

$$g_i(z)dz = \sum_{j=0}^{i-1} p_{i,j} X^j \frac{dX}{Y} \qquad (p_{i,i-1} \ne 0).$$

So

$$g_1(z) = p_{1,0} \frac{dX}{Y}, \qquad g_2(z) = (p_{2,0} + p_{2,1}X) \frac{dX}{Y},$$

$$x = \frac{g_2(z)}{g_1(z)} = \frac{p_{2,0} + p_{2,1}X}{p_{1,0}}.$$

Because X is of degree two, x is also of degree two. □

Finally, we construct y as follows: we have shown dx/y has zeros of degree $g-1$ at $\overline{i\infty}$. By Lemma 1 we have

$$\frac{dx}{y} = 2\pi i(t_1 f_1(z)dz + \cdots + t_g f_g(z)dz), \qquad t_1, \cdots, t_g \in \boldsymbol{Q}.$$

Considering the order at $\overline{i\infty}$, we have

$$\frac{dx}{y} = c \times 2\pi i g_1(z)dz, \qquad c \in \boldsymbol{C}^*.$$

We set

$$x := \frac{g_2}{g_1}, \qquad y := \frac{q}{g_1} \frac{dx}{dq}.$$

(This construction of x, y is an extension of that of [M] to higher genus.) Thus from the Fourier expansions of x and y, namely

$$y^2 = q^{-(2g+2)} + \cdots, \qquad x = q^{-1} + \cdots,$$

we can recursively determine the coefficients $a_1, a_2, \cdots$ of a defining equation as follows:

$$\begin{cases} y^2 - x^{2g+2} = a_1 q^{-2g-1} + \cdots \\ y^2 - x^{2g+2} - a_1 x^{2g+1} = a_2 q^{-2g} + \cdots \\ \quad \cdots\cdots . \end{cases}$$

Thus we have a defining equation of $X_0(N)$, namely

$$y^2 = x^{2g+2} + a_1 x^{2g+1} + \cdots + a_{2g+2} .$$

REMARK 1. To get a normal form, we have only to know $\{s_{1,g+1}, \cdots, s_{1,3g+3}, s_{2,g}, \cdots, s_{2,3g+2}\}$.

2.2. Non-hyperelliptic case. Let $\omega_1, \cdots, \omega_g$ be a basis of $H^0(X_0(N), \Omega^1)$. The canonical map which is induced by $\omega_1, \cdots, \omega_g$

$$\begin{aligned} \Phi\colon X_0(N) &\longrightarrow P^{g-1} \\ z &\longmapsto (1, \omega_2/\omega_1, \cdots, \omega_g/\omega_1) \end{aligned}$$

gives an embedding of $X_0(N)$ in P^{g-1}, because $X_0(N)$ is non-hyperelliptic. The image of this map is called a canonical curve.

By Lemma 1, we have $F_i := f_i/f_1 = \omega_i/\omega_1$. It is clear that

$$L((\omega_1)) = \langle F_1 = 1, F_2, \cdots, F_g \rangle .$$

Here, and in what follows, $L(D) := \{f \in \boldsymbol{C}(C) \mid \operatorname{div}(f) \geq -D\}$ and $l(D) := \dim_{\boldsymbol{C}}(L(D))$. (Let C be a complete non-singular curve of genus g, K_C be the canonical divisor of C and D be any divisor on C.)

Let n (>1) be a positive integer. To get a system of defining equations of $X_0(N)$, we will find the relations of the monomials of degree not exceeding n which are generated by $F_1, \cdots, F_g$. But there are two problems. One is the number of linearly independent monomials, and the other is the upper bound of the degree n sufficient for our purpose. Petri's theorem solves these problems. By the proof of Petri's theorem we find that the rank of the monomials of degree not exceeding n which are generated by $F_1, \cdots, F_g$ is $l(n(\omega_1))$. On the other hand, it is clear that the monomials belong to $L(n(\omega_1))$.

In the following, we deal with the monomials of degree n which are generated by f_i's instead of the monomials of degree not exceeding n which are generated by F_i's. It is obvious that

$$\#\{\text{monomials of degree } n \text{ which are generated by } f_i\text{'s}\} = {}_gH_n ,$$

$$l(n(\omega_1)) = n \deg(\omega_1) + 1 - g = n(2g-2) + 1 - g = (2n-1)(g-1)$$

(by the Riemann-Roch theorem).

THEOREM 2 (Petri's Theorem) (cf. [ACGH]). *Let C be a canonical curve of genus ≥ 4. Then C can be given by the intersection of some quadratic hypersurfaces, unless C is* *trigonal or isomorphic to a smooth plane quintic curve in these exceptional C is given by the intersection of some quadratic and cubic hypersurfaces.*

By this theorem, we have only to find the quadratic and cubic relations.

- Quadratic relations.

$$\#\{\text{the monomials of degree } 2\} = {}_gH_2\,, \qquad l(2(\omega_1)) = (2\times 2-1)(g-1)\,.$$

So the number of quadratic relations is

$${}_gH_2 - (2\times 2-1)(g-1) = (g-2)(g-3)/2\,.$$

Let these relations be $P_1(f_1, \cdots, f_g)=0, \cdots, P_{(g-2)(g-3)/2}(f_1, \cdots, f_g)=0$.

- Cubic relations.

The rank of cubic relations
$= \#\{\text{the monomials of degree 3 generated by } f_i\text{'s}\} - l(3(\omega_1))$
$-$(the rank of cubic relations obtained by the quadratic relations),

The rank of cubic relations obtained by the quadratic relations
$=$ the rank $\langle x_i P_j(x_1, \cdots, x_g)\rangle \qquad (1\le i\le g,\ 1\le j\le (g-2)(g-3)/2)$.

We apply these to $X_0(N)$ of genus $g=3, 4, 5, 6$.

- Case $g=3$. There is no quadratic and cubic relations. The number of quartic relations is ${}_3H_4-(2\times 4-1)(3-1)=1$. So the defining equation of $X_0(N)$ is obtained by the quartic relation.
- Case $g=4$. The number of quadratic and cubic relations are one each.
- Case $g=5$. The number of quadratic relations is three.
- Case $g=6$. The number of quadratic relations is six. There are no cubic relations: because ${}_6H_3=56$, $l(3(\omega_1))=(2\times 3-1)(6-1)=25$ and the number of cubic relations which come from quadratic relations is 31, the number of cubic relations is $56-25-31=0$.

When $g\ge 7$, we can calculate the relations by the same way.

3. Covering.

If $X_0(N)$ has genus one, its defining equation can not be obtained by the method in Section 2.2. But some of them can be obtained by means of a covering between modular curves.

For example, We have a defining equation of $X_0(20)$ as follows. A defining equation of $X_0(40)$ is

$$y^2 = x^8+8x^6-2x^4+8x^2+1 = \prod_{i=1}^{4}(x+\alpha_i)(x-\alpha_i)\,.$$

Let

$$Y^2=F(X)=\prod_{i=1}^{4}(X-\beta_i)\,, \qquad \deg F(X)=4\,, \quad F(X)\in \boldsymbol{Q}[X, Y]$$

be a defining equation of $X_0(20)$ and

$$\phi: X_0(40) \longrightarrow X_0(20)$$

$$(x, y) \longmapsto \left(\frac{p_2(x, y)}{p_1(x, y)}, \frac{q_2(x, y)}{q_1(x, y)}\right)$$

be a covering map from $X_0(40)$ to $X_0(20)$. This map can be taken $\boldsymbol{Q}$-rational. Moreover, by modifying the equation of $X_0(20)$ we can assume $p_1, p_2, q_1, q_2 \in \boldsymbol{Z}[x, y]$ and monic. The next diagram commutes:

$$\begin{array}{ccc} \Gamma_0(40)\backslash\mathfrak{H}^* & \longrightarrow & \Gamma_0(20)\backslash\mathfrak{H}^* \\ \downarrow l & & \downarrow l \\ X_0(40) & \xrightarrow{\phi} & X_0(20) \\ \downarrow x & & \downarrow X \\ \boldsymbol{P}^1 & \longrightarrow & \boldsymbol{P}^1 . \end{array}$$

Hence $p_1(x, y), p_2(x, y) \in \boldsymbol{Z}[x]$. Since $(\Gamma_0(20):\Gamma_0(40))=2$, the degrees of p_1 and p_2 are at most two.

The next rational function has the factor $y^2-f(x)$ in its numerator.

(1) $$\left(\frac{q_2}{q_1}\right)^2 - F\left(\frac{p_2}{p_1}\right).$$

It leads to the following form by comparing the coefficients:

$$\phi: (x, y) \longmapsto \left(\frac{p_2}{p_1}, \frac{y}{p_1^2}\right), \qquad p_1, p_2 \in \boldsymbol{Z}[x]\,.$$

The Weierstrass points of $X_0(40)$ are sent to those of $X_0(20)$ by ϕ. $\boldsymbol{Q}(\alpha_i)/\boldsymbol{Q}$ is of degree eight, $\boldsymbol{Q}(\alpha_i^2)/\boldsymbol{Q}$ is of degree four and $\boldsymbol{Q}(\beta_i)/\boldsymbol{Q}$ is of degree at most four. Therefore, we see

$$\frac{p_2(\pm\alpha_i)}{p_1(\pm\alpha_i)}=\beta_i\,, \qquad i=1, 2, 3, 4\,.$$

Further, p_1 and p_2 cannot have terms of degree 1.

Hence ϕ must be

$$\phi:(x,y)\longmapsto(x^2,y)\,,\quad \text{or}\quad \phi:(x,y)\longmapsto\left(\frac{1}{x^2},\frac{y}{x^4}\right)$$

(up to linear transformation for X). In both cases we have

$$X_0(20):\ Y^2=X^4+8X^3-2X^2+8X+1\,.$$

By (1), we find $y^2=f(x)=F(x^2)$ or $y^2=x^8F(1/x^2)=f(x)$.

There are three curves which are given in this way. They are

$$X_0(14)\,,\quad X_0(20)\,,\quad X_0(24)\,.$$

The next example is a covering not between modular curves, but from a modular curve to an elliptic curve. There exists a covering

$$X_0(43)\to E_{43}:Y^2+Y=X^3+X^2\,.$$

The curve E_{43} is an elliptic curve with conductor 43. We will explicitly give this map by using the main involution.

The action of the main involution to cusp forms with weight 2 is given by the Brandt matrix (cf. [P]). The invariant part of the function field of

$$X_0(43):\ 2x^3y+\cdots=0 \qquad \text{(see Table 7)}$$

under the action of the main involution is generated by

$$\xi=z-\frac{x+y}{2}\,,\qquad \eta=xy\,.$$

The covering will be obtained by describing the defining equation of $X_0(43)$ with ξ and η:

$$X_0(43)\longrightarrow E_{43}$$

$$(y,z)\longmapsto(X,Y)\,,$$

$$X=\frac{y}{2}\,,\qquad Y=\frac{4y+3y^2-4z-4yz+4z^2}{4}\,.$$

Here the defining equation of $X_0(43)$ is dehomogenized by setting $x=1$.

4. Covering between modular curves.

Let M, N be positive integers such that N divides M and put $n=(\Gamma_0(N):\Gamma_0(M))$. Then there is a covering map from $X_0(M)$ to $X_0(N)$. We will explicitly give this covering. There are three cases according as $X_0(M)$ and $X_0(N)$ are hyperelliptic or not.

- Case 1: both $X_0(M)$ and $X_0(N)$ are hyperelliptic. There are only one such pair (M,N): $M=46$, $N=23$.

$$X_0(46):\ y^2=(x^3-2x^2+3x-1)(x^3+x^2-x+7)$$
$$\cdot(x^6-x^5+4x^4-x^3+2x^2+2x+1)\,,$$
$$X_0(23):\ y^2=(x^3-x+1)(x^3-8x^2+3x-7)\,,$$
$$(x,y)\longmapsto\left(\frac{x^3-x^2+2x}{x-1},\frac{y(x^3-2x^2+x-1)}{(x-1)^3}\right).$$

This is obtained by giving a rational map which maps the Weierstrass points of $X_0(46)$ to those of $X_0(23)$.

• Case 2: both $X_0(M)$ and $X_0(N)$ are non-hyperelliptic. We can take a basis $\{f_1,\cdots,f_{g_n},\cdots,f_{g_m}\}$ of $S_2(\Gamma_0(M))$, such that $\{f_1,\cdots,f_{g_n}\}$ is a basis of $S_2(\Gamma_0(N))$. Further, let (x_i) be the coordinate corresponding to $\{f_i\}$. Then

$$X_0(M)\longrightarrow X_0(N)$$
$$(x_1,\cdots,x_{g_n},\cdots,x_{g_m})\longmapsto(x_1,\cdots,x_{g_n})$$

is a covering, because the following diagram commutes:

$$\begin{array}{ccc}\Gamma_0(M)\backslash\mathfrak{H}^* & \xrightarrow{proj.} & \Gamma_0(N)\backslash\mathfrak{H}^* \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l} \\ X_0(M) & \xrightarrow{proj.} & X_0(N) \\ \downarrow & & \downarrow \\ \boldsymbol{P}^{g_m} & \xrightarrow{proj.} & \boldsymbol{P}^{g_n}\,. \end{array}$$

• Case 3: $X_0(M)$ is non-hyperelliptic, but $X_0(N)$ is hyperelliptic. Let $\{f_1,\cdots,f_{g_m}\}$ and (x_i) be the same as in case 2 and let f_1 and f_2 be a basis of $S_2(\Gamma_0(N))$ satisfying the following equation:

$$X_0(N):\ y^2=f(x)\,,\ x=f_2/f_1\,.$$

If

$$\phi:\ X_0(M)\longrightarrow X_0(N)$$
$$(x_1,\cdots,x_{g_m})\longmapsto\left(\frac{x_2}{x_1},\frac{p_1(x_1,\cdots,x_{g_m})}{p_2(x_1,\cdots,x_{g_m})}\right)$$

is a covering, then $\phi^*(dx/y)$ is a holomorphic 1-form on $X_0(M)$ and the degree of p_1, p_2 are chosen to be at most n. By Lemma 1 we obtain

$$\phi^*\left(\frac{dx}{y}\right)=2\pi i(a_1f_1+\cdots+a_{g_m}f_{g_m})dz \qquad (a_i\in\mathbf{Q}),$$

$$(\phi^*y)(a_1f_1+\cdots+a_{g_m}f_{g_m})=\frac{1}{2\pi i}\frac{dx}{dz}=q\frac{dx}{dq},$$

$$p_1(a_1f_1+\cdots+a_{g_m}f_{g_m})$$
$$=\sum_{i_1+\cdots+i_{g_m}=n+1} a_{i_1,\cdots,i_{g_m}}f_1^{i_1}\cdots f_{g_m}^{i_{g_m}}=p_2q\frac{dx}{dq}$$
$$=\left(\sum_{j_1+\cdots+j_{g_m}=n} b_{j_1,\cdots,j_{g_m}}f_1^{j_1}\cdots f_{g_m}^{j_{g_m}}\right)q\frac{dx}{dq}.$$

We know the Fourier expansions of f_i's and dx/dq. Therefore we can decide p_1, p_2 by comparing their Fourier coefficients.

Example:

$$X_0(44) \longrightarrow X_0(22)$$

$$(x, y, z, w)\longmapsto(xy^2, xw(x+4y+8z), y^3)=(y^2, w(x+4y+8z), z(x+4y+4z)).$$

TABLE 1. Example of Fourier expansions ($X_0(46)$, $g=5$)

f_1	$q-q^3-q^4-2q^6+2q^7-q^8+2q^9+2q^{10}-4q^{11}+3q^{12}+3q^{13}+2q^{14}-4q^{15}+2q^{17}+\cdots$
f_2	$q^2-2q^3-q^4+2q^5+q^6+2q^7-2q^8-2q^{10}-2q^{11}+q^{12}+2q^{15}+\cdots$
f_3	$q^2-q^6-q^8-2q^{12}+2q^{14}-q^{16}+2q^{18}+\cdots$
f_4	$q^4-2q^6-q^8+2q^{10}+q^{12}+2q^{14}-2q^{16}-2q^{20}+\cdots$
f_5	$q-q^2+q^4+4q^5-4q^7-q^8-3q^9-4q^{10}+2q^{11}-2q^{13}+4q^{14}+q^{16}-2q^{17}+\cdots$
g_1	$q^5+q^6-q^7-q^9-2q^{10}+q^{11}-q^{12}-q^{13}+q^{15}+q^{16}-q^{17}+\cdots$
g_2	$q^4-2q^6-q^8+2q^{10}+q^{12}+2q^{14}-2q^{16}-2q^{20}+\cdots$
x	$q^{-1}-1-q^2+q^3-q^4+q^5-q^6+2q^7-2q^8+2q^9-2q^{10}+3q^{11}-3q^{12}+\cdots$
y	$-q^{-6}+q^{-5}-2q^{-4}+q^{-3}-q^{-2}-3q^{-1}+8-22q+47q^2-92q^3+160q^4-275q^5+459q^6-\cdots$

The following tables are our defining equations of modular curves. After Table 7, our defining equations are given by the intersection of some hypersurfaces.

TABLE 2. ($g=2$)

$X_0(N)$	$y^2=f(x)$	Discriminant of $f(x)$
$X_2(22)$	$y^2=x^6+6x^5+11x^4+24x^3+11x^2+18x-7$	$-2^{24}\cdot 11^4$
$X_0(23)$	$y^2=x^6-8x^5+2x^4+2x^3-11x^2+10x-7$	$-2^{12}\cdot 23^6$
$X_0(26)$	$y^2=x^6-8x^5+8x^4-18x^3+8x^2-8x+1$	$-2^{20}\cdot 13^3$
$X_0(28)$	$y^2=x^6+10x^4+25x^2+28$	$2^{28}\cdot 7^3$
$X_0(29)$	$y^2=x^6-4x^5-12x^4+2x^3+8x^2+8x-7$	$-2^{12}\cdot 29^5$
$X_0(31)$	$y^2=x^6-8x^5+6x^4+18x^3-11x^2-14x-3$	$-2^{12}\cdot 31^4$
$X_0(37)$	$y^2=x^6+8x^5-20x^4+28x^3-24x^2+12x-4$	$-2^{12}\cdot 37^3$
$X_0(50)$	$y^2=x^6-4x^5-10x^3-4x+1$	$-2^{16}\cdot 5^5$

TABLE 3. ($g=3$, hyperelliptic)

$X_0(N)$	$y^2=f(x)$	Discriminant of $f(x)$
$X_0(30)$	$y^2=x^8+6x^7+9x^6+6x^5-4x^4-6x^3+9x^2-6x+1$	$2^{28}\cdot 3^6\cdot 5^4$
$X_0(33)$	$y^2=x^8+10x^6-8x^5+47x^4-40x^3+82x^2-44x+33$	$2^{16}\cdot 3^{12}\cdot 11^6$
$X_0(35)$	$y^2=x^8-4x^7-6x^6-4x^5-9x^4+4x^3-6x^2+4x+1$	$2^{16}\cdot 5^8\cdot 7^6$
$X_0(39)$	$y^2=x^8-6x^7+3x^6+12x^5-23x^4+12x^3+3x^2-6x+1$	$2^{16}\cdot 3^8\cdot 13^4$
$X_0(40)$	$y^2=x^8+8x^6-2x^4+8x^2+1$	$2^{40}\cdot 5^4$
$X_0(41)$	$y^2=x^8-4x^7-8x^6+10x^5+20x^4+8x^3-15x^2-20x-8$	$-2^{16}\cdot 41^6$
$X_0(48)$	$y^2=x^8+14x^4+1$	$2^{40}\cdot 3^4$

TABLE 4. ($g=4$, hyperelliptic)

$X_0(N)$	$y^2=f(x)$	Discriminant of $f(x)$
$X_0(47)$	$y^2=x^{10}-6x^9+11x^8-24x^7+19x^6-16x^5-13x^4$ $+30x^3+38x^2+28x-11$	$-2^{20}\cdot 47^8$

TABLE 5. ($g=5$, hyperelliptic)

$X_0(N)$	$y^2=f(x)$	Discriminant of $f(x)$
$X_0(46)$	$y^2=x^{12}-2x^{11}+5x^{10}+6x^9-26x^8+84x^7-113x^6$ $+134x^5-64x^4+26x^3+12x^2+8x-7$	$-2^{54}\cdot 23^7$
$X_0(59)$	$y^2=x^{12}-8x^{11}+22x^{10}-28x^9+3x^8+40x^7-62x^6$ $+40x^5-3x^4-24x^3+20x^2-4x-8$	$-2^{24}\cdot 59^9$

TABLE 6. ($g=6$, hyperelliptic)

$X_0(N)$	$y^2=f(x)$	Discriminant of $f(x)$
$X_0(71)$	$y^2=x^{14}-10x^{13}+37x^{12}-66x^{11}+66x^{10}-48x^9+15x^8$ $+40x^7-66x^6+66x^5-58x^4+40x^3-27x^2+6x-7$	$-2^{28}\cdot 71^{10}$

TABLE 7. ($g=3$, non-hyperelliptic)

$X_0(34)$	$x^4+y^4-z^4+x^3y+xy^3-2x^2y^2+3xyz^2=0$
$X_0(43)$	$2x^3y+6x^2y^2+11xy^3+9y^4-x^3z-6x^2yz-14xy^2z-12y^3z+2x^2z^2+8xyz^2$ $+10y^2z^2-xz^3-4yz^3+z^4=0$
$X_0(45)$	$x^4+y^4+81z^4-2x^2y^2-2x^2z^2-18y^2z^2-16xy^2z=0$
$X_0(64)$	$x^4+y^4-z^4=0$

TABLE 8. ($g=4$, non-hyperelliptic)

$X_0(38)$	$4x^2+16y^2+9z^2-5w^2-20xy-8xw-16yw=0$ $28xy^2-xz^2+5xw^2-16y^3-11yz^2+15yw^2+16y^2w-5z^2w+w^3-16xyw=0$
$X_0(44)$	$x^2+8y^2+16z^2-w^2+4xy+4xz+16yz=0$ $y^3+16yz^2+8y^2z+16z^3-zw^2=0$
$X_0(53)$	$x^2-y^2+11z^2+7w^2+2yz-2yw-10zw=0$ $yz^2+5yw^2-zw^2+y^2w+2zw^2+6w^3-yzw=0$
$X_0(54)$	$z^2-w^2-4xy=0$ $x^3+8y^3-w^3-3xyw=0$
$X_0(61)$	$x^2-y^2+3z^2-5w^2+2yz-6yw+6zw=0$ $yz^2+5yw^2+5zw^2+y^2w+4z^2w+6w^3+yzw=0$
$X_0(81)$	$xy-w^2=0$ $x^3+27y^3-z^3+9xyw=0$

TABLE 9. ($g=5$, non-hyperelliptic)

$X_0(42)$	$2x^2+6xy+4zw-zu+2wu-u^2=0$ $-x^2+9y^2+zu-2wu=0$ $4x^2+3z^2+20zw+12w^2-2zu+4wu-5u^2=0$
$X_0(51)$	$8x^2+36xy+72y^2-14z^2-xw-3yw+7w^2+15wu=0$ $-20xy+6z^2+xw+3yw-7w^2+8xu+24yu-7wu=0$ $-12xy+2z^2-xw-3yw-w^2-wu+8u^2=0$
$X_0(52)$	$11x^2-16xy+32y^2+9z^2-4xu+16yu-16u^2=0$ $x^2-4xy+3z^2+12zw-4xu=0$ $-19x^2+16xy-33z^2+96w^2+36xu-80yu+16u^2=0$
$X_0(55)$	$2x^2-106xy+50y^2-7z^2+5w^2+22wu+27u^2=0$ $-196xy-21z^2+4xw+20yw+17w^2+50wu+49u^2=0$ $-84xy-7z^2+7w^2+4xu+20yu+10wu+15u^2=0$
$X_0(56)$	$4y^2-w^2+u^2=0$ $4xy+16xz+16yz-4yw+w^2-u^2=0$ $4x^2-32xz+64z^2+8yw+3w^2-7u^2=0$
$X_0(57)$	$12x^2+15xy+3z^2-8zw+w^2+2zu-10wu=0$ $-3x^2+27y^2+4zw-zu=0$ $36x^2+9z^2-24zw+8w^2+6zu-40wu+5u^2=0$
$X_0(63)$	$x^2-2xy+9y^2+4zw-u^2=0$ $-4xy-z^2-2zw+3w^2+u^2=0$ $-4xy-2z^2-2zw+xu+3yu+u^2=0$
$X_0(65)$	$x^2+y^2+2yz+3z^2-2w^2=0$ $-4x^2+5y^2-14yz-w^2+6wu=0$ $x^2-2y^2+2yz+w^2+3u^2=0$
$X_0(67)$	$110x^2-225xy+110y^2+52z^2-108zw-54w^2+46zu+187wu=0$ $2yz+3yw-5xu+6yu=0$ $25xy+4z^2+12zw-16w^2+34zu-99wu+66u^2=0$
$X_0(72)$	$x^2+2xy+9y^2-u^2=0$ $-4xy+z^2+4w^2-u^2=0$ $-4xy+2z^2-xu+3yu-u^2=0$
$X_0(73)$	$15x^2-87xy+30y^2+44z^2+24zw-83w^2+46zu-71wu=0$ $-2yz-yw+3xu+2yu=0$ $9xy-8z^2-8zw+16w^2-2zu+17wu+10u^2=0$
$X_0(75)$	$24xy-z^2+4w^2-3u^2=0$ $12x^2+300y^2+z^2-16zw+3u^2=0$ $13z^2-20zw-20w^2+12xu+60yu+15u^2=0$

TABLE 10. ($g=6$, non-hyperelliptic)

$X_0(58)$	$-16x^2-24xy-16y^2+40yz+12z^2+40xw-u^2+6uv+11v^2=0$ $12x^2+8xy-8y^2+10xz-14z^2+40yw+2u^2-7uv-7v^2=0$ $-4x^2+4xy+16y^2+8z^2+20zw+u^2+4uv-v^2=0$ $x^2-4y^2-z^2+4w^2-uv=0$ $-3xu-8yu+3zu+8wu+3xv-8yv+7zv=0$ $xu-zu-xv+3zv+8wv=0$
$X_0(79)$	$-791x^2+791y^2-1582yz+1029z^2+3658yw+13257zw+2027w^2$ $-32719yu-2222zu-13597yv+30287zv=0$ $399x^2-399y^2+798yz+1203z^2+128yw+506zw+5580yu$ $+4547zu+2027wu+3576yv+1369zv=0$ $-6x^2+6y^2-12yz+241z^2+120yw+221zw-343yu+1349zu$ $+2027u^2+312yv+650zv=0$ $97x^2-97y^2+194yz-180z^2+87yw-3235zw+7910yu+826zu$ $+7118yv-7130zv+2027wv=0$ $-101x^2+101y^2-202yz-335z^2-7yw-4zw-1382yu-1278zu$ $-829yv+131zv+2027uv=0$ $16x^2-16y^2+32yz+33z^2-320yw+762zw-1788yu-219zu$ $-2859yv+1645zv+2027v^2=0$

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