TOKYO J. MATH.
VOL. 19, NO. 1, 1996

Uniqueness for an Inverse Problem for the Wave Equation in the Half Space

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(Communicated by T. Suzuki)

1. Introduction.

Let us consider the initial boundary value problem:

$$u_{tt}-\Delta_x u+b(x)u_t+a(x)u=0 \qquad \text{in } \mathbf{R}^n_+\times(0, T)\,, \tag{1.1}$$

$$u=u_t=0 \qquad \text{on } \mathbf{R}^n_+\{0\}\,, \tag{1.2}$$

$$\partial_z u(y, 0, t)=f(y, t) \qquad \text{on } \mathbf{R}^{n-1}\times(0, T)\,. \tag{1.3}$$

Here $\mathbf{R}^n_+$ $(n>1)$ is the half space:

$$\mathbf{R}^n_+=\{x=(y, z)\in\mathbf{R}^n : y\in\mathbf{R}^{n-1}, z\in\mathbf{R}, z>0\}\,,$$

and $a\in L^\infty(\mathbf{R}^n_+)$, $b\in C^2(\overline{\mathbf{R}^n_+})$. Throughout this article we assume that

$$a \text{ and } b \text{ are constants on } \{x : |x|>r\}\,. \tag{1.4}$$

We are interested in uniqueness results of a, b from the Neumann to Dirichlet map:

$$\Lambda(a, b):\ f\mapsto u(y, 0, t) \qquad \text{on } \mathbf{R}^{n-1}\times(0, T)\,.$$

In the case that $b\equiv 0$, Rakesh [6] showed that $\Lambda(a, 0)$ uniquely determines a if T is large enough. Some authors studied the problem stated above in the bounded domain case instead of the half space ([2], [3]). Rakesh pointed out that the proof for the bounded domain case does not apply to the half space one, but he succeeded to prove the uniqueness mentioned above by using the results of x-ray transform obtained by Hamaker, Smith, Solmon, and Wagner ([1]). In this article we show that the result of Rakesh can be extended for the mixed problem (1.1)–(1.3) by the methods which are used in [3]. By u_j and $\Lambda_j\equiv\Lambda(a_j, b_j)$, we denote solutions and the Neumann to Dirichlet maps to the problem (1.1)–(1.3) corresponding to $a=a_j$, $b=b_j$ $(j=1, 2)$ respectively.

Received October 24, 1994
Revised January 12, 1995

The following is our main theorem.

THEOREM 1. *For $T>(\pi+1)r$, suppose that $\Lambda_1=\Lambda_2$, then*

$$a_1=a_2 \quad \text{and} \quad b_1=b_2 \qquad \text{on } \mathbf{R}^n_+ .$$

In the next section, we collect some lemmas which will be used in the proof of the theorem.

2. Lemmas.

We first state the results of the x-ray transform in [1].

LEMMA 2.1 ([1]). *Suppose $p(x)$ is a bounded function on $\mathbf{R}^n$ with compact support, and A an infinite subset of $\mathbf{R}^n$ bounded away from the convex hull of the support of p. If the* x-*ray transform*

$$\int_0^{+\infty} p(a+s\omega)ds=0 \qquad \text{for any } a\in A \text{ and any } \omega\in\mathbf{S}^{n-1} ,$$

then $p=0$.

From (1.4), a_j, b_j $(j=1, 2)$ are constants (possibly different) on $\{x:|x|>r\}$. We claim that $a_1=a_2$ and $b_1=b_2$ on $\{x:|x|>r\}$ by the same argument in [6]. We consider the case that the spatial dimension is equal to one.

LEMMA 2.2. *Assume a_j, b_j $(j=1, 2)$ are real constants, and u_j $(j=1, 2)$ satisfy*

$$u^j_{tt}-u^j_{zz}+b_j u^j_t+a_j u^j=0 \qquad z\geq 0\,,\ t\in(0, T)\,,$$
$$u^j(z, 0)=u^j_t(z, 0)=0 \qquad z\geq 0\,,$$
$$u^j_z(0, t)=f(t) \qquad t\in(0, T)\,.$$

If $u^1(0, t)=u^2(0, t)$ for any $f\in C^2(0, T)$ and $f\neq 0$, then $a_1=a_2$ and $b_1=b_2$.

PROOF OF LEMMA 2.2. Consider the mixed problem

$$u_{tt}-u_{zz}+bu_t+au=0 \qquad z\geq 0\,,\ t\in(0, T)\,,$$
$$u(z, 0)=u_t(z, 0)=0 \qquad z\geq 0\,,$$
$$u_z(0, t)=f(t) \qquad t\in(0, T)\,.$$

Then the following cases occur: (1) $a-b^2/4>0$, (2) $a-b^2/4<0$, (3) $a-b^2/4=0$.

Case (1). Set

$$w(k, z, t)=e^{(ik\sqrt{a-b^2/4}+(b/2)t)}u(z, t) \qquad k\in\mathbf{R}\,.$$

Then w solves the following problem:

$$w_{tt}-w_{zz}-w_{kk}=0 \qquad z\geq 0\,,\ k\in\mathbf{R}\,,\ t\in(0,T)\,,$$

$$w(k,z,0)=w_t(k,z,0)=0 \qquad z\geq 0\,,\ k\in\mathbf{R}\,,$$

$$w_z(k,0,t)=e^{(ik\sqrt{a-b^2/4}+(b/2)t)}f(t) \qquad k\in\mathbf{R}\,,\ t\in(0,T)\,.$$

It is known that w has the following representation ([8])

$$w(k,z,t)=-\frac{1}{2\pi}\iint_{\sqrt{(k-\zeta)^2+z^2+\xi^2}\leq t}\frac{w_2(\zeta,0,t-\sqrt{(k-\zeta)^2+z^2+\xi^2})}{\sqrt{(k-\zeta)^2+z^2+\xi^2}}\,d\xi\,d\zeta\,.$$

Thus we have

$$\begin{aligned}u(0,t)&=e^{-(b/2)t}w(0,0,t)\\&=e^{-(b/2)t}\left(-\frac{1}{2\pi}\right)\int_0^t\int_0^{2\pi}e^{ir\cos\theta\sqrt{a-b^2/4}+(b/2)(t-r)}f(t-r)\,dr\,d\theta\\&=\int_0^t f(r)e^{(b/2)(r-t)}\phi((t-r)\sqrt{m})\,dr\,,\end{aligned}$$

where $m=a-b^2/4$, $\phi(s)=-\frac{1}{2\pi}\int_0^{2\pi}e^{is\cos\theta}d\theta$. Noting that $\phi(0)=-1$, $\phi'(0)=0$, $\phi''(0)=1/2$, we obtain

$$\begin{aligned}u_{tt}(0,t)=-f'(t)+\frac{b}{2}f(t)+\int_0^t f(r)e^{(b/2)(r-t)}\Big\{\Big(-\frac{b}{2}\Big)^2\phi((t-r)\sqrt{m})\\-b\phi'((t-r)\sqrt{m})\sqrt{m}+\phi''((t-r)\sqrt{m})m\Big\}dr\,,\end{aligned}$$

$$\begin{aligned}u_{ttt}(0,t)=-f''(t)+\frac{b}{2}f'(t)-\Big(-\frac{b}{2}\Big)^2f(t)+\frac{m}{2}f(t)\\+\int_0^t\Big(-\frac{b}{2}\Big)f(r)e^{(b/2)(r-t)}\Big\{\Big(-\frac{b}{2}\Big)^2\phi((t-r)\sqrt{m})\\-b\phi'((t-r)\sqrt{m})\sqrt{m}+\phi''((t-r)\sqrt{m})m\Big\}dr\\+\int_0^t f(r)e^{(b/2)(r-t)}\Big\{\Big(-\frac{b}{2}\Big)^2\phi'((t-r)\sqrt{m})\sqrt{m}\\-b\phi''((t-r)\sqrt{m})m+\phi'''((t-r)\sqrt{m})m^{3/2}\Big\}dr\,.\end{aligned}$$

If $u^1(0,t)=u^2(0,t)$, then $u^1_{tt}(0,t)=u^2_{tt}(0,t)$. Hence we have

$$\frac{b_1}{2}f(t)+\int_0^t f(r)e^{(b_1/2)(r-t)}\left\{\left(-\frac{b_1}{2}\right)^2\phi((t-r)\sqrt{m_1})\right.$$
$$\left.-b_1\phi'((t-r)\sqrt{m_1})\sqrt{m_1}+\phi''((t-r)\sqrt{m_1})m_1\right\}dr$$
$$=\frac{b^2}{2}f(t)+\int_0^t f(r)e^{(b_2/2)(r-t)}\left\{\left(-\frac{b_2}{2}\right)^2\phi((t-r)\sqrt{m_2})\right.$$
$$\left.-b_2\phi'((t-r)\sqrt{m_2})\sqrt{m_2}+\phi''((t-r)\sqrt{m_2})m_2\right\}dr,$$

where $m_j=a_j-b_j^2/4, j=1, 2$. From this equality, there exists a constant $C_1>0$ such that

$$|b_1-b_2||f(t)|\le C_1\int_0^t |f(r)|dr.$$

Suppose that $b_1\ne b_2$, then there exists a constant $C_2>0$ such that

$$|f(t)|\le C_2\int_0^t |f(r)|dr.$$

From Gronwall's inequality, we see that $|f(t)|=0$. This contradicts the hypothesis $f\ne 0$. So we conclude that $b_1=b_2$.

If $u^1(0, t)=u^2(0, t)$, then $u^1_{ttt}(0, t)=u^2_{ttt}(0, t)$. Using $b_1=b_2$, we see that the following inequality is obtained by the same procedures in the case that $b_1=b_2$.

$$|m_1-m_2||f(t)|\le C_3\int_0^t |f(r)|dr,$$

where C_3 is a positive constant. So we conclude $m_1=m_2$ by the same argument used above. From the definition m_j $(j=1, 2)$, if $m_1=m_2$ and $b_1=b_2$, then we have $a_1=a_2$.

Case (2). Set

$$w(k, z, t)=-e^{(ik\sqrt{b^2/4-a}+(b/2)t)}u(z, t),$$

then it is easily seen that $a_1=a_2$ and $b_1=b_2$ by the same procedures those are used in the Case (1).

Case (3). Set

$$v(z, t)=e^{(b/2)t}u(z, t),$$

then v solves the following problem:

$$v_{tt}-v_{zz}=0 \qquad z\geq 0\,,\ t\in(0, T)\,,$$
$$v(z, 0)=v_t(z, 0)=0 \qquad z\geq 0\,,$$
$$v_z(0, t)=e^{(b/2)t}f(t) \qquad t\in(0, T)\,.$$

It is known that v has the following representation ([8])

$$\begin{aligned} v(z, t) &= -H(t-|z|)\int_0^{t-z} v_z(0, t')dt' \\ &= -\int_0^{t-z} e^{(b/2)t'}f(t')dt' \qquad \text{for } t>z\,, \end{aligned}$$

where H is the Heaviside function. Hence we obtain

$$u(0, t)=e^{-(b/2)t}v(0, t)=-e^{-(b/2)t}\int_0^t e^{(b/2)r}f(r)dr\,.$$

So we have

$$u_{tt}(0, t)=-\frac{b^2}{4}e^{-(b/2)t}\int_0^t e^{(b/2)r}f(r)dr+\frac{b}{2}f(t)-f'(t)\,.$$

If $u^1(0, t)=u^2(0, t)$, then $u^1_{tt}(0, t)=u^2_{tt}(0, t)$. Hence there exists a constant $C_4>0$ such that

$$|b_1-b_2|\,|f(t)|\leq C_4\int_0^t |f(r)|dr\,.$$

So we conclude $b_1=b_2$ by the same procedures in the case (1). Recalling $a_j=b_j^2/4$ $(j=1, 2)$, we see that $a_1=a_2$. The proof is complete.

We need the following unique continuation result.

LEMMA 2.3. *Suppose a, b are real numbers, ρ, l, T are positive real numbers, and $u(y, z, t)$ is a distribution on $\mathbf{R}^{n+1}$ satisfying*

$$u_{tt}-\Delta_y u-u_{zz}+bu_t+au=0$$

on

$$\{(y, z, t): |y|<\rho, 0\leq z<l, |t|<T\}\,.$$

If u and u_z are zero on

$$\{(y, 0, t): |y|<\rho, |t|<T\}\,,$$

then u is zero on

$$\{(y, z, t): |y|<\rho, 0\leq z<l, 2z+|t|<T\}\,.$$

PROOF OF LEMMA 2.3. As in the proof of lemma 2.2, the equation

$$u_{tt}-\Delta_y u-u_{zz}+bu_t+au=0$$

is transformed to the following equation

$$w_{tt}-\Delta_y w-w_{zz}-w_{kk}=0\,.$$

So we can apply Lemma 2 in [6], then we obtain lemma 2.3.

3. Proof of theorem.

PROOF OF THEOREM 1. We first claim that $a_1=a_2$ and $b_1=b_2$ if $|x|>r$. If $\Lambda_1=\Lambda_2$ for all $f\in C_0^\infty(\mathbf{R}^{n-1}\times(0,T))$, then $u_1(0,0,t)=u_2(0,0,t)$ for all $f\in C_0^\infty(\mathbf{R}^{n-1}\times(0,T))$ with $\operatorname{supp}f\subset\{y:|y|<T\}\times(0,T)$. From the domain of dependence of u_j, we see that $u_1(0,0,t)=u_2(0,0,t)$ for all f which are independent of y i.e. for all $f\in C^2(0,T)$. If f is independent of y then so are u_j $(j=1,2)$. Hence we can apply lemma 2.2 to our case. So we have $a_1=a_2$ and $b_1=b_2$ on $\{x:|x|>r\}$. From now on we may assume that $a_1=a_2$ and $b_1=b_2$ on $\{x:|x|>r\}$. Set $w=u_1-u_2$. If $\Lambda_1=\Lambda_2$, then we have

$$\begin{aligned} &w_{tt}-\Delta_x w+b_1 w_t+a_1 w=Bu_{2t}+Au_2 \quad \text{in } \mathbf{R}^n_+\times(0,T)\,,\\ &w(x,0)=w_t(x,0)=0 \quad \text{on } \mathbf{R}^n_+\,,\\ &w(y,0,t)=0\,,\quad w_z(y,0,t)=f-f=0 \quad \text{on } \mathbf{R}^{n-1}\times(0,T)\,, \end{aligned} \tag{3.1}$$

where $A=a_2-a_1$, $B=b_2-b_1$. Since $a_1=a_2$ and $b_1=b_2$ on $\{x:|x|>r\}$, we have

$$\begin{aligned} &w_{tt}-\Delta_x w+b_1 w_t+a_1 w=0 \quad x\in\mathbf{R}^n_+\,,\ |x|>r\,,\ t\in(0,T)\,,\\ &w(y,0,t)=0\,,\quad w_z(y,0,t)=0 \quad |y|>r\,,\ t\in(0,T)\,. \end{aligned} \tag{3.2}$$

From (1.4), a_j, b_j $(j=1,2)$ are constants on $\{x:|x|>r\}$. Applying lemma 2.3, we conclude the following by the same procedures in [6]:

$$w(x,t)=0 \quad \text{for } r\le|x|\le r+\varepsilon\,,\ z>0\,,\ t\in(0,T^*)\,,$$

where $\varepsilon=(T-(\pi+1)r)/6$, $T^*=T-\pi(r+\varepsilon)$. Hence, using (3.2), we obtain

$$w(x,t)=0\,,\quad \partial_\nu w(x,t)=0 \quad \text{for } (x,t)\in\partial H\times(0,T^*)\,. \tag{3.3}$$

Here H is the hemiball $H\equiv\{x\in\mathbf{R}^n_+:|x|<r\}$, and ν is the outer normal to ∂H.
Assume that $v(x,t)$ is a smooth function satisfying

$$v_{tt}-\Delta_x v-b_1 v_t+a_1 v=0 \quad (x,t)\in H\times(0,T^*)\,, \tag{3.4}$$

$$v(x,T^*)=v_t(x,T^*)=0 \quad x\in H\,. \tag{3.5}$$

Using (3.1), (3.3) and the divergence theorem, we have

$$\int_{H\times(0,T^*)} (Bu_{2t}+Au_2)v\,dxdt=\int_{H\times(0,T^*)} (w_{tt}-\Delta_x w+b_1 w_t+a_1 w)v\,dxdt$$
$$=\int_H [w_t v-w v_t]_0^{T^*}dx\,.$$

From the initial conditions of v and w, we conclude that

$$\int_{H\times(0,T^*)} (Bu_{2t}+Au_2)v\,dxdt=0\,. \tag{3.6}$$

To continue the proof we need the following lemma due to Isakov [2].

LEMMA 3.1 ([2]). *For any $\omega\in\mathbf{S}^{n-1}$, any $\tau>0$, any function $\phi\in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp}\phi\subset\{x\in\mathbf{R}^n: x_n<-\varepsilon, |x|<2\varepsilon\}\equiv N_\varepsilon$, there exist the solution u_2 to* (1.1)–(1.2) *and the solution v to* (3.4)–(3.5) *of the form*

$$u_2=\phi(x+t\omega)e^{-(1/2)\int_0^t b_2(x+s\omega)ds}e^{i\tau(x\cdot\omega+t)}+r(x,t)\,, \tag{3.7}$$

$$v=\phi(x+t\omega)e^{(1/2)\int_0^t b_1(x+s\omega)ds}e^{-i\tau(x\cdot\omega+t)}+R(x,t)\,. \tag{3.8}$$

Moreover

$$\begin{cases} r(x,0)=r_t(x,0)=0 & x\in\mathbf{R}^n_+\,,\\ r_z(y,0,t)=0 & (y,t)\in\mathbf{R}^{n-1}\times(0,T^*)\,,\\ \tau\|r\|_{L^2(\mathbf{R}^n_+\times(0,T^*))}+\|r_t\|_{L^2(\mathbf{R}^n_+\times(0,T^*))}\le C_1\,, \end{cases} \tag{3.9}$$

$$\begin{cases} R(x,T^*)=R_t(x,T^*)=0 & x\in H\,,\\ \partial_\nu R=0 & (x,t)\in\partial H\times(0,T^*)\,,\\ \tau\|R\|_{L^2(\mathbf{R}^n_+\times(0,T^*))}+\|R_t\|_{L^2(\mathbf{R}^n_+\times(0,T^*))}\le C_2\,, \end{cases} \tag{3.10}$$

where C_j $(j=1,2)$ are positive constants.

PROOF OF THEOREM 1 CONTINUED. Inserting (3.7) and (3.8) into (3.6), we have

$$0=i\tau\int_{H\times(0,T^*)} B(x)\phi^2(x+t\omega)e^{-(1/2)\int_0^t B(x+s\omega)ds}dxdt+\text{Remainder}\,.$$

From (3.9) and (3.10), we see that there exists a constant $C>0$ independent of τ such that

$$|\,\text{Remainder}\,|\le C\,.$$

Hence we have

$$0=\int_{H\times(0,T^*)} B(x)\phi^2(x+t\omega)e^{-(1/2)\int_0^t B(x+s\omega)ds}dxdt+\lim_{\tau\to+\infty}\left(\frac{\text{Remainder}}{i\tau}\right)$$

$$=\int_{H\times(0,T^*)} B(x)\phi^2(x+t\omega)e^{-(1/2)\int_0^t B(x+s\omega)ds}dxdt$$

$$=\int_{\mathbf{R}^n}\phi^2(x)\left(\int_0^{T^*} B(x-t\omega)e^{-(1/2)\int_0^t B(x-t\omega+s\omega)ds}dt\right)dx\,,$$

where we used the fact that B is zero outside H. Since the class of functions $\phi^2(x)$ is dense in $L^2(N_\varepsilon)$, we have

$$\int_0^{T^*} B(x-t\omega)e^{-(1/2)\int_0^t B(x-t\omega+s\omega)ds}dt=0 \qquad \text{for any } x\in N_\varepsilon, \text{ any } \omega\in \mathbf{S}^{n-1}\,.$$

Now let us change the variable of integration from s to l by the relation $s-t=-l$, then

$$\int_0^t B(x-t\omega+s\omega)ds=\int_0^t B(x-l\omega)dl\,.$$

Hence

$$0=\int_0^{T^*} B(x-t\omega)e^{-(1/2)\int_0^t B(x-l\omega)dl}dt$$

$$=\int_0^{T^*} -2\frac{d}{dt}\{e^{-(1/2)\int_0^t B(x-l\omega)dl}\}dt$$

$$=2\{1-e^{-(1/2)\int_0^{T^*} B(x-l\omega)dl}\}\,.$$

Therefore we conclude that

$$\int_0^{T^*} B(x-l\omega)dl=0 \qquad \text{for any } x\in N_\varepsilon, \text{ any } \omega\in \mathbf{S}^{n-1}\,.$$

Note that any point which is at a distance greater than T^* from some point of N_ε is outside H. Since B is zero outside H, we obtain

$$\int_0^{+\infty} B(x-l\omega)dl=0 \qquad \text{for any } x\in N_\varepsilon, \text{ any } \omega\in \mathbf{S}^{n-1}\,.$$

Applying lemma 2.1, we conclude that $B=b_2-b_1=0$.

From (3.4), using $B=0$, we have

$$0=\int_{H\times(0,T^*)} Au_2 v dxdt .$$

So, by using (3.7)–(3.10), we obtain

$$0=\int_{H\times(0,T^*)} A(x)\phi^2(x+t\omega)dxdt .$$

Hence, by the same argument used above, we conclude that $A=a_2-a_1=0$. This completes the proof of our main theorem.

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