TOKYO J. MATH.
VOL. 19, NO. 2, 1996

Critical Blow-Up for Quasilinear Parabolic Equations in Exterior Domains

Ryuichi SUZUKI*

Kokushikan University

(Communicated by T. Ishikawa)

Abstract. We consider nonnegative solutions to the exterior Dirichlet problem for quasilinear parabolic equations $u_t=\Delta u^m+u^p$ with $p=m+2/N$ and $m\geq 1$. In this paper we show that when $N\geq 3$ all nontrivial solutions to above problem blow up in finite time. For this aim, it is important to study the asymptotic behavior of solutions to the exterior Dirichlet problem for the quasilinear parabolic equations $u_t=\Delta u^m$.

1. Introduction.

This paper is continued from the previous our work with K. Mochizuki "Critical exponent and critical blow-up for quasilinear parabolic equations" [8].

Let $N\geq 2$ and let Ω be an exterior domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$. In the work [8] we considered the initial-boundary value problem

$$\partial_t u=\Delta u^m+u^p \quad \text{in} \quad (x, t)\in\Omega\times(0, T) \tag{1.1}$$

$$u(x, 0)=u_0(x) \quad \text{in} \quad x\in\Omega \tag{1.2}$$

$$u(x, t)=0 \quad \text{on} \quad (x, t)\in\partial\Omega\times(0, T) \tag{1.3}$$

where $p>m\geq 1$ and $u_0(x)\geq 0$ and showed that

$$p_m^*=m+2/N \tag{1.4}$$

is the critical exponent for the above initial boundary-value problem. Namely, the following results hold:

(I) If $m<p<p_m^*$, then all nontrivial nonnegative weak solutions of (1.1)–(1.3) blow up in finite time.

(II) If $p>p_m^*$, then all global solutions of (1.1)–(1.3) exist when the initial data are sufficiently small.

Received February 16, 1995

* Partially supported by Grant-in-Aid for Scientific Research (No. 05640241), Ministry of Education, Science and Culture, Japan.

Case (I) is called the blow-up case; (II) is called the global existence case. The definition of a nonnegative weak solutions of (1.1)–(1.3) is referred to [8].

It is not yet established in [8] whether or not $p=p_m^*$ is in the blow-up case (also refer to Levine [7] which is a survey of such results and related problems). It is the purpose of this paper to answer this problem. More precisely, our result is as follows:

THEOREM 1.1. *If* $N\geq 3$ *and* $p=p_m^*$, *then all nonnegative nontrivial solutions of* (1.1)–(1.3) *blow up in finite time.*

In the following, we assume

(1.5) $$p=p_m^* .$$

Further, without loss of generality, we assume

(1.6) $$\Omega=E_R\equiv\{x \mid |x|>R\} \qquad (R>0)$$

and $u_0(x)$ has the compact support in $\bar{\Omega}$:

(1.7) $$u_0(x)\in C_0(\bar{\Omega}) .$$

Let $u(x, t)$ be a global weak solution to (1.1)–(1.3) with $\Omega=E_R$ and $p=p_m^*$. Then we see that $u(\cdot, t)\in L^1(\Omega)$ for each $t\in(0, T)$. In order to show Theorem 1.1 with $\Omega=E_R$, we need the following L^1-estimates for the global weak solution $u(x, t)$ when $N\geq 3$, which is obtained by [8]:

(1.8) $$\int_{E_R} u(x, t)\rho_R(|x|)dx\leq C(N) \qquad \text{for any} \quad t\geq 0$$

where $C(N)=\pi^{N/2}(2N+4)^{1/(p-m)}$ and

(1.9) $$\rho_R(r)=(r-R)/r .$$

This inequality and equation (1.1) imply another inequality

(1.10) $$\int_0^\tau\int_{E_R} u(x, t)^p\rho_R(|x|)dxdt\leq C(N) \qquad \text{for any} \quad \tau>0$$

and then $u(x, t)\equiv 0$ is concluded by reduction to absurdity.

In case $\Omega=\mathbf{R}^N$, (1.10) holds with $\rho_R\equiv 1$ and we directly found a subsolution $v\leq u$ of (1.1)–(1.3) (which is a Barenblatt-Pattle solution; see [8]) to satisfy

(1.11) $$\int_0^\infty\int_{\mathbf{R}^N} v(x, t)^p dxdt=\infty .$$

But in case $R>0$, it is difficult to find such subsolution. We need another consideration. For a global solution u of (1.1)–(1.3) with $\Omega=E_R$, put

(1.12) $$u_k(x, t)=k^N u(kx, k^{N/l}t) \qquad \text{where} \quad l=(p_m^*-1)^{-1} .$$

Then u_k becomes also a global solution of the same system with E_R and $u_0(x)$ replaced by $E_{R/k}$ and $k^N u_0(kx)$ respectively, and u_k satisfies the inequality

$$\int_0^\tau \int_{E_{R/k}} u_k(x, t)^p \rho_{R/k}(|x|) dxdt \le C(N) \tag{1.13}$$

for any $\tau > 0$ and $k \ge 1$. Assume $u_k \not\equiv 0$. Then we can find a subsolution $v_k \le u_k$ of (1.1) to satisfy

$$\liminf_{k\to\infty} \int_0^\tau \int_{E_{R/k}} v_k(x, t)^p \rho_{R/k} dxdt = \infty \tag{1.14}$$

and hence we can reduce to the contradiction. More precisely, we choose $v_k(x, t)$ as

$$v_k(x, t) = k^N v(kx, k^{N/l} t) \tag{1.15}$$

where $v(x, t)$ is a unique weak solution of the initial-boundary value problem

$$\begin{cases} \partial_t v = \Delta v^m & (x, t) \in \Omega \times (0, T) \\ v(x, 0) = u_0(x) & x \in \Omega \\ v(x, t) = 0 & \text{on} \quad x \in \Omega, \quad t > 0 \end{cases} \tag{1.16}$$

with $\Omega = E_R$.

Therefore, in order to show (1.14) it becomes very important to study the asymptotic behavior of $v_k(x, t)$ as $k \to \infty$, and this is the main contents of this paper.

Let $V_m(x, t; L)$ be a unique weak solution of

$$\begin{cases} \partial_t v = \Delta v^m & (x, t) \in \mathbf{R}^N \times (0, \infty) \\ v(x, 0) = L\delta(x) & x \in \mathbf{R}^N \end{cases} \tag{1.17}$$

where $L \ge 0$ and $\delta(x)$ is Dirac's δ-function. Then we can show that for some $L > 0$

$$v_k \to V_m(x, t; L) \qquad \text{as} \quad k \to \infty, \tag{1.18}$$

locally uniformly in $\{\mathbf{R}^N \setminus \{0\}\} \times (0, \infty)$, and hence (1.14) follows from

$$\int_0^\tau \int_{\mathbf{R}^N} V_m^p dxdt = \infty . \tag{1.19}$$

In case $\Omega = \mathbf{R}^N$, (1.18) was shown by Friedman-Kamin [6]. Then the convergence of v_k is locally uniform convergence in $\mathbf{R}^N \times (0, \infty)$. The methods of the proof of (1.18) are same as those in [6], namely, are based on the self-similarity of equation (1.16) and the equicontinuity of the solution to (1.16) (see also Alikakos-Rostamian [1]).

The rest of the paper is organized as follows. In the next Section 2 we define a weak solution of (1.16) and prepare several preliminary lemmas to show (1.18). In Section 3, by using these lemmas we show (1.18). Finally, in Section 4 we prove Theorem 1.1. In appendix we mention the asymptotic behavior of a solution to (1.16) as direct applications of the results of Section 3.

Finally, we note that when $N=2$ it is still unsolved whether or not $p=p_m^*$ is in the blow-up case.

ACKNOWLEDGEMENT. The author wishes to express his gratitude to Professor K. Mochizuki for his valuable suggestions and helpful encouragement.

2. Preliminaries.

In this and next section we consider the initial-boundary value problem

$$\partial_t v = \Delta v^m \quad \text{in} \quad (x, t) \in E_R \times (0, \infty) \tag{2.1}$$

$$v(x, 0) = v_0(x) \quad \text{in} \quad x \in E_R \tag{2.2}$$

$$v(x, t) = 0 \quad \text{on} \quad x \in E_R, \quad t > 0 \tag{2.3}$$

where $m \geq 1$ and $v_0(x) \geq 0$, and prepare several lemmas for the proof of (1.18). The definition of a nonnegative weak solution $v(x, t) \in BC(\bar{E}_R \times [0, \infty))$ (bounded continuous functions) to (2.1)–(2.3) is referred to [8].

For this aim we need a concrete expression of the elementary solution of the initial value problem (1.17). Let

$$l = (m-1+2/N)^{-1} = (p_m^* - 1)^{-1}, \tag{2.4}$$

$$G_m(s) = \begin{cases} (4\pi)^{-n/2} e^{-s^2/4} & m=1 \\ [A - Bs^2]_+^{1/(m-1)} & m>1 \end{cases} \tag{2.5}$$

where $[a]_+ = \max\{a, 0\}$, $B = (m-1)l/(2mN)$ and $A > 0$ is chosen to satisfy

$$\int_{R^N} G_m(|x|)dx = 1 .$$

LEMMA 2.1. *The weak solution of* (1.17) *is given by*

$$V_m(x, t; L) \equiv L(L^{m-1}t)^{-l} G_m((L^{m-1}t)^{-l/N}|x|) \tag{2.6}$$

and it is self-similar in the following sense: For any $k > 0$

$$k^N V_m(kx, k^{N/l}t; L) = V_m(x, t; L) . \tag{2.7}$$

Furthermore, it satisfies

$$\int_{R^N} V_m dx = L \quad \text{and} \quad \int_0^\tau \int_{R^N} V_m^m dx dt < \infty \tag{2.8}$$

and, if $L > 0$ *and* $p = p_m^*$,

$$\int_0^\tau \int_{\mathbf{R}^N} V_m^p \, dx dt = \infty \tag{2.9}$$

for each $\tau > 0$.

PROOF. If $m = 1$, (2.6) gives the usual heat kernel. (2.6) with $m > 1$ is also well known as the Barenblatt-Pattle solution to the porous media equation (1.17) (see e.g. Barenblatt [2], Pattle [10]). (2.7), (2.8) and (2.9) follow from the concrete expression (2.6). □

This weak solution of (1.17) is unique in the following sense:

LEMMA 2.2. *Let* $v(x, t)$, $v(x, t)^m \in L^1(\mathbf{R}^N \times (0, T)) \cap L^\infty(\mathbf{R}^N \times (\tau, T))$ *for any* $\tau \in (0, T)$ *and* $v(x, t) \geq 0$. *If* $v(x, t)$ *satisfies the identity*

$$\int_{\mathbf{R}^N} v\zeta \, dx \bigg|_{t=\tau} = \int_0^\tau \int_{\mathbf{R}^N} (v\zeta_t + v^m \Delta\zeta) \, dx dt + L\zeta(0, 0) \tag{2.10}$$

for any $\tau \in (0, T)$ *and* $\zeta \in C_0^\infty(\mathbf{R}^N \times [0, T))$, *then*

$$v(x, t) \equiv V_m(x, t; L) \quad \text{in} \quad \mathbf{R}^N \times (0, T). \tag{2.11}$$

PROOF. See Pierre [9]. But the assumptions in this lemma are stronger than those in [9]. □

Let $v(x, t)$ be a weak solution to the initial-boundary value problem (2.1)–(2.3) with $v_0(x) \in C_0(\bar{E}_R)$. In the rest of this section we consider the one-parameter family of functions

$$v_k(x, t) = k^N v(kx, k^{N/l} t) \qquad k \geq 1. \tag{2.12}$$

Then $v_k(x, t)$ is self-similar, namely, $v_k(x, t)$ is a weak solution to the initial-boundary value problem

$$\begin{cases} \partial_t v = \Delta v^m & (x, t) \in E_{R/k} \times (0, \infty) \\ v(x, 0) = k^N v_0(kx) & x \in E_{R/k} \\ v(x, t) = 0 & \text{on } |x| = R/k, \quad t > 0. \end{cases} \tag{2.13}$$

Since $v_0(x) \leq V_m(x, \tau; L)$ in E_R for some $\tau > 0$ and $L > 0$, the next lemma follows immediately from (2.7).

LEMMA 2.3.

$$v_k(x, t) \leq V_m\left(x, t + \frac{\tau}{k^{N/l}}; L\right) \leq \frac{L^{2l/N}}{t^l} G_m(0) \tag{2.14}$$

for $|x| \geq R/k$, $t \geq 0$, *and hence for any* $t_1 > 0$

(2.15)
$$v_k(x,t) \le \frac{L^{2l/N}}{t_1^l} G_m(0)$$

for $|x| \ge R/k$, $t \ge t_1$.

PROOF. (2.14) and (2.15) are obvious by the comparison theorem (which is referred to Proposition 2.1 of [8]). □

Therefore, for any $\delta > 0$

(2.16)
$$v_k(x,t) \le C_\delta \quad \text{for} \quad |x| \ge \delta, \quad t \ge \delta, \quad k \ge 1$$

where C_δ is a constant depending on δ. Applying the continuity result of DiBenedetto [5] and Caffarelli-Friedman [3] [4], we reduce that

(2.17) $v_k(x,t)$ are equicontinuous in $|x| \ge \delta$, $t \ge \delta$ for $k \ge 1$.

Hence, the next lemma holds.

LEMMA 2.4. *For any sequence* $\{k_i^*\} \uparrow \infty$, *there exists subsequence* $\{k_i\} \subset \{k_i^*\}$ *and* $w(x,t) \in C(\{\mathbf{R}^N \backslash \{0\}\} \times (0,\infty))$ *such that*

(2.18)
$$v_{k_i}(x,t) \to w(x,t) \quad as \quad k_i \to \infty$$

uniformly in (x,t) *in any compact subset of* $\{\mathbf{R}^N \backslash \{0\}\} \times (0,\infty)$. *Further*

(2.19)
$$w(x,t) \le V_m(x,t;L) \quad for \quad (x,t) \in \{\mathbf{R}^N \backslash \{0\}\} \times (0,\infty).$$

PROOF. Let $\delta > 0$ be fixed. Applying Ascoli-Arzelà theorem to v_k, from any sequence $\{k_i^*\} \uparrow \infty$ we can extract a subsequence $\{k_i\} \subset \{k_i^*\}$ such that

(2.20)
$$v_{k_i}(x,t) \to w(x,t) \quad \text{as} \quad k \to \infty$$

uniformly in $\delta \le |x| \le 1/\delta$, $\delta \le t \le 1/\delta$. Using the diagonal methods, we can choose a subsequence $\{k_i\}$ uniformly with respect to $\delta > 0$, and finish the proof of (2.18). (2.19) follows soon from (2.14). □

The limit function w may a priori depend on the sequence $\{k_i^*\}$. If we show that for some $L > 0$

(2.21)
$$w(x,t) \equiv V_m(x,t;L),$$

then we can conclude (1.18).

3. Asymptotic behavior of $v_k(x,t)$.

Let $v_{k_i}(x,t)$ and $w(x,t)$ be as in Lemma 2.4. In this section we shall show (2.21) and so (1.18) when $N \ge 3$. For this aim we shall use Lemma 2.2. Namely, we shall show (2.10) with $v(x,t)$ replaced by $w(x,t)$ for some $L > 0$.

First, since v_k is a weak solution to (2.13), it satisfies the integral identity

$$\int_{E_{R/k}} v_k(x, T)\varphi(x, T)\,dx - \int_{E_{R/k}} k^N v_0(kx)\varphi(x, 0)\,dx = \int_0^T \int_{E_{R/k}} \{v_k \partial_t \varphi + v_k^m \Delta\varphi\}\,dxdt \tag{3.1}$$

for any $T>0$ and $\varphi(x, t) \in C_0^\infty(\bar{E}_{R/k} \times [0, \infty))$ (see (2.3) of [8]).

Let $\zeta(x, t)$ be a C^∞-function with support in $\mathbf{R}^N \times (-\infty, \infty)$ and put

$$K_R(r; N) = \frac{r^{N-2} - R^{N-2}}{r^{N-2}} \qquad (N \geq 3)\,. \tag{3.2}$$

If we choose $\varphi(x, t) = K_{R/k}(|x|; N)\zeta(x, t)$ in (3.1), then we have

$$\begin{aligned}
&\int_{E_{R/k}} v_k(x, T) K_{R/k}(|x|)\zeta(x, T)\,dx \\
&= \int_0^T \int_{E_{R/k}} v_k K_{R/k} \zeta_t\,dxdt + 2(N-2)\left(\frac{R}{k}\right)^{N-2} \int_0^T \int_{E_{R/k}} v_k^m |x|^{-N} x \cdot \nabla\zeta\,dxdt \\
&\quad + \int_0^T \int_{E_{R/k}} v_k^m K_{R/k} \Delta\zeta\,dxdt + \int_{E_{R/k}} k^N v_0(kx) K_{R/k}(|x|)\zeta(x, 0)\,dx \\
&\equiv J_1 + J_2 + J_3 + J_4\,.
\end{aligned} \tag{3.3}$$

Here we have used

$$\Delta K_{R/k}(|x|; N) = 0\,, \tag{3.4}$$

$$\nabla K_{R/k}(|x|; N) = (N-2)\left(\frac{R}{k}\right)^{N-2} |x|^{-N} x\,. \tag{3.5}$$

In the following, we shall estimate the both sides of (3.3).

LEMMA 3.1. *If $k = k_i \to \infty$, then*

$$\textit{the left side of } (3.3) \to \int_{\mathbf{R}^N} w(x, T)\zeta(x, T)\,dx \tag{3.6}$$

and

$$J_4 \to \zeta(0, 0) I_N\,, \tag{3.7}$$

where

$$I_N = \int_{E_R} v_0(x) K_R(|x|; N)\,dx\,. \tag{3.8}$$

PROOF. If we note (2.14), (2.18) and the inequality

$$K_{R/k}(|x|; N) \leq 1 \qquad \text{for} \quad x \in E_{R/k}, \tag{3.9}$$

then (3.6) follows from the Lebesgue dominated convergence theorem. (3.7) similarly follows, since we have

$$J_4 = \int_{E_R} v_0(x) K_R(|y|; N) \zeta\left(\frac{y}{k}, 0\right) dy. \tag{3.10}$$

□

LEMMA 3.2.

$$J_3 \to \int_0^T \int_{R^N} w \Delta\zeta \, dxdt \qquad as \quad k = k_i \to \infty. \tag{3.11}$$

PROOF. Let δ be a positive real number and put

$$\begin{aligned} J_3 &= \int_0^\delta \int_{E_{R/k}} v_k^m K_{R/k} \Delta\zeta \, dxdt + \int_\delta^T \int_{E_{R/k}} v_k^m K_{R/k} \Delta\zeta \, dxdt \\ &\equiv J_{3,\delta}^+ + J_{3,\delta}^-. \end{aligned} \tag{3.12}$$

Set

$$t_k(\tau) = t + \tau / k^{N/l}. \tag{3.13}$$

Then, it follows from (2.14) and (3.9) that

$$\begin{aligned} |J_{3,\delta}^+| &\leq C \int_0^\delta \int_{R^N} V_m(x, t_k(\tau); L)^m \, dxdt \\ &= CL^{2lm/n} \int_0^\delta t_k^{-ml} \int_{\mathbf{R}^N} G_m((L^{m-1} t_k)^{-l/N} |x|)^m \, dxdt \end{aligned} \tag{3.14}$$

where C is a constant independent of δ and k. Here, we have used the equality

$$2l/N + l(m-1) = 1. \tag{3.15}$$

Put $y = (L^{m-1} t_k)^{-l/N} x$. Then, noting $t_k \geq t$ and (3.15), we have

$$\begin{aligned} |J_{3,\delta}^+| &\leq CL^{l(m-1+2m/N)} \int_0^\delta t_k^{-l(m-1)} dt \int_{\mathbf{R}^N} G_m(|y|)^m dy \\ &\leq C_1 \int_0^\delta t^{-l(m-1)} dt = \frac{N}{2l} C_1 \delta^{2l/N} \end{aligned} \tag{3.16}$$

where $C_1 = CL^{l(m-1+2m/N)} \int_{\mathbf{R}^N} G_m(|y|)^m dy$. Hence, we obtain

$$J_{3,\delta}^+ \to 0 \qquad \text{as} \quad \delta \downarrow 0 \tag{3.17}$$

uniformly with respect to k.

On the other hand, in view of Lebesgue dominated convergence theorem, we have

$$J_{3,\delta}^{-}\to\int_{\delta}^{T}\int_{\mathbf{R}^N} w\Delta\zeta\,dxdt \qquad \text{as} \quad k=k_i\to\infty\,. \tag{3.18}$$

Therefore, combining this and (3.17) we obtain (3.11). The proof is complete. □

Similarly, since

$$\begin{aligned}\left|\int_0^{\delta}\int_{E_{R/k}} v_k K_{R/k}\zeta_t\,dxdt\right| &\le C\int_0^{\delta}\int_{\mathbf{R}^N} V_m(x,t_k(\tau);L)\,dxdt\\ &= C\delta L\to 0 \qquad \text{as} \quad \delta\downarrow 0\end{aligned} \tag{3.19}$$

(see (2.8)) uniformly with respect to k, we obtain

LEMMA 3.3.

$$J_1\to\int_0^{T}\int_{\mathbf{R}^N} w\zeta_t\,dxdt \qquad as \quad k=k_i\to\infty\,. \tag{3.20}$$

PROOF. The proof is similar to that of Lemma 3.2. We omit it. □

Finally we consider J_2.

LEMMA 3.4.

$$J_2\to 0 \qquad as \quad k\to\infty\,. \tag{3.21}$$

PROOF. Similarly as above proofs, we have

$$\begin{aligned}|J_2| &\le 2C(N-2)\left(\frac{R}{k}\right)^{N-2}\int_0^T\int_{\mathbf{R}^N} V_m(x,t_k(\tau);L)^m|x|^{-N+1}\,dxdt\\ &= C_1 k^{-(N-2)}\int_0^T \{t_k(\tau)\}^{-l(m-1/N)}\,dt\end{aligned} \tag{3.22}$$

where C is a positive constant independent of δ and k, and

$$C_1=2C(N-2)L^{l(3m-1)/N}R^{N-2}\int_{\mathbf{R}^N}|y|^{-N+1}G_m(|y|)^m\,dy\,.$$

We note $1-l(m-1/N)=-l(N-3)/N$. Hence, when $N>3$ we get

$$\begin{aligned}|J_2| &\le \frac{C_1N}{l(N-3)}\,k^{-(N-2)}[\tau^{-l(N-3)/N}k^{N-3}-(T+\tau k^{-N/l})^{-l(N-3)/N}]\\ &\le \frac{C_1N}{l(N-3)}\,\tau^{-l(N-3)/N}k^{-1}\to 0 \qquad \text{as} \quad k\to\infty\,.\end{aligned} \tag{3.23}$$

Similarly, when $N=3$, noting $l(m-1/N)=1$ we get

$$(3.24)\qquad |J_2| \le C_1 k^{-1}(\log(T+\tau k^{-N/l}) - \log(\tau k^{-N/l})) \le C_1 k^{-1}\left(\frac{N}{l}\log k + +\log\frac{T+\tau}{\tau}\right) \to 0 \quad \text{as} \quad k\to\infty\,.$$

The proof is complete. □

Combining these four lemmas, if $k=k_i\to\infty$ in (3.3), then we get the following integral identity for the limit function $w(x,t)$:

$$(3.25)\qquad \int_{\mathbf{R}^N} w\zeta\,dx\Big|_{t=T} = \int_0^T\int_{\mathbf{R}^N}(w\zeta_t + w^m\Delta\zeta)\,dxdt + I_N\zeta(0,0)$$

for any $T>0$ and $\zeta\in C_0^\infty(\mathbf{R}^N\times[0,\infty))$, where I_N is defined by (3.8). Hence, since $w(x,t)$, $w(x,t)^m\in L^1(\mathbf{R}^N\times(0,T)\cap L^\infty(\mathbf{R}^N\times(\tau,T))$ for any $T>0$ and $\tau\in(0,T)$ by (2.8) and (2.19), it follows from Lemma 2.2 that

$$(3.26)\qquad w\equiv V_m(x,t;I_N) \quad \text{in} \quad \mathbf{R}^N\times[0,\infty)\,.$$

Thus, we obtain the following result.

PROPOSITION 3.5. *Assume $N\ge 3$. Let $v(x,t)$ be a nonnegative weak solution to* (2.1)–(2.3) *with $v_0(x)\in C_0(\bar{E}_R)$, and put $v_k(x,t)=k^N v(kx, k^{N/l}t)$. Then*

$$(3.27)\qquad v_k(x,t)\to V_m(x,t;I_N) \quad \text{as} \quad k\to\infty$$

locally uniformly in $\{\mathbf{R}^N\backslash\{0\}\}\times(0,\infty)$ where I_N is defined by (3.8).

PROOF. This proposition follows from Lemma 2.4 and (3.26). □

4. Proof of Theorem 1.1.

In this section we prove Theorem 1.1. The next result due to K. Mochizuki-R. Suzuki [8] plays an important role in the proof of it.

LEMMA 4.1. *Assume $N\ge 3$. Let $u(x,t)$ be a global weak solution of* (1.1)–(1.3) *with $\Omega=E_R$ and $u_0(x)\in C_0(\bar{E}_R)$. Then*

$$(4.1)\qquad u(\cdot,t)\in L^1(E_R) \quad \text{for} \quad t\ge 0$$

and if $p=p_m^*$, then

$$(4.2)\qquad \int_{E_R} u(x,t)\rho_R(|x|)\,dx \le C(N) \quad \text{for any} \quad t\ge 0$$

where $C(N)=\pi^{N/2}(2N+4)^{1/(p-m)}$ and

$$(4.3)\qquad \rho_R(r)=(r-R)/r\,.$$

PROOF. (4.1) is in Proposition 2.2 of [8]. (4.2) follows from Lemma 4.3 of [8] if $\varepsilon \to 0$. □

This inequality and equation (1.1) imply the following lemma.

LEMMA 4.2. *Let $u(x, t)$ be as in Lemma 4.1. Then, if $p=p_m^*$,*

$$\int_0^\tau \int_{E_R} u(x, t)^p \rho_R(|x|)\,dxdt \le C(N) \qquad \text{for any} \quad \tau>0\,. \tag{4.4}$$

PROOF. Since $u(\cdot, t) \in L^1(E_R)$, we can choose $\rho_R(|x|)$ as a test function in the integral identity satisfied by u (see (2.3) of [8]). Then we have

$$\int_{E_R} u(x, \tau)\rho_R(|x|)\,dx \ge \int_0^\tau \int_{E_R} u(x, t)^p \rho_R(|x|)\,dxdt + \int_{E_R} u_0(x)\rho_R(|x|)\,dx\,. \tag{4.5}$$

Here we have used

$$\Delta\rho_R(|x|) = \left(\partial_r^2 + \frac{N-1}{r}\partial_r\right)\rho_R(|x|) \ge 0 \qquad (N \ge 3)\,. \tag{4.6}$$

Therefore, (4.2) and (4.5) is reduced to (4.4). □

PROOF OF THEOREM 1.1. (Special case) Let $u(x, t)$ be as in the above lemma and $u_k(x, t) = k^N u(kx, k^{N/l}t)$ where $l = (p_m^* - 1)^{-1}$. Then, when $p = p_m^*$, u_k is a global weak solution of the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta u^m + u^p & (x, t) \in E_{R/k} \times (0, \infty) \\ u(x, 0) = k^N u_0(kx) & x \in E_{R/k} \\ u(x, t) = 0 & \text{on } |x| = R/k, \quad t>0\,. \end{cases} \tag{4.7}$$

Applying Lemma 4.2 to u_k, we have

$$\int_0^\tau \int_{E_{R/k}} u_k^p \rho_{R/k}\,dxdt \le C(N) \qquad \text{for any} \quad \tau > 0\,. \tag{4.8}$$

If we set $v_k(x, t) = k^N v(kx, k^{N/l}t)$ where $v(x, t)$ is a weak solution to (2.1)–(2.3) with $v_0(x) = u_0(x)$, then $v_k(x, t) \le u_k(x, t)$ in $E_{R/k} \times (0, \infty)$ by the comparison theorem (see Proposition 2.1 of [8]). Hence, we obtain

$$\int_0^\tau \int_{E_{R/k}} v_k^p \rho_{R/k}\,dxdt \le C(N) \qquad \text{for any} \quad \tau > 0\,. \tag{4.9}$$

Suppose $u_0(x) \not\equiv 0$ and let $k \to \infty$ in (4.9). Then, since $\rho_{R/k}(r) \to 1$ as $k \to \infty$ in $r > 0$, it follows from Proposition 3.5 and Fatau's lemma that

$$\int_0^\tau \int_{\mathbf{R}^N} V_m(x, t; I_N)^p\,dxdt \le C(N) \qquad \text{for} \quad \tau > 0 \tag{4.10}$$

where I_N (>0) is defined by (3.8) with $v_0(x)=u_0(x)$. This is a contradiction to (2.9) and so $u_0(x)\equiv 0$. The proof is complete.

(General case) The methods of the proof are same as those in [8]. We omit it. □

Appendix.

In this appendix, applying Proposition 3.5 directly, we shall study the asymptotic behavior of the solution $v(x, t)$ of the initial-boundary value problem (2.1)–(2.3) as $t\to\infty$. We shall show the following theorem.

THEOREM A.1. *Assume* $N\geq 3$. *Let* $v(x, t)$ *be a weak solution to* (2.1)–(2.3) *with* $\Omega=E_R$ *and* $v_0(x)\in C_0(\bar{E}_R)$. *Then*

$$t^l|v(x, t)-V_m(x, t; I_N)|\to 0 \quad \text{as} \quad t\to\infty \tag{A.1}$$

uniformly on sets

$$P_{\delta,C}(t)=\{x\in\mathbf{R}^N \mid \delta t^{l/N}\leq|x|\leq Ct^{l/N}\} \qquad (C>\delta>0), \tag{A.2}$$

where V_m *and* I_N *are as in Proposition* 3.5. *Further*

$$\int_{E_R} v(x, t)dx\to I_N \quad \text{as} \quad t\to\infty \tag{A.3}$$

and so

$$t^l \sup_{x\in E_R} v(x, t)\to I_N^{2l/N}G_m(0) \quad \text{as} \quad t\to\infty. \tag{A.4}$$

PROOF. (A.1) follows soon from Proposition 3.5 (see Friedman-Kamin [6]). (A.3) follows from the Lebesgue dominated convergence theorem as follows:

$$\int_{E_R} v(x, t)dx=\int_{E_{R/k}} v_k(x, 1)dx\Big|_{k=t} \to \int_{\mathbf{R}^N} V_m(x, 1; I_N)dx=I_N \qquad (\text{as } t\to\infty). \tag{A.5}$$

For any $\tau>0$, let $\tilde{v}_\tau(x, t)$ be a weak solution to the Cauchy problem

$$\begin{cases}\partial_t\tilde{v}=\Delta\tilde{v}^m & (x, t)\in\mathbf{R}^N\times(\tau, T)\\ \tilde{v}(x, \tau)=v(x, \tau) & x\in\mathbf{R}^N.\end{cases} \tag{A.6}$$

By the comparison theorem we have

$$v(x, t)\leq\tilde{v}_\tau(x, t) \quad \text{in} \quad (x, t)\in E_R\times(\tau, \infty). \tag{A.7}$$

If we recall the result [6] such that

$$t^l \sup_{x\in\mathbf{R}^N} v_\tau(x, t)\to J_\tau^{2l/N}G_m(0) \quad \text{as} \quad t\to\infty \tag{A.8}$$

where $J_\tau=\int_{E_R} v(x,\tau)\,dx$, then

$$\limsup_{t\to\infty}\left\{t^l \sup_{x\in E_R} v(x,t)\right\}\le J_\tau^{2l/N} G_m(0)\,. \tag{A.9}$$

Hence, since J_τ converges I_N as $\tau\to\infty$ by (A.3), we get

$$\limsup_{t\to\infty}\left\{t^l \sup_{x\in E_R} v(x,t)\right\}\le I_N^{2l/N} G_m(0)\,. \tag{A.10}$$

On the other hand, by virtue of (A.1) we obtain

$$I_N^{2l/N} G_m(0)\le \liminf_{t\to\infty}\left\{t^l \sup_{x\in E_R} v(x,t)\right\}. \tag{A.11}$$

Thus, (A.10) and (A.11) imply (A.4). The proof is complete. □

REMARK A.2. By the same methods as those in Friedman-Kamin [6], we can show Proposition 3.5 and (A.1) under the condition $v_0(x)\in C_0(\bar{E}_R)$ replaced by $v_0(x)\in L^1(E_R)$. The proof is omitted.

References

[1] N. ALIKAKOS and R. ROSTAMIAN, On the uniformization of the solutions of the porous medium equation in $\mathbf{R}^N$, Israel J. Math. **47** (1984), 270–290.

[2] G. I. BARENBLATT, On some unsteady motions in a liquid on a gas in a porous medium, Prikl. Mat. Mekh. **16** (1952), 67–78.

[3] L. A. CAFFARELLI and A. FRIEDMAN, Continuity of the density of a gas flow in a porous medium, Trans. Amer. Math. Soc. **252** (1979), 99–113.

[4] L. A. CAFFARELLI and A. FRIEDMAN, Regularity of the free boundary of a gas flow in an n-dimensional porous medium, Indiana Univ. Math. J. **29** (1980), 361–391.

[5] E. DIBENEDETTO, Continuity of weak solutions to a general porous media equation, Indiana Univ. Math. J. **32** (1983), 83–118.

[6] A. FRIEDMAN and S. KAMIN, The asymptotic behavior of gas in an n-dimensional porous medium, Trans. Amer. Math. Soc. **262** (1980), 551–563.

[7] H. A. LEVINE, The role of critical exponents in blowup theorems, SIAM Review **32** (1990), 262–288.

[8] K. MOCHIZUKI and R. SUZUKI, Critical exponent and critical blow-up for quasilinear parabolic equations. Israel J. Math., to appear.

[9] M. PIERRE, Uniqueness of the solutions of $u_t-\Delta\varphi(u)=0$ with initial datum a measure, Nonlinear Anal. TMA **6** (1982), 175–187.

[10] R. E. PATTLE, Diffusion form an instantaneous point source with concentration-dependent coefficient, Quart. J. Mech. Appl. Math. **12** (1959), 407–409.

Present Address:

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING (TSURUKAWA), KOKUSHIKAN UNIVERSITY, HIROBAKAMA-CHO, MACHIDA, TOKYO, 195 JAPAN.