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On the Banach Algebra $M(p, q)$ $(1 \leq p < q \leq \infty)$

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1. Introduction.

Let G be an infinite compact abelian group, Γ (or $\hat{G}$) the dual group, $M(G)$ the convolution measure algebra on G, and $L^p(G)$ the L^p-space with respect to the Haar measure m_G on G for $1 \leq p \leq \infty$. Also for $1 \leq p \leq q \leq \infty$ let $M(p, q)$ be the set of all translation invariant bounded linear operators from $L^p(G)$ to $L^q(G)$. For $\mu \in M(G)$, μ is called an L^p-improving measure, if $\mu \in M(r, s)$ for some $1 \leq r < s \leq \infty$ (cf. [5]). When $p < q$, $M(p, q)$ is a commutative Banach algebra without unit with the operator norm, and $M(p, p)$ is a commutative Banach algebra with unit.

The purpose of this paper is an investigation of Fourier multiplier algebra $M(p, q)$ $(1 \leq p < q \leq \infty)$.

Hatori [10] characterized $\Lambda(p)$-sets on $\hat{G}$ by using the Banach algebra $M(p, q)$ $(1 < p < q < \infty)$ (cf. [3], [4]). Also, he characterized the maximal ideal space of $M(p, q)$ $(1 < p < q < \infty)$. These results are showed by applying Stone-Čech's compactification.

In § 2, we give proofs of Theorems 2.2 and 2.5 that are simple proofs of his results, by the method of [12] and [20] without using Stone-Čech's compactification.

Igari-Sato [12] studied the operating function of $M(p, q)$ $(1 \leq p \leq q \leq \infty)$. The domain of the operating function is $[-1, 1]$. In § 3, we investigate the operating function whose domain is the complex plane. When G is the unit circle, our result is an extension of Rider's result [14] (cf. [16]).

There are many papers [5], [15], etc. about L^p-improving measures. But it seems that it is unknown about L^p-improving measures on thin sets (cf. [5; Open questions]). In § 4, we construct non L^p-improving measures on some independent set, that are in $M_0(G)$ that is the set of all bounded regular Borel measures whose Fourier-Stieltjes transforms vanish at infinity.

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In § 5, we give a remark about generalized Riesz products. Throughout this paper, we denote p' for any $p \geq 1$ such that $1/p + 1/p' = 1$.

2. On Hatori's theorems.

In this section, we give alternative proofs of Hatori's two theorems [10].

DEFINITION 2.1 (cf. [3]). Let $1 < r < \infty$, and $E \subset \Gamma$. E is called a $\Lambda(r)$-set, if there exist $1 < s < r$ and $B > 0$ such that $\|f\|_r \leq B\|f\|_s$ for any trigonometric polynomial f with $supp \hat{f} \subset E$.

Then it is known that E is a $\Lambda(r)$-set, if and only if for any $1 \leq s < r$, there exists a constant $A_s > 0$ such that $\|f\|_r \leq A_s\|f\|_s$ for any trigonometric polynomial f with $supp \hat{f} \subset E$.

THEOREM 2.2 *Let G be an infinite compact abelian group with the dual Γ, $1 < p < q < \infty$, and $E \subset \Gamma$. Then the next conditions* (1), (2), (3) *and* (4) *are equivalent*:

(1) *E is a $\Lambda(r)$-set, where $r = \max(p', q)$.*

(2) *$\chi_E \in M(p, q)^\wedge$, where χ_E is the characteristic function on E and $M(p, q)^\wedge = \{\hat{T} \mid T \in M(p, q)\}$.*

(3) *$M(p, q)^\wedge|_E = l^\infty(E)$.*

(4) *There exists $T \in M(p, q)$ such that* $\inf\{|\hat{T}(\gamma)| \mid \gamma \in E\} > 0$.

PROOF. We prove (1)⇒(2)⇒(3)⇒(4)⇒(1).

(1)⇒(2): Let $1 < p < q \leq 2$ or $2 \leq p < q < \infty$. For $2 \leq p < q < \infty$, the proof is similar to the case $1 < p < q \leq 2$. Then we may assume $1 < p < q \leq 2$. By the assumption, $q \leq 2 < p'$. So we have $r = p'$. Since E is a $\Lambda(r)$-set, there exists a constant $C > 0$ such that $\|f\|_{p'} \leq C\|f\|_2$ for any trigonometric polynomial f with $supp \hat{f} \subset E$. Putting $\hat{T} = \chi_E$, we obtain $\|Tf\|_{p'} \leq C\|f\|_2$ for any trigonometric polynomial f on G. Then $\|Tf\|_q \leq C\|f\|_p$ for any trigonometric polynomial f on G. Thus we have $\chi_E \in M(p, q)^\wedge$.

Next let $1 < p < 2 \leq q < \infty$. Then we may have $p' \geq q$, and $r = p'$. Putting $\hat{T} = \chi_E$, by the assumption there exists a constant $C > 0$ such that $\|Tf\|_{p'} \leq C\|f\|_2$ for any trigonometric polynomial f. So $\|Tf\|_q \leq C\|f\|_2$. Thus there exists a constant $C' > 0$ such that $\|Tf\|_{p'} \leq C'\|f\|_{q'}$ for any trigonometric polynomial f. Therefore, $\chi_E \in M(q', p')^\wedge = M(p, q)^\wedge$.

(2)⇒(3): Suppose $1 < p < q \leq 2$ or $2 \leq p < q < \infty$. Then we may assume $1 < p < q \leq 2$. Also it is sufficient to prove $l^\infty(E) \subset M(p, q)^\wedge|_E$. Since $\hat{T} = \chi_E \in M(p, q)^\wedge$ by the assumption, there exists a natural number $N \geq 1$ such that $\widehat{T^N} \in M(p, 2)$ (cf. [9]) and $\widehat{T^N} = \chi_E = \hat{T}$. Let ϕ be in $l^\infty(E)$, and f a trigonometric polynomial. Then

$$\begin{aligned}\|\sum \phi(\gamma)\hat{f}(\gamma)\gamma\|_2 &\leq \|\phi\|_\infty \|\sum \chi_E \hat{f}(\gamma)\gamma\|_2 \\ &\leq \|\phi\|_\infty \|Tf\|_2 \leq \|\phi\|_\infty \|T\|_{M(p,2)} \|f\|_p .\end{aligned}$$

Hence, there exists $S \in M(p, 2)$ such that $\hat{S} = \phi$. Therefore we have $l^\infty(E) \subset M(p, q)^\wedge|_E$.

Next suppose $1 < p < 2 < q < \infty$. For any $\phi \in l^\infty(E)$, let $\alpha(\gamma)$ ($\in \mathbf{R}$) be $\phi(\gamma) = e^{i\alpha(\gamma)}|\phi(\gamma)|$, and $\Psi(\gamma) = e^{i\alpha(\gamma)/2}\sqrt{|\phi(\gamma)|}$. Then $\Psi^2 = \phi$. Since $\chi_E \in M(p, q)^\wedge \subset M(p, 2)^\wedge$, we obtain $\chi_E \in M(p, 2)^\wedge$. Hence, for any trigonometric polynomial f, we have

$$\begin{aligned}\|\sum \hat{f}(\gamma)\Psi(\gamma)\gamma\|_2 &\le \|\Psi\|_\infty \|\sum \chi_E \hat{f}(\gamma)\gamma\|_2 \\ &\le \|\Psi\|_\infty \|T\|_{M(p,2)} \|f\|_p = \|\phi\|_\infty^{1/2} \|T\|_{M(p,2)} \|f\|_p .\end{aligned}$$

Therefore there exists $T_1 \in M(p, 2)$ such that $\hat{T}_1 = \Psi$. Similarly, we obtain $T_2 \in M(2, q)$ with $\hat{T}_2 = \Psi$. Defining $S = T_2 T_1$, we have $S \in M(p, q)$, and $\hat{S} = \Psi^2 = \phi$. Then $l^\infty(E) \subset M(p, q)^\wedge|_E$.

(3)$\Rightarrow$(4): trivial.

(4)$\Rightarrow$(1): Let $\varepsilon = \inf\{|\hat{T}(\gamma)| \mid \gamma \varepsilon E\}$, and $E(\eta) = \{\gamma \mid |\hat{T}(\gamma)| \ge \eta\}$ for any $\eta > 0$. First suppose $1 < p < q \le 2$ or $2 \le p < q < \infty$. We may assume $1 < p < q \le 2$. So we have $r = p'$. Then there exists a natural number $N \ge 1$ such that $T^N \in M(p, 2)$ (cf. [12]). So $T^N \in M(2, p')$. By Hare [8], $\{\gamma \mid |\hat{T}^N(\gamma)| \ge \varepsilon^N\}$ is a $\Lambda(p')$-set. Hence, $E(\varepsilon)$ is a $\Lambda(p')$-set i.e. $\Lambda(r)$-set. Since E is a subset of $E(\varepsilon)$, E is a $\Lambda(r)$-set.

Next suppose $1 < p < 2 \le q < \infty$. We may assume $p' \ge q$, and $r = p'$. By $T \in M(p, 2)$, we have $T \in M(2, p')$. So $E(\varepsilon)$ is a $\Lambda(p')$-set. Since E is a subset of $E(\varepsilon)$, E is a $\Lambda(p')$-set, i.e. $\Lambda(r)$-set. q.e.d.

Corollary 2.3. *Let $r = \max(p', q)$, and E a $\Lambda(r)$-set in Γ. Also let $\bar{E}$ be the closure of E in the maximal ideal space of the commutative Banach algebra $M(r, r)$, and $\tilde{T} \in M(p, q)^\sim$ the Gelfand representation of $T \in M(p, q)$. Then $\bar{E}$ is the Stone-Čech's compactification of E.*

Proof. By $M(p, q) \subset M(r, r)$, we have $\tilde{T}|_{\bar{E}} \in C(\bar{E})$ for any $T \in M(p, q)$, where $C(\bar{E})$ is the set of all continuous functions on $\bar{E}$. Then, by Theorem 2.2 $M(p, q)^\wedge|_E = l^\infty(E)$. Thus $\bar{E}$ is the Stone-Čech's compactification of E. q.e.d.

Hatori [10] gave a characterization of the maximal ideal space of $M(p, q)$ ($1 < p < q < \infty$) by using Stone-Čech's compactification. We prove the same result without applying Stone-Čech's compactification.

Definition 2.4. Suppose $1 < r \le s < \infty$. We define $\Delta M(r, s)$ the maximal ideal sapce of $M(r, s)$.

Theorem 2.5. *Suppose $1 < p < q < \infty$ and $r = \max(p', q)$. Then we have*

$$\Delta M(p, q) = \bigcup_{E:\Lambda(r)\text{-set}} \bar{E},$$

where $\bar{E}$ is the closure of E ($\subset \Gamma$) in $\Delta M(r, r)$. Moreover, $\Delta M(p, q)$ is an open subset of $\Delta M(r, r)$.

Proof. Let A be the commutative Banach algebra that is the adjunction of a unity $M(p, q)$, and $\Delta(A)$ the maximal ideal space of A. Then ϕ ($\in \Delta(A)$) is a nontrivial complex homomorphism on $M(p, q)$ or $\phi=0$ on $M(p, q)$. Here we show the following:

LEMMA 2.6. *Let ϕ be a nontrivial complex homomorphism on $M(p, q)$. Then ϕ has a unique extension $\tilde{\phi}$ which is a nontrivial complex homomorphism on $M(r, r)$. Let Δ_0 be the set of all $\tilde{\phi}$. Then we have*

$$\Delta_0=\{\psi \in \Delta M(r, r) \mid \psi(T_0) \neq 0 \text{ for some } T_0 \in M(p, q)\} .$$

PROOF OF LEMMA 2.6. For $T \in M(r, r)$ and $T_0 \in M(p, q)$, we remark that TT_0 is in $M(p, q)$. Let ϕ be in $\Delta M(p, q)$, and T_0 be in $M(p, q)$ with $\phi(T_0) \neq 0$. Also let

$$\tilde{\phi}(T)=\frac{\phi(TT_0)}{\phi(T_0)} \quad \text{for any} \quad T \in M(r, r) .$$

It is well-defined. In fact, let T_1 be in $M(p, q)$ with $\phi(T_1) \neq 0$. Then

$$\phi(T_1)\phi(TT_0)=\phi(TT_0T_1)=\phi(T_0)\phi(TT_1) ,$$

and

$$\frac{\phi(TT_0)}{\phi(T_0)}=\frac{\phi(TT_1)}{\phi(T_1)} .$$

Therefore for T, $S \in M(r, r)$,

$$\tilde{\phi}(TS)=\frac{\phi(TST_0^2)}{\phi(T_0^2)}=\frac{\phi(TT_0)}{\phi(T_0)}\frac{\phi(ST_0)}{\phi(T_0)}=\tilde{\phi}(T)\tilde{\phi}(S) ,$$

and $\tilde{\phi} \in \Delta M(r, r)$. Also by the construction, $\tilde{\phi}$ is a unique extension of ϕ. Moreover, Δ_0 is an open subset of $\Delta M(r, r)$. In fact, suppose $\tilde{\phi}_0 \in \Delta_0$. Then there exists $T_0 \in M(p, q)$ such that $\tilde{\phi}_0(T_0)=1$. Putting

$$V=\{\tilde{\phi} \in \Delta M(r, r) \mid |\tilde{\phi}(T_0)-\tilde{\phi}_0(T_0)|<1/2\} ,$$

we have $V \subset \Delta_0$. Since V is an open subset of $\Delta M(r, r)$, Δ_0 is open in $\Delta M(r, r)$. q.e.d. of Lemma 2.6.

The following is easy to be proved by the construction of the commutative Banach algebra A and Lemma 2.6.

LEMMA 2.7. *Let $\phi_{\{0\}}$ be the nontrivial complex homomorphism on A such that $\phi_{\{0\}}=0$ on $M(p, q)$. Then Δ_0 is homeomorphic to $\Delta M(p, q)=\Delta(A)\setminus\{\phi_{\{0\}}\}$ by the natural mapping in Lemma 2.6, and $\Delta(A)$ is 1-point compactification of Δ_0.*

After that, we identify Δ_0 as $\Delta M(p, q)$.

LEMMA 2.8. *$\Delta_0 \subset \bar{\Gamma}^{\text{in } \Delta M(r,r)}$, where $\bar{\Gamma}^{\text{in } \Delta M(r,r)}$ is the closure of Γ in $\Delta M(r, r)$.*

PROOF OF LEMMA 2.8. *First Step.* $\Delta_0 \subset \bar{\Gamma}^{\text{in } \Delta(A)}$. In particular, $\Delta(A)=\bar{\Gamma}^{\text{in } \Delta(A)}$, where $\bar{\Gamma}^{\text{in }\Delta(A)}$ is in the closure of Γ in $\Delta(A)$. In fact, let ϕ_0 be in Δ_0 with $\phi_0 \notin \bar{\Gamma}^{\text{in }\Delta(A)}$. Then for any $\phi \in \bar{\Gamma}^{\text{in }\Delta(A)}$, there exists $T \in A$ such that $\phi(T) \neq 0$ and $\phi_0(T) = 0$, since $\phi \neq \phi_0$ and A has the unit. Also let

$$V_\phi = \{\phi' \in \Delta(A) \mid |\phi'(T) - \phi(T)| < \tfrac{1}{2}|\phi(T)|\}.$$

V_ϕ is an open neighborhood of $\phi \in \Delta(A)$. Since

$$\bigcup_{\phi \in \bar{\Gamma}^{\text{in }\Delta(A)}} V_\phi \supset \bar{\Gamma}^{\text{in }\Delta(A)}$$

and $\bar{\Gamma}^{\text{in }\Delta(A)}$ is a compact subset of $\Delta(A)$, there exist $\phi_j \in \bar{\Gamma}^{\text{in }\Delta(A)}$ $(1 \leq j \leq n)$ such that $\bigcup_{j=1}^n V_{\phi_j} \supset \bar{\Gamma}^{\text{in }\Delta(A)}$. Hence, by the definition of V_{ϕ_j}, there exist $T_j \in A$ $(1 \leq j \leq n)$ such that $\phi_j(T_j) \neq 0$ and $\phi_0(T_j) = 0$. Then for any $\phi \in V_{\phi_j}$, we have

$$|\phi(T_j)| \geq \tfrac{1}{2}|\phi_j(T_j)|.$$

Here, for any $\gamma \in \Gamma$, defining $\phi_\gamma(T) = \hat{T}(\gamma)$, we have

$$\sum_{j=1}^n \phi_\gamma(T_j T_j^*) \geq \frac{1}{4} \min_{1 \leq j \leq n} |\phi_j(T_j)|^2 \quad (>0)$$

for all $\gamma \in \Gamma$, where $T^* \in M(p, q)$ is defined by $\hat{T^*} = \bar{\hat{T}}$ for any $T \in M(p, q)$. Suppose $T_0 = \sum_{j=1}^n T_j T_j^*$ $(\in A)$. By Sato [20] (cf. [12]),

$$|\phi(T_0)| \geq \frac{1}{4} \min_{1 \leq j \leq n} |\phi_j(T_j)|^2 > 0 \qquad \text{for any} \quad \phi \in \Delta(A),$$

since the spectrum of T_0 in A is the closure of $\hat{T}(\Gamma)$. On the other hand, we have

$$\phi_0(T_0) = \sum_{j=1}^n \phi_0(T_j)\phi_0(T_j^*) = 0.$$

This is a contradiction to the above result. Therefore we have the desired result.

Second Step. $\Delta_0 \subset \bar{\Gamma}^{\text{in }\Delta M(r,r)}$. In fact, let $\phi_0 \in \Delta_0$. By First Step, there exists a net $\{\gamma_\alpha\} \subset \Gamma$ such that $\lim_{\alpha \to \infty} \hat{T}(\gamma_\alpha) = \phi_0(T)$ for all $T \in M(p, q)$. Since ϕ_0 is in Δ_0, there exists $T_0 \in M(p, q)$ such that $\phi_0(T_0) \neq 0$. Then for any $T \in M(r, r)$,

$$\begin{aligned} \lim_{\alpha \to \infty} \hat{T}(\gamma_\alpha) &= \lim_{\alpha \to \infty} \frac{\hat{T}(\gamma_\alpha)\hat{T}_0(\gamma_\alpha)}{\hat{T}_0(\gamma_\alpha)} = \frac{\lim_{\alpha \to \infty} \widehat{TT_0}(\gamma_\alpha)}{\lim_{\alpha \to \infty} \hat{T}_0(\gamma_\alpha)} \\ &= \phi_0(TT_0)/\phi_0(T_0) = \phi_0(T). \end{aligned}$$

Therefore $\lim_{\alpha \to \infty} \hat{T}(\gamma_\alpha) = \phi_0(T)$ for all $T \in M(r, r)$. q.e.d. of Lemma 2.8.

Now we succeed the proof of Theorem 2.5. For the proof, it is sufficient to prove that

$$\Delta M(p, q) = \bigcup_{E : A(r)\text{-set}} \bar{E},$$

where $\bar{E}$ is the closure of E in $\Delta M(r, r)$.

In fact, by Lemma 2.8 we have $\Delta_0 \subset \bar{\Gamma}^{\text{in } \Delta M(r,r)}$. Then for any $\phi_0 \in \Delta_0$, there exists $T_0 \in M(p, q)$ such that $\phi_0(T_0)=1$, and $\{\phi \in \Delta_0 \,|\, |\phi(T_0)| \geq 1/2\} \cap \Gamma \neq \emptyset$. Here, let $E=\{\gamma \in \Gamma \,|\, |\hat{T}_0(\gamma)| \geq 1/2\}$. By Hare [8], E is a $\Lambda(r)$-set. Also $\phi_0 \in \bar{E}$. Hence,

$$\Delta M(p, q) \subset \bigcup_{E:\Lambda(r)\text{-set}} \bar{E}.$$

Conversely, let E be a $\Lambda(r)$-set such that $\phi_0 \in \bar{E}$. Since E is a $\Lambda(r)$-set, by Theorem 2.2 there exist $T \in M(p, q)$ and $\varepsilon > 0$ such that $\inf\{|\hat{T}(\gamma)| \,|\, \gamma \in E\} \geq \varepsilon$. On the other hand, by $\phi_0 \in \bar{E}$ there exists a net $\{\gamma_\alpha\} \subset E$ such that $\gamma_\alpha \to \phi_0$ in $\Delta M(r, r)$. Then $|\phi_0(T)| \geq \varepsilon$, and $\phi_0 \in \Delta M(p, q)$. q.e.d. of Theorem 2.5.

3. Operating functions on $M(p, q)$ $(1 \leq p < q \leq \infty)$.

In this section, we shall investigate operating functions on $M(p, q)$, which are defined on the complex plane.

DEFINITION 3.1. A function $\Phi(z)$ on the complex plane $\mathbf{C}$ is said to operate on $M(p, q)$, if $\Phi(T) \in M(p, q)$ for every $T \in M(p, q)$, where $\Phi(T)^\wedge(\gamma) = \Phi(\hat{T}(\gamma))$ for all $\gamma \in \Gamma$.

The following is an analogy of Igari-Sato [12; Theorem 1].

THEOREM 3.2. *Let $1 \leq p < \infty$ and Φ_0 be a function on $\mathbf{C}$. Assume that Φ_0 is bounded near the origin if $p=1$ or $q=\infty$ and bounded on every bounded domain if $p>1$.*

(a) *Suppose $1 \leq p < q \leq 2$ or $2 \leq p < q \leq \infty$. Let $\beta_0 = (1/q - 1/2)/(1/p - 1/q)$ or $(1/2 - 1/p)/(1/p - 1/q)$ respectively and n_0 be the smallest integer such that $n_0 \geq \beta_0$. Then for any constants $\alpha_1, \alpha_2, \cdots, \alpha_{2n_0-1}, \alpha_{2n_0}$*

$$\Phi(z) = \alpha_1 z + \alpha_2 \bar{z} + \cdots + \alpha_{2n_0-1} z^{n_0} + \alpha_{2n_0} \bar{z}^{n_0} + |z|^{\beta_0+1} \Phi_0(z)$$

operates on $M(p, q)$.

(b) *Suppose $1 \leq p < 2 \leq q \leq \infty$. Let $\beta_1 = \min\{(1/2 - 1/q)/(1/p - 1/2),\ (1/p - 1/2)/(1/2 - 1/q)\}$. Then for any constants α_1, α_2*

$$\Phi(z) = \alpha_1 z + \alpha_2 \bar{z} + |z|^{\beta_1+1} \Phi_0(z)$$

operates on $M(p, q)$.

The proof of the above theorem is similar to that of [12; Theorem 1] (cf. [1]). The proofs for $p=1$ or $q=\infty$ are given by Hausdorff-Young's theorem and the duality of $M(p, q)$ (cf. [3], [16]). We omit the details.

Next, in Theorem 3.3, we shall show that the converse of (b) holds for $G=\mathbf{T}$ (the unit circle). The following result is also an analogy of Igari-Sato [12; Theorem 3].

THEOREM 3.3. *Let $1 \leq p < 2 \leq q \leq \infty$ and Φ be a function on $\mathbf{C}$. If Φ operates on $M(p, q)$, then Φ is of the form*

$$\Phi(z)=\alpha_1 z+\alpha_2 \bar{z}+|z|^{\beta_1+1}\Phi_0(z)\,,$$

where α_1, α_2 are any complex numbers and β_1 is the number given in Theorem 3.2 and Φ_0 is a function on $\mathbf{C}$ bounded near the origin if $p=1$ or $q=\infty$ and bounded on every bounded domain if $p>1$.

PROOF. We shall prove the theorem, following to Igari-Sato [9; Theorem 3] (cf. [1]). We may assume $q\le p'$, i.e. $\beta_1=(1/2-1/q)/(1/p-1/2)$. Also we remark that the proof for $q=\infty$ will be derived from that for $p=1$ by the duality (cf. [3], [14], [16]).

First Step. If Φ operates on $M(p, q)$, then there exist two constants $C>0$ and $\eta>0$ such that if $\|T\|_{M(p,q)}<\eta$, then $\|\Phi(T)\|_{M(p,q)}\le C$. The proof is similar to that of [12; Theorem 3] (cf. [1]).

Second Step. Let $p>1$ and $q<\infty$. Then Φ is a bounded function on every bounded domain. In fact, assume that Φ is unbounded on a bounded set K. Then there exist $r_K>0$ and a sequence $\{z_n\}$ such that $|z_n|\le r_K$ $(n\ge 1)$ and $|\Phi(z_n)|\to\infty$ as $n\to\infty$. Let

$$m(k)=\begin{cases} z_n & (k=2^n) \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, there exists a positive number C_1 such that

$$\left\|\sum_{k=1}^{\infty} m(k)\hat{f}(k)e^{ikx}\right\|_{p'}\le C_1\left\|\sum_{k=1}^{\infty} m(k)\hat{f}(k)e^{ikx}\right\|_2\le C_1 r_K\|f\|_2$$

for every trigonometric polynomial f, since $\{2^n\,|\,n\ge 1\}$ is a Sidon set (cf. [3], [17]). Hence, $\{m(k)\}_{k=1}^{\infty}\in M(2, p')^\wedge=M(p, 2)^\wedge$. Therefore, we have

$$\left\|\sum_{k=1}^{\infty} m(k)\hat{f}(k)e^{ikx}\right\|_2\le C_1 r_K\|f\|_p$$

for every trigonometric polynomial f. Similarly, we have

$$\left\|\sum_{k=1}^{\infty} m(k)\hat{f}(k)e^{ikx}\right\|_q\le C_2\left\|\sum_{k=1}^{\infty} m(k)\hat{f}(k)e^{ikx}\right\|_2$$

for some $C_2>0$. Then $\{m(k)\}\in M(p, q)^\wedge$. Therefore, we get

$$\{\Phi(m(k))\}\in M(p, q)^\wedge\subset M(2, 2)^\wedge=l^\infty(\boldsymbol{Z})\,.$$

This contradicts the choice of $\{m(k)\}$. So we proved Second Step.

For $p=1$, we remark that Φ is bounded near the origin, by $M(1, q)=L^q(\mathbf{T})$ $(2\le q\le\infty)$ and Riemann-Lebesgue's lemma.

Now it is sufficient to prove Theorem 3.3 when Φ is even or odd.

LEMMA 3.4 ([14]). *Let r be a prime number and $\alpha=\exp(2\pi i/r)$. There is a sequence $\{\varepsilon_r(n)\}$ with $\varepsilon_r(n)$ having for each n one of the values $1, \alpha, \cdots, \alpha^{r-1}$ such that for $t=1, 2, \cdots, r-1$*

$$\left|\sum_{n=1}^{N}(\varepsilon_r(n))^t e^{inx}\right| < r(1+\sqrt{r})\sqrt{N} \qquad (0 \le x \le 2\pi,\ N \ge 1)\,.$$

LEMMA 3.5 ([14]). *For a prime number r there is a sequence $\{\delta_r(n)\}$ with $\delta_r(n)=r-1$ or -1 such that*

$$\left|\sum_{n=1}^{N}\delta_r(n)e^{inx}\right| < (r-1)r(1+\sqrt{r})\sqrt{N} \qquad (0 \le x \le 2\pi,\ N \ge 1)\,.$$

By Second Step, Lemmas 3.4 and 3.5, as Φ is even, we can prove Theorem 3.3 in the same way [12; Theorem 3]. So we shall prove Theorem 3.3 as Φ is odd. Next we shall omit the following proofs from Third Step to Sixth Step, since we can prove the results by the same method of [12; Theorem 3].

Third Step. For any $r>0$, there exists $C_3>0$ such that $|\Phi(z)| \le C_3|z|$ for all $|z| \le r$.

Then, we may assume $\beta_1>0$.

Fourth Step. For any $r>0$, there exists $C_4>0$ such that $|\Phi(z)-2\Phi(z/2)| \le C_4|z|^{1+\beta_1}$ for all $|z|\le r$. Here, for any $z\in\mathbf{C}$ we define

$$\Phi_1(z)=\lim_{n\to\infty}2^n\Phi(z/2^n)\,, \qquad \Phi_2(z)=\Phi(z)-\Phi_1(z)\,.$$

Fifth Step. For any $r>0$, there exists $C_5>0$ such that $|\Phi_2(z)|\le C_5|z|^{1+\beta_1}$ for all $|z|\le r$. Then Φ_2 and Φ_1 operate on $M(p,q)$ by Theorem 3.2.

Sixth Step. Φ_1 is continuous on $\mathbf{C}$.

Seventh Step. $\Phi_1(te^{ix})=\alpha_1(t)e^{ix}+\alpha_2(t)e^{-ix}$ $(0<t<\infty, 0\le x\le 2\pi)$ for some continuous functions α_1, α_2. For the proof, we prepare the following whose proof is easy by Lemma 3.4 and Riesz-Thorin's interpolation theorem.

LEMMA 3.6. *For a prime number r we have*

$$\left\|\sum_{n=1}^{N}(\varepsilon_r(n))^t e^{inx}\right\|_{M(p,q)} \le r(1+\sqrt{r})N^{1/p-1/2} \qquad (N\ge 1,\ t=1,\cdots,r-1)\,.$$

Proof of Seventh Step. Let $r\ge 2$ be a natural number. We assume that

$$(*) \qquad \sum_{j=1}^{r}\Phi_1(ze^{2\pi ij/r})=0 \qquad \text{for all} \quad z\in\mathbf{C}\,.$$

Then, we shall show Seventh Step. For any t we define $G_t(e^{ix})=\Phi_1(te^{ix})$. By Sixth Step $G_t(e^{ix})$ is continuous with respect to x. Also by $(*)$ and $r\ne 1$,

$$\sum_{j=1}^{r}G_t(e^{i(x+2\pi j/r)})=\sum_{j=1}^{r}\Phi_1(te^{i(t+2\pi j/r)})=0\,.$$

Hence, $G_t(e^{ix})=-\sum_{j=1}^{r-1}G_t(e^{i(x+2\pi j/r)})$. For $n\in\mathbf{Z}$, we have

$$\hat{G}_t(n) = -\frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{r-1} G_t(e^{i(x+2\pi j/r)})e^{-inx}dx$$

$$= -\frac{1}{2\pi}\sum_{j=1}^{r-1}\int_{2\pi j/r}^{2\pi+2\pi j/r} G_t(e^{ix})e^{-inx}e^{2\pi ijn/r}dx = \left(-\sum_{j=1}^{r-1} e^{2\pi ijn/r}\right)\hat{G}_t(n)\,,$$

and $\hat{G}_t(n)(\sum_{j=0}^{r-1} e^{2\pi ijn/r}) = 0$ for every $r \geq 2$. Then, if $n=0$ or $|n| \geq 2$, we obtain $\hat{G}_t(n)=0$, and

$$\hat{G}_t(e^{ix}) = \alpha_1(t)e^{ix} + \alpha_2(t)e^{-ix}\,.$$

Here, $\alpha_1(t)$ and $\alpha_2(t)$ are continuous on $0<t<\infty$, by Sixth Step. So it is sufficient to prove $(*)$. Suppose that $(*)$ is proved for any prime r. For any r such that $r=pq$ (p is prime), we have

$$\sum_{j=1}^{r} \Phi_1(ze^{2\pi ij/r}) = \sum_{s=1}^{q}\sum_{j=1}^{p} \Phi_1(ze^{2\pi i((j-1)q+s)/(pq)})$$

$$= \sum_{s=1}^{q}\sum_{j=1}^{p} \Phi_1((ze^{2\pi i(s-q)/(pq)})(e^{2\pi ij/p})) = 0\,.$$

Hence, we show $(*)$ for a prime $r \geq 2$. Let $N_m = [2^{m/(1/p-1/2)}m^{-1/(1/p-1/2)}]$, and $T_m^t(e^{ix}) = (1/2^m)\sum_{n=1}^{N_m}(\varepsilon_r(n))^t e^{inx}$ $(t=1,\cdots,r-1)$. Then by Lemma 3.6, $\|T_m^t\|_{M(p,q)} \leq (1/2^m)r(1+\sqrt{r})2^m m^{-1}$, and $\|T_m^t\|_{M(p,q)} = O(1/m)$. Also let $\beta = \sum_{j=1}^{r}\Phi_1(ze^{2\pi ij/r})$, and

$$H_m = \sum_{t=1}^{r-1}[\Phi_1(zT_m^t) + (\beta/r - \Phi_1(z))T_m^t]\,.$$

Then, applying the proof of First Step since Φ_1 operates on $M(p, q)$ by Fifth Step, we have

$$\|H_m\|_{M(p,q)} \leq \sum_{t=1}^{r-1}\|\Phi_1(zT_m^t)\|_{M(p,q)} + |\beta/r - \Phi_1(z)|\sum_{t=1}^{r-1}\|T_m^t\|_{M(p,q)}$$

$$\leq (r-1)C_6 + (1/m)|\beta/r - \Phi_1(z)|C_7$$

for some $C_6, C_7 > 0$. Hence, $\{\|H_m\|_{M(p,q)}\}_{m=1}^{\infty}$ is uniformly bounded. Here, by the Fourier coefficients of H_m we have

$$H_m(e^{ix}) = \sum_{n=1}^{N_m}\frac{1}{2^m}\left(1-\frac{1}{r}\right)\beta e^{inx}\,.$$

So

$$\|H_m\|_{M(p,q)} = \left(1-\frac{1}{r}\right)\frac{|\beta|}{2^m}\left\|\sum_{n=1}^{N_m} e^{inx}\right\|_{M(p,q)} \geq \left(1-\frac{1}{r}\right)\frac{|\beta|}{2^m}C_8 N_m^{(1/p-1/q)}$$

for some $C_8 > 0$. Then we obtain $\beta = 0$, and the desired result. q.e.d. of Seventh Step.

Eighth Step. $\alpha_j(t)=t\alpha_j(1)$ $(0<t<\infty, j=1,2)$. Proof. Let $0<t<\infty$, $0\leq\phi\leq 2\pi$ be fixed, and r a prime number. Also let $N_m=[2^{m/(1/p-1/2)}m^{-1/(1/p-1/2)}]$, and

$$T_m=\frac{te^{i\phi}}{(r-1)2^m}\sum_{n=1}^{N_m}\delta_r(n)e^{inx}.$$

Then we have

$$(**)\qquad \Phi_1(T_m)=\frac{1}{2^m r}\left\{\Phi_1(te^{i\phi})-(r-1)\Phi_1\left(\frac{te^{i\phi}}{r-1}\right)\right\}\sum_{n=1}^{N_m}e^{inx}$$
$$+\frac{1}{2^m r}\left\{\Phi_1(te^{i\phi})+\phi_1\left(\frac{te^{i\phi}}{r-1}\right)\right\}\sum_{n=1}^{N_m}\delta_r(n)e^{inx}.$$

Also we have $\|\Phi_1(T_m)\|_{M(p,q)}\leq C_9$ for some $C_9>0$ by applying the proof of First Step, since $\|T_m\|_{M(p,q)}\leq tr(1+\sqrt{r})/m$ by Lemma 3.5. Moreover, we have

$$\frac{1}{2^m}\left\|\sum_{n=1}^{N_m}\delta_r(n)e^{int}\right\|_{M(p,q)}\leq C_{10}$$

for some $C_{10}>0$ by Lemma 3.5 and Riesz-Thorin's interpolation. Therefore, by (**) there are $C_{11}>0$ such that

$$\frac{1}{2^m r}\left|\Phi_1(te^{i\phi})+\Phi\left(\frac{te^{i\phi}}{r-1}\right)\right|\left\|\sum_{n=1}^{N_m}e^{inx}\right\|_{M(p,q)}$$
$$\leq C_9+\left|\Phi_1(te^{i\phi})+\Phi_1\left(\frac{te^{i\phi}}{r-1}\right)\right|\frac{1}{2^m r}\left\|\sum_{n=1}^{N_m}\delta_r(n)e^{inx}\right\|_{M(p,q)}\leq C_{11}\,,\quad\text{and}$$
$$2^{-m}N^{(1/p-1/q)}\left|\Phi_1(te^{i\phi})-(r-1)\Phi_1\left(\frac{te^{i\phi}}{r-1}\right)\right|\leq C_{12}$$

for some $C_{12}>0$. Since we have $2^{-m}N_m^{(1/p-1/q)}\geq 1/(3m^{(1/p-1/q)/(1/p-1/2)})2^{m\beta_1}\to\infty$ $(m\to\infty)$ by $\beta_1>0$,

$$\Phi_1(te^{i\phi})-(r-1)\phi_1\left(\frac{te^{i\phi}}{r-1}\right)=0\,.$$

Therefore we have $\alpha_j(t)=(r-1)\alpha_j(t/(r-1))$ $(0<t<\infty, j=1,2)$ by Seventh Step. Thus for all prime numbers $u>r$ and $n=1,2,\cdots$ we have

$$\alpha_j\left(\left(\frac{r-1}{u-1}\right)^n\right)=\left(\frac{r-1}{u-1}\right)^n\alpha_j(1)\qquad(j=1,2)\,.$$

Remark that the set

$$\{((r-1)/(u-1))^n\,|\,r\text{ and }u\text{ are prime and }n=1,2,\cdots\}$$

is dense in $(0, \infty)$ and that α_j $(j=1, 2)$ are continuous, then we have $\alpha_j(t)=t\alpha_j(1)$ $(0<t<\infty)$. q.e.d. of Eighth Step.

We succeed the proof of Theorem 3.3. By Seventh Step and Eighth Step, we have

$$\Phi_1(te^{ix})=\alpha_1(1)te^{ix}+\alpha_2(1)te^{-ix} \qquad (0\leq x\leq 2\pi, 0<t<\infty).$$

Then, by First Step we have

$$\Phi(z)=\Phi_1(z)+\Phi_2(z)=\alpha_1 z+\alpha_2\bar{z}+|z|^{\beta_1+1}\Phi_0(z)$$

for some $\alpha_1, \alpha_2\in\mathbf{C}$. This proves our desired result. q.e.d. of Theorem 3.3.

REMARK 1. Suppose that a function $\Phi(z)$ on $|z|<1$ operates on $M(p, q)$ with $1<p<q<\infty$. Then Φ_0 is bounded in Theorem 3.3 in the same way of the above proof.

REMARK 2 ([9]). Hatori [9; Cor. 3] characterized the operating functions on $M(p, 2)$ $(1<p<2)$ for any infinite compact abelian groups.

4. Non L^p-improving measures on some independent set.

There are many papers Graham-Hare-Ritter [5], Hare [7], Ritter [15], etc. with respect to L^p-improving measures. But it seems that there is little papers about L^p-improving measures or non L^p-improving measures on thin sets. In this section, we shall construct some non L^p-improving measures on some independent set.

For $F\subset G$, we denote $(F)_1=F\cup(-F)$, and $M_0^+(F)$ the set of all positive measures on F, whose Fourier-Stieltjes transform vanishes at infinity. Also for $E\subset G$, $Gp(E)$ denotes the subgroup of G which is algebraically generated by E.

A compact set K $(\subset G)$ is called a Kronecker set, if $f\in C(K)$ with $|f|=1$ on K is uniformly approximated on K by a continuous character $\gamma\in\Gamma$ (cf. [13], [17]). A non zero measure on a Kronecker set is a non L^p-improving measure.

In fact, let G be an infinite compact abelian group, and K a Kronecker set. Also let μ be a non zero measure in $M(K)$. Then,

$$\|\mu\|=\limsup_{\gamma\to\infty}|\hat{\mu}(\gamma)|,$$

since K is a Kronecker set (cf. [13], [17]). Here, we assume that μ is an L^p-improving measure. By Graham-Hare-Ritter [5; Corollary 3.2],

$$\|\mu\|=\limsup_{\gamma\to\infty}|\hat{\mu}(\gamma)|\leq(2-2/r)^{1/2}\|\mu\|$$

for some $1<r<2$, where $\|\mu\|$ is the total variation norm of μ. This contradicts $\mu\neq 0$. Therefore, μ is not an L^p-improving measure.

Now it is known that $M_0^+(K)=\{0\}$ if K is a Kronecker set. We shall construct a nonzero non L^p-improving measure in $M_0(G)$ whose support is an independent set (cf.

[17]), by the method of Körner [13], Saeki [18].

THEOREM 4.1. *Let G be an infinite compact metrizable abelian group, and $C \subset G$ a compact set. Also let $0 \neq \lambda_0 \in M_0^+(G)$ be an L^p-improving measure with $\lambda_0(x+C)=0$ for all $x \in G$. Then there exist $x_0 \in G$ and $\mu \in M_0^+(\mathrm{supp}\,\lambda_0)$ such that*
(i) $\lambda_0(x+C+(\mathrm{supp}\,\mu)_1)=0$ *for all* $x \in G$,
(ii) μ *is not an L^p-improving measure, and*
(iii) $\mathrm{supp}\,\mu - x_0$ *is an independent set.*

For the proof, we prepare some lemmas.

LEMMA 4.2 ([18]). *Let λ_0 be a measure in $M_c^+(G)$, and D a compact subset such that $\lambda_0(x+D)=0$ for all $x \in G$, where $M_c^+(G)$ denotes the set of all positive continuous measures on G. Then, for each finite set F in G and $\varepsilon>0$, there exists a neighborhood V of $0 \in G$ such that*

$$\lambda_0[x+D+(F+V)_1]<\varepsilon \qquad (x \in G).$$

LEMMA 4.3 ([18]). *Suppose that G is metrizable and λ_0 a nonzero measure in $M_0^+(G)$. Then there exist a point $x_0 \in G$ and a nonempty, totally disconnected, compact, perfect subset K_0 of $\mathrm{supp}\,\lambda_0$ with the following three properties:*
(a) *Every nonempty (relatively) open subset of K_0 has positive λ_0-measure.*
(b) *The elements of $K_0 - x_0$ have the same order, say q_0.*
(c) *If $V_1, V_2, \cdots, V_m$ are m-disjoint, nonempty, open subsets of K_0, there exist m points $x_j \in V_j$ such that $x_1 - x_0, x_2 - x_0, \cdots, x_m - x_0$ are independent.*

PROOF OF THEOREM 4.1. Let K_0 and x_0 be in Lemma 4.3, and $\Gamma = \bigcup_{n=1}^{\infty} \Lambda_n$, where Λ_n is a finite set such that $\{\Lambda_n\}$ increases to Γ as $n \to \infty$. Putting $\lambda(E) = \lambda_0[(E+x_0) \cap K_0]$ for a Borel set E, we have $\mathrm{supp}\,\lambda = K_0 - x_0$, and any element of $\mathrm{supp}\,\lambda$ has the same order q_0 (in Lemma 4.3). Since λ_0 is an L^p-improving measure, there exist $1<r<2$ and $C>0$ such that $\|\lambda_0 * f\|_2 \leq C\|f\|_r$ for all trigonometric polynomial f, by Riesz-Thorin's interpolation (cf. [7]). The norm of λ_0 in $M(r, 2)$ is denoted by $\|\lambda_0\|_{M(r,2)}$ (cf. §2, §3). Then, λ has the same norm as $\|\lambda_0\|_{M(r,2)}$.

Here, let $\{r(n)\}_{n=1}^{\infty}$ be a sequence such that $1<r<r(n)<r(n+1)<2$ and $\lim_{n\to\infty} r(n)=2$. Hereafter, by induction we shall construct $\{n_p\}_{p=1}^{\infty} \subset \mathbf{N}$ and $\{f_n, \mathscr{F}_n, \mu_n, \Gamma_n\}_{n=1}^{\infty}$ with the following properties, where $f_n \in L^2(G)$, $\mathscr{F}_n$ is a finite collection of disjoint clopen subset of $K_0 - x_0$, μ_n a probability measure in $L^1(\lambda)$, and Γ_n a finite subset of Γ:

(1) $$\mu_n = \sum_{I \in \mathscr{F}_n} a_I \lambda_I\,,$$

where $a_I>0$, $\lambda_I = \lambda|_I$, and $\|\mu_n\| = \sum_{I \in \mathscr{F}_n} a_I \lambda(I) = 1$. We remark $\|\lambda_I\|_{M(r,2)} < \infty$ (cf. [5]).

(2) $$\sup\{|\widehat{\mu_n|_I}(\gamma)|\,|\,\gamma \in \Gamma \setminus \Gamma_n\} < 2^{-n}\mu_n(I) \qquad \text{for all} \quad I \in \mathscr{F}_n\,.$$

$$\Lambda_n \subset \Gamma_n \,. \tag{3}$$

Now $n_1 = 1$, and f_1, $\mathscr{F}_1$, μ_1, Γ_1 may be arbitrary. Suppose that n_p, f_{n_p}, $\mathscr{F}_{n_p}$, μ_{n_p} and Γ_{n_p} have been constructed for some $p \in \mathbf{N}$. Let $l_p = \mathrm{Card}(\mathscr{F}_{n_p})$, and write $\mathscr{F}_{n_p} = \{I_i\}_{i=1}^{l_p} = \{I_i^p\}_{i=1}^{l_p}$. Also let M_p be the largest natural number such that $\max\{\mu_{n_p}(I) \mid I \in \mathscr{F}_{n_p}\} \le M_p^{-2}$. We define

$$\mathscr{T}_p = \{A \subset \mathscr{F}_{n_p} \mid 1 \le \mathrm{Card}(A) \le M_p\} = \{A_r\}_{r=1}^{s_p} \,.$$

We may assume $A_r = \{I_j\} = \{I_j^r\}_{j=1}^{l_p}$. We shall inductively construct f_n, $\mathscr{F}_n$ $(n_p < n \le n_{p+1})$. Suppose that f_n, $\mathscr{F}_n$, μ_n, and Γ_n have been constructed for some $n = n_p + r - 1$ $(r = 1, 2, \cdots, s_p)$, and put

$$\mathscr{K}_n = \{I \in \mathscr{F}_n \mid I \subset J \text{ for some } J \in A_r\} \,.$$

Then we have the following:

LEMMA 4.4. *There exist $\{a_j^K\}_j$ real numbers, $f_{n+1} \in L^2(G)$, $\|f_{n+1}\|_{r(n+1)} = 1$ and collections $\{L_j^K\}_j$ of disjoint clopen subset of $K \in \mathscr{K}_n$ such that*

$$0 < a_j^K \mu_n(L_j^K) < 2^{-1} \mu_n(K) \qquad (K \in \mathscr{K}_n) \,, \tag{4}$$

$$\sum_j a_j^K \mu_n(L_j^K) = \mu_n(K) \qquad (K \in \mathscr{K}_n) \,, \tag{5}$$

$$\left| \sum_j a_j^K \widehat{\mu_n|_{L_j^K}}(\gamma) - \widehat{\mu_n|_K}(\gamma) \right| < 2^{-n} \mu_n(K) \qquad (\gamma \in \Gamma_n,\ K \in \mathscr{K}_n) \,, \tag{6}$$

$$\sum_j \mathrm{dia}(L_j^K) < n^{-1} \qquad (K \in \mathscr{K}_n) \,, \tag{7}$$

$$\text{the } \{L_j^K\}_{K,j} \text{ are } M_p\text{-independent} \,, \tag{8}$$

where we say that $L_1, L_2, \cdots, L_n$ $(\subset G)$ are M-independent if and only if $\sum_{j=1}^n m_j x_j \ne 0$ whenever $m_j \in \mathbf{Z}$, $|m_j| < q_0$, $x_j \in L_j$ $(j = 1, \cdots, n)$ and $0 \ne \sum_{j=1}^n |m_j| < M$ (cf. [13], [18; p. 232]).

$$\sup_x \lambda_0[x + C + (\textstyle\bigcup_j L_j^K)_1] < (n l_p)^{-1} \qquad (K \in \mathscr{K}_n) \,, \tag{9}$$

$$\left\| f_j * \left(\sum_{I \in \mathscr{F}_n \setminus \mathscr{K}_n} a_I \lambda_I + \sum_{K \in \mathscr{K}_n} \sum_i a_i^K \mu_n|_{L_i^K} \right) - f_j * \sum_{I \in \mathscr{F}_n} a_I \lambda_I \right\|_2 < 2^{-(n+1)} \tag{10}$$

for $1 \le j \le n$, and

$$\left\| f_{n+1} * \left(\sum_{I \in \mathscr{F}_n \setminus \mathscr{K}_n} a_I \lambda_I + \sum_{K \in \mathscr{K}_n} \sum_i a_i^K \mu_n|_{L_j^K} \right) \right\|_2 > n + 1 \,. \tag{11}$$

PROOF OF LEMMA 4.4. If we choose $L_j^K \subset K$ such that $\mathrm{dia}(L_j^K)$ (the diameter of

L_j^K) is "sufficiently small", then we shall have (7), (8) and (9) by Lemmas 4.2 and 4.3.

First Step. For $\varepsilon>0$, there are $\{K_j\}_j$ a partition of $K\in\mathscr{K}_n$, $y_j^K\in K_j$, and $f_{n+1}\in L^2(G)$ with $\|f_{n+1}\|_{r(n+1)}=1$ such that

(i) $$\left\|\sum_{K\in\mathscr{K}_{n,j}}\mu_n(K_j)\delta_{y_j^K}*f_i-\sum_{K\in\mathscr{K}_n}\mu_n|_K*f_i\right\|_2<\varepsilon \qquad (1\le i\le n),$$

(ii) $\{y_j^K\}_{j,K}$ are M_p-independent (cf. [10]),

(iii) each K_j is a relatively clopen set of K, $\mathrm{dia}(K_j)<\varepsilon$, and $|\gamma(x)-\gamma(y)|<\varepsilon$ for all $\gamma\in\Gamma_n$, $x, y\in K_j$ $(1\le j\le n)$,

and putting $\nu=\sum_{K\in\mathscr{K}_{n,j}}\mu_n(K_j)\delta_{y_j^K}$,

(iv) $$\|\nu*f_{n+1}\|_2>n+3+\left\|\sum_{I\in\mathscr{F}_n\setminus\mathscr{K}_n}a_I\mu_n|_I\right\|_{M(r,2)},$$

(v) $$\mu_n(K_j)\le 2^{-1}\mu(K) \qquad \text{for all } j.$$

Proof. We obtain (i) by the following:

LEMMA 4.5. *Let f be in $L^2(G)$, L a relatively clopen subset of K_0-x_0, and $\varepsilon>0$. Then, there exist $\{K_j\}$ a partition of L such that each K_j is a relatively clopen set, and $\{y_j\}$ $(y_i\in K_j)$ such that*

$$\left\|\sum_j\mu_n(K_j)\delta_{y_j}*f-\mu_n|_L*f\right\|_2<\varepsilon.$$

PROOF OF LEMMA 4.5. Let $\eta>0$ with $\eta\mu_n(L)<\varepsilon$. Then there exist $\{K_j\}$, $\{y_j\}$ $(y_j\in K_j)$ such that $\|\tau_{y_j}f-\tau_y f\|_2<\eta$ for any y, $y_j\in K_j$, where $\tau_z f(x)=f(x-z)$. We remark that

$$\sum\mu_n(K_j)\delta_{y_j}*f(x)-\mu_n|_L*f(x)=\sum_j\int_{K_j}(f(x-y_j)-f(x-y))d\mu_n(y).$$

Then, by Schwarz's inequality we have

$$\left\|\sum_j\mu_n(K_j)\delta_{y_j}*f-\mu_n|_L*f\right\|_2$$
$$\le\sum_j\mu_n(K_j)^{1/2}\left(\int_{K_j}\|\tau_{y_j}f-\tau_y f\|_2^2d\mu_n(y)\right)^{1/2}\le\eta\mu_n(L)<\varepsilon.$$

q.e.d. of proof of Lemma 4.5.

Now we have (ii) and (iii) in First Step by Lemmas 4.3 and 4.5. Also we obtain (iv) in First Step by the following: Let $\nu=\sum_{j,K}\mu_n(K_j)\delta_{y_j^K}$. Since $\{y_j^K\}$ are all distinct, there exists V a compact neighborhood of $0\in G$ such that $\{V+y_j^K\}_{j,K}$ are pairwise disjoint. Then we have

$$\|\nu * f\|_2^2 = \left(\sum_{j,K} (\mu_n(K_j))\right)^2 \|f\|_2^2 \qquad (f \in L^2(V))\,.$$

Since G is nondiscrete and $L^{r(n+1)}(V) \neq L^2(V)$, there exists $f_{n+1} \in L^{r(n+1)}(V)$ such that

(a) $$\|f_{n+1}\|_{r(n+1)} = 1\,, \qquad supp f_{n+1} \subset V\,, \quad \text{and}$$

(b) $$\|\nu * f_{n+1}\|_2 > n+3+\left\|\sum_{I \in \mathscr{F}_n \backslash \mathscr{K}_n} a_I \lambda_I\right\|_{M(r,2)}.$$

q.e.d. of proof of First Step.

Second Step. For any $\varepsilon > 0$, there exist L_j^K $(\subset K_j)$ which is a relatively clopen set with $y_j^K \in L_j^K$ such that

(i) $$\{L_j^K\}_{j,K} \text{ are } M_p\text{-independent,}$$

(ii) $$\left\|\sum_{j,K} \mu_n(K_j)\delta_{y_j^K} * f_i - \sum_{j,K} \frac{\mu_n(K_j)}{\mu_n(L_j^K)} \mu_n|_{L_j^K} * f_i\right\|_2 < \varepsilon \qquad (1 \leq i \leq n)\,,$$

(iii) $$\left\|\sum_{j,K} \frac{\mu_n(K_j)}{\mu_n(L_j^K)} \mu_n|_{L_j^K} * f_{n+1}\right\|_2 > n+2+\left\|\sum_{I \in \mathscr{F}_n \backslash \mathscr{K}_n} a_I \lambda_I\right\|_{M(r,2)},$$

(iv) $$\left|\sum_j \frac{\mu_n(K_j)}{\mu_n(L_j^K)} \widehat{\mu_n|_{L_j^K}}(\gamma) - \widehat{\mu_n|_K}(\gamma)\right| < 2\varepsilon\mu_n(K) \qquad (\gamma \in \Gamma_n,\ K \in \mathscr{K}_n)\,.$$

Proof. We may have (i) by Lemma 4.3 (cf. [13]). Also we can obtain (ii) by the method of proof of Lemma 4.5 (iii): When $\mathrm{dia}(L_j^K)$ are "sufficiently small," we have

$$\left\|\sum_{j,K} \frac{\mu_n(K_j)}{\mu_n(L_j^K)} \mu_n|_{L_j^K} * f_{n+1} - \sum_{j,K} \mu_n(K_j)\delta_{y_j^K} * f_{n+1}\right\|_2 \leq 1$$

in the same way as in the proof of (ii). Then, by (iv) in First Step, we have

$$\begin{aligned}\left\|\sum_{j,K} \frac{\mu_n(K_j)}{\mu_n(L_j^K)} \mu_n|_{L_j^K} * f_{n+1}\right\|_2 &\geq \left\|\sum_{j,K} \mu_n(K_j)\delta_{y_j^K} * f_{n+1}\right\|_2 - 1 \\ &\geq n+2+\left\|\sum_{I \in \mathscr{F}_n \backslash \mathscr{K}_n} a_I \lambda_I\right\|_{M(r,2)}.\end{aligned}$$

(iv): Since $|\int_{K_j} \gamma d\mu_n - \gamma(y_j^K)\mu_n(K_j)| \leq \varepsilon\mu_n(K_j)$ by First Step, we have

$$\left|\gamma(y_j^K)\mu_n(K_j) - \int \frac{\mu_n(K_j)}{\mu_n(L_j^K)} \gamma(y) d\mu_n|_{L_j^K}\right| < \varepsilon\mu_n(K_j)\,.$$

Then,

$$\left|\widehat{\mu_n|_K}(\gamma) - \sum_j \frac{\mu_n(K_j)}{\mu_n(L_j^K)} \widehat{\mu_n|_{L_j^K}}(\gamma)\right| \leq \left|\sum_j \widehat{\mu_n|_{K_j}}(\gamma) - \sum_j \gamma(y_j^K)\mu_n(K_j)\right|$$

$$+\left|\sum_j\left(\gamma(y_j)\mu_n(K_j)-\frac{\mu_n(K_j)}{\mu_n(L_j^K)}\int\gamma(y)d\mu_n|_{L_j^K}\right)\right|$$

$$\leq\varepsilon\sum_j\mu_n(K_j)+\varepsilon\sum_j\mu_n(L_j^K)\ \leq 2\varepsilon\mu_n(K)$$

for all $K\in\mathscr{K}_n$. q.e.d. of proof of Second Step.

In Second Step, let $a_j^K=\mu_n(K_j)/\mu_n(L_j^K)$. Then we have (4), (5), (6), (7), (8), (9) and (10) in Lemma 4.4.

Hence, it is sufficient for the proof of Lemma 4.4 to prove (11) in Lemma 4.4. By Second Step (iii), we have

$$\left\|f_{n+1}*\left(\sum_{I\in\mathscr{F}_n\backslash\mathscr{K}_n}a_I\lambda_I+\sum_{K\in\mathscr{K}_n}\sum_j a_j^K\mu_n|_{L_j^K}\right)\right\|_2$$

$$\geq\left\|f_{n+1}*\left(\sum_{j,K}a_j^K\mu_n|_{L_j^K}\right\|_2-\|f_{n+1}\|_r\left\|\sum_{I\in\mathscr{F}_n\backslash\mathscr{K}_n}a_I\lambda_I\right\|_{M(r,2)}$$

$$\geq\left\|f_{n+1}*\left(\sum_{j,K}a_j^K\mu_n|_{L_j^K}\right)\right\|_2-\|f_{n+1}\|_{r(n+1)}\left\|\sum_{I\in\mathscr{F}_n\backslash\mathscr{K}_n}a_I\lambda_I\right\|_{M(r,2)}\geq n+1\,.$$

q.e.d. of proof of Lemma 4.4.

Now we succeed the proof of Theorem 4.1. Let

$$\mathscr{F}_{n+1}=(\mathscr{F}_n\diagdown\mathscr{K}_n)\cup\bigcup_{K\in\mathscr{K}_n}\{L_j^K\}_j\,,$$

$$\mu_{n+1}=\sum_{I\in\mathscr{F}_n\backslash\mathscr{K}_n}a_I\lambda_I+\sum_{K\in\mathscr{K}_n}\sum_j a_j^K\mu_n|_{L_j^K}\,.$$

Then, by the choice of $\{a_j^K\}$, μ_{n+1} is a probability measure. Moreover, let Γ_{n+1} be a finite subset of Γ such that $\Gamma_{n+1}\supset\Lambda_n\cup\Gamma_n$ and $|\widehat{\mu_{n+1}|_I}(\gamma)|\leq 2^{-(n+1)}\mu_{n+1}(I)$ for all $I\in\mathscr{F}_{n+1}$ and $\gamma\in\Gamma\backslash\Gamma_{n+1}$. Thus, we obtain f_{n+1} (in Lemma 4.4), $\mathscr{F}_{n+1}$, μ_{n+1} and Γ_{n+1}, and $\{f_n,\mathscr{F}_n,\mu_n,\Gamma_n\}_{n_p<n\leq n_{p+1}}$.

We repeat the above process with n_p replaced by $n_{p+1}=n_p+s_p$, which completes our induction. Let μ_∞ be a weak*-cluster point of $\{\mu_n\}$ in $M(G)$, and $\mu\in M^+(G)$ defined by $\mu(E)=\mu_\infty(E-x_0)$ for all Borel sets in G. We claim that μ has the required properties.

In fact, in the same way of Saeki [18; Lemma 4], we can show that $\mu\in M_0^+(supp\,\lambda_0)$, $\mu(G)=1$, $\lambda_0(x+C+(supp\,\mu)_1)=0$ for all $x\in G$, and $supp\,\mu-x_0$ is an independent set. Then, it is sufficient to prove that μ is not an L^p-improving measure. Also by $\mu_\infty=\mu*\delta_{-x_0}$, we may show that μ_∞ is not L^p-improving. Suppose that μ_∞ is an L^p-improving measure. Then there exist $1<r<r(n_0)<2$ and $C>0$ such that

$$\|\mu_\infty*f\|_2\leq C\|f\|_{r(n_0)}\qquad\text{for any } f\in L^{r(n_0)}(G)\,.$$

On the other hand, for any $n\in\mathbf{N}$ there exist f_n such that $\|f_n\|_{r(n)}=1$, $\|\mu_n*f_n\|_2>n+1$, and

$$\|\mu_m * f_n - \mu_{m+1} * f_n\|_2 < 2^{-(m+1)} \qquad \text{for all } m \geq n\,,$$

by the construction of μ_n and (10), (11) in Lemma 4.4. Then, for a fixed $n \in \mathbf{N}$, $\{\mu_m * f_n\}_{m=n}^{\infty}$ converges to g_n for some g_n in $L^2(G)$. Also we have $\mu_\infty * f_n = g_n$, since μ_∞ is a weak*-cluster point of $\{\mu_n\}$ in $M(G)$. Then, we have

$$\|g_n\|_2 = \|\mu_\infty * f_n\|_2 \leq C\|f_n\|_{r(n_0)} \leq C \qquad \text{for some } C > 0 \quad (n \geq n_0)\,.$$

Moreover, since

$$\|\mu_\infty * f_n - \mu_n * f_n\|_2 = \lim_{m \to \infty} \|\mu_m * f_n - \mu_n * f_n\|_2 \leq \sum_{m=n}^{\infty} 2^{-m} \leq 1\,,$$

we have

$$\|\mu_\infty * f_n\|_2 \geq \|\mu_n * f_n\|_2 - 1 \geq n$$

for all $n \geq n_0$. This contradicts the above result. Therefore, Theorem 4.1 is completely proved. q.e.d. of proof of Theorem 4.1.

THEOREM 4.6. *Let G be an infinite compact abelian group, C a compact subset of G, and $0 \neq \lambda_0 \in M_0^+(G)$ and L^p-improving measure such that*

$$\lambda_0(x + C) = 0 \qquad \textit{for all } x \in G\,.$$

Then there exists a non L^p-improving measure $\mu \in M_0^+(\operatorname{supp} \lambda_0)$ such that

$$\lambda_0(x + C + (\operatorname{supp} \mu)_1) = 0 \qquad \textit{for all } x \in G\,.$$

If, in addition, G is metrizable, such a measure μ can be taken so that $\operatorname{supp} \mu - x_0$ is independent for some $x_0 \in G$.

PROOF OF THEOREM 4.6. When G is metrizable, we have Theorem 4.6 by Theorem 4.1. To prove the general case, we need a Lemma.

LEMMA 4.7 ([18]). *Let λ_0 and C be as in Theorem 4.6. Then, given a countable subset Λ_0 of Γ, we can find an infinite countable subgroup Γ_0 of Γ such that $\Lambda_0 \subset \Gamma_0$ and*

$$\lambda_0[x + C + H_{\Gamma_0}] = 0 \qquad \textit{for all } x \in G\,,$$

where H_{Γ_0} denotes the annihilator of Γ_0.

Now we succeed the proof of Theorem 4.6. Let λ_0 and C be in Theorem 4.6, and $\Lambda_0 = \{\gamma \in \Gamma \mid \hat{\lambda}_0(\gamma) \neq 0\}$. For Λ_0, take a Γ_0 as in Lemma 4.7. Setting $H = H_{\Gamma_0}$, we denote by π and m_H the natural mapping of G onto $G_0 = G/H$ and the normalized Haar measure of H, respectively. Then G_0 is metrizable. For each $\nu \in M(G)$, define a measure $\nu' \in M(G_0)$ by setting

$$\int_G f d\nu' = \int_G f \circ \pi d\nu \qquad (f \in C(G_0)). \tag{1}$$

Identifying Γ_0 with $\hat{G}_0$ in the usual way, we see $\hat{\nu}' = \hat{\nu}|_{\Gamma_0}$ for all $\nu \in M(G)$. Then, we obtain $0 \neq \lambda'_0 \in M_0(G_0)$, and that λ'_0 is L^p-improving (cf. [7]). On the other hand, we have

$$\lambda'_0(x' + C') = 0 \qquad \text{for all } x' \in G_0 \tag{2}$$

by Lemma 4.7 and (1), where $C' = \pi(C)$. Therefore we can apply our result for metrizable groups to find a measure $\mu' \in M_0^+(supp\,\lambda'_0)$ such that

$$\lambda'_0[x' + C' + (supp\,\mu')_1] = 0 \qquad \text{for all } x' \in G_0, \quad \text{and} \tag{3}$$

$$\mu' \text{ is not an } L^p\text{-improving measure}. \tag{4}$$

Now define a measure $\mu \in M(G)$ by setting

$$\int_G f d\mu = \int_{G_0} \left\{ \int_H f(x+t) dm_H(t) \right\} d\mu'(x') \qquad (f \in C(G)). \tag{5}$$

Since $\mu * m_H = \mu$ and $\lambda_0 * m_H = \lambda_0$, we have $supp\,\mu = \pi^{-1}(supp\,\mu')$ and $supp\,\lambda_0 = \pi^{-1}(supp\,\lambda'_0)$. Then, we have

$$\lambda_0[x + C + (supp\,\mu)_1] = \lambda'_0[x' + C' + (supp\,\mu')_1] = 0$$

for all $x \in G$. Hence, it is sufficient to prove that μ is not L^p-improving. Let $\{r(n)\}_{n=1}^{\infty}$ be an increasing sequence such that $1 < r(n) < r(n+1) < 2$ and $\lim_{n\to\infty} r(n) = 2$. Then for any $n \in \mathrm{N}$, there exist f'_n a trigonometric polynomial on G_0 such that

$$\|f'_n * \mu'\|_{L^2(G_0)} > n \quad \text{and} \quad \|f'_n\|_{L^{r(n)}(G_0)} = 1. \tag{6}$$

Here, we regard f'_n on G_0 as f_n a trigonometric polynomial on G. Then, we have

$$\begin{aligned} \|f_n * \mu\|^2_{L^2(G)} &= \int \left| \int_G f_n(x-y) d\mu(y) \right|^2 dm_G(x) \\ &= \|f'_n * \mu'\|^2_{L^{r(n)}(G_0)} > n^2. \end{aligned} \tag{7}$$

Since $\|f_n\|^{r(n)}_{L^{r(n)}(G_0)} = \|f'_n\|^{r(n)}_{L^{r(n)}(G_0)} = 1$, we have $\mu \in M_0^+(supp\,\lambda_0)$ such that

$$\lambda_0(x + C + (supp\,\mu)_1) = 0 \qquad \text{for all } x \in G, \quad \text{and} \tag{8}$$

$$\mu \text{ is not } L^p\text{-improving}. \tag{9}$$

Therefore the proof of Theorem 4.6 is completed.

DEFINITION 4.8. (1) $q = q(G) = \sup\{s \mid \text{every neighborhood of } 0 \in G \text{ contains an element of order } \geq s\}$.

(2) $(G \supset)\ E$ is strongly independent, if (i) any $x \in E$ has order q, and (ii) for $\{x_j\}_{j=1}^n \subset E$ distinct and $\sum_j m_j x_j = 0$, $m_j = 0 \pmod q$ for all j, where $m = 0 \pmod \infty$

means $m=0$.

COROLLARY 4.9. *Let G be an infinite compact abelian group. Then there exists* $\mu \in M_0^+(G)$ *a non* L^p*-improving measure so that* $m_G(Gp(supp\,\mu))=0$. *In particular, if G is metrizable,* $supp\,\mu$ *is taken so that* $supp\,\mu$ *is strongly independent.*

PROOF. Let $\lambda_0 = m_G$. Then if we apply the proofs of Theorems 4.1 and 4.6, we shall obtain the desired result. We omit the details (cf. [13]). q.e.d.

REMARK 4.10. (i) μ in Corollary 4.9 is a strongly continuous measure (cf. [6]).

(ii) In Corollary 4.9, let $E = supp\,\mu$ and $H = Gp(E)$. Then H is a proper subgroup of G. Moreover, $\mu \in M(H)$ is strongly continuous by (i), but not L^p-improving (cf. [5; Open questions (iv)]).

(iii) There are many measures μ in $M_0^+(G)$ such that μ is L^p-improving, for example, any measure in $L^r(G)$ for some $r>1$, and some Riesz products (cf. [2]), etc.

5. A remark on generalized Riesz products.

Ritter [15] has showed that Riesz products are L^p-improving on the unit circle. In this section, we remark that generalized Riesz products are not necessarily L^p-improving.

DEFINITION 5.1. (1) Let $\Theta = \{I_j\}_{j=1}^{\infty}$ be a set in which $I_j \subset \Gamma$ is a finite symmetric set with $0 \in I_j$. Then Θ is called a dissociate set, if for any $w \in \Gamma$, w is represented by $w = \sum_{\text{finite}} \theta_j$ $(\theta_j \in I_j \setminus \{0\})$, where the representation is at most one.

(2) Let $\Theta = \{I_j\}_{j=l}^{\infty}$ be a dissociate set, and $\{P_j\}_{j=l}^{\infty}$ nonnegative trigonometric polynomials such that $\|P_j\|_1 = \hat{P}_j(0) = 1$ and $supp\,\hat{P}_j \subset I_j$. Then we get $\mu \in M(G)$ in the weak*-limit of $\{\prod_{j=1}^{n} P_j\}_{n=1}^{\infty}$, which is called a generalized Riesz product. We remark that Riesz product is a generalized Riesz product.

We obtain the next result with respect to generalized Riesz products.

THEOREM 5.2. *Let G be an infinite compact abelian group with the dual* Γ, *and* δ *a positive number with* $\delta<1$. *Then there exists* μ_δ *a non* L^p*-improving generalized Riesz product with* $\sup_{\gamma \in \Gamma \setminus \{0\}} |\hat{\mu_\delta}(\gamma)| \leq \delta$.

REMARK 5.3. μ_δ in Theorem 5.2 is a strongly continuous singular measure (cf. [11]).

PROOF OF THEOREM 5.2. *Case* 1: Let G be the circle group $\mathbf{T}$, and ε a positive number with $\varepsilon = 1-\delta$. Also let $\{r(n)\}$ be an increasing sequence such that $1<r(n)<r(n+1)<2$ $(n \geq 1)$ and $\lim_{n\to\infty} r(n) = 2$. Since $L^{r(n)}(G) \neq L^2(G)$, there exist $\{g_n\}_n$ trigonometric polynomials such that $\|g_n\|_{r(n)} = 1$ and $\|g_n\|_2 > \varepsilon^{-n}$ $(n \geq 1)$. Then we choose $\{N_n\}_n$ a sequence such that $f_n(x) = K_{N_n}(x)$ the Fejer kernel with degree N_n such that $supp\,\hat{g_n} \subset \{r \mid 1-|r|/(1+N_n) > \varepsilon\}$ and $\|f_n * g_n\|_2 > \varepsilon^{-n}$ $(n \geq 1)$. We remark that $\|f_n\|_1 = 1$

and $f_n \geq 0$. Also we choose $\{s_n\}_n$ rapidly increasing sequence such that

$$\sum_{j=1}^{k-1} \varepsilon_j \, supp\widehat{f_j(s_j x)} \cap \varepsilon_k \, supp\widehat{f_k(s_k x)} = \{0\} \qquad (k \geq 2)\,,$$

where $\varepsilon_j = 0, \pm 1, \pm 2$ $(j \geq 1)$, and $\sum_{j=1}^{m} \varepsilon_j E_j = \{\sum_{j=1}^{m} \varepsilon_j e_j \,|\, e_j \in E_j\}$ for $E_j \subset \Gamma$ $(j=1, \cdots, m)$.

Putting

$$f_0(x) = \sum_{k=1}^{\infty} f_k(s_k x) \varepsilon^k\,,$$

we have $\|f_0\|_1 = \varepsilon(1-\varepsilon)^{-1}$ and $f_0 \geq 0$. Moreover, by the choice of $\{f_j\}_j$, $\{I_j = supp\widehat{f_j(s_j x)}\}_j$ is a dissociate set. Then we define $P_j = \varepsilon^{-1}(1-\varepsilon) f_0 * \widetilde{K}_{N_j}(x)$, where $\widetilde{K}_{N_j}(x) = f_j(s_j x)$, and obtain that $supp\hat{P}_j \subset I_j$, $\|P_j\|_1 = 1$, and $P_j \geq 0$. Therefore μ in the weak*-limit of $\{\prod_{j=1}^{n} P_j\}_n$ is a generalized Riesz product.

Now let $Q_n(x) = g_n(s_n x)$ $(n \geq 1)$. Then

$$\begin{aligned}\widehat{\mu * Q_n}(m) &= \hat{\mu}(s_n r)\hat{g}_n(r) = \varepsilon^{-1}(1-\varepsilon)\hat{f}_0(s_n r)\hat{K}_{N_n}(r)\hat{g}_n(r) \\ &= (1-\varepsilon)\varepsilon^{n-1}(\hat{K}_{N_n}(r))^2 \hat{g}_n(r)\,,\end{aligned}$$

where $m = s_n r$, $r \neq 0$, $\hat{g}_n(r) \neq 0$. So we get

$$\sum_m |\widehat{\mu * Q_n}(m)|^2 \geq (1-\varepsilon)\varepsilon^{n+1}\|g_n\|_2^2\,, \quad \text{and}$$

$$\|\mu * Q_n\|_2 \geq \sqrt{(1-\varepsilon)\varepsilon^{1-n}} \qquad (n \geq 1)\,.$$

If μ is L^p-improving, there exist $r(n_0)$ $(1 < r(n_0) < 2)$ and $C > 0$ such that $C\|g_n\|_{r(n)} \geq \|\mu * Q_n\|_2$ for $n \geq n_0$. Hence we obtain $C \geq \sqrt{(1-\varepsilon)\varepsilon^{1-n}}$ for $n \geq n_0$. This is a contradiction. Thus μ is not L^p-improving. Also let m be a nonzero integer with $m = \sum \varepsilon_j s_j r_j$ for some $|r_j| \leq N_j$. Then there exists $\varepsilon_{j_0} \neq 0$ for some j_0 such that

$$|\hat{\mu}(m)| = \hat{\mu}(s_{j_0} r_{j_0})\hat{\mu}\left(\sum_{j \neq j_0} s_j r_j\right) \leq (1-\varepsilon)\varepsilon^{-1}\varepsilon^{j_0} \leq 1-\varepsilon\,.$$

Case 2: Let Γ be $\Delta(q)$ $(q \geq 2)$, which is the weak direct group of the q-cyclic group $\mathbf{Z}(q)$. We put

$$G(m, n) = \prod_{k=m+1}^{n} \mathbf{Z}(q)\,, \qquad \Gamma(m, n) = \prod_{k=m+1}^{n} \widehat{\mathbf{Z}(q)}\,,$$

and let $\{n_k\}_{k=0}^{j}$ be the strictly increasing sequence, and $\{g_k\}_{k=1}^{j-1}$ the non-negative trigonometric polynomials on G. Also let $\{r(j)\}$ be the same sequence and $\varepsilon = 1-\delta$ in Case 1. Since $L^{r(j)}(G(n_j, \infty)) \neq L^2(G(n_j, \infty))$, we can choose the trigonometric polynomials g_j on $G(n_j, \infty)$ such that $\|g_j\|_{r(j)} = 1$ and $\|g_j\|_2 > \varepsilon^{-j}$ $(j \geq 1)$. Then there exist a natural integer n_{j+1} $(> n_j)$ and a nonnegative trigonometric polynomial f_j such that

$\|f_j\|_1 = 1$, $\|f_j * g_j\|_2 > \varepsilon^{-j}$, $|\hat{f_j}| \geq \varepsilon$ on $supp\,\hat{g_j}$, and $supp\,\hat{f_j} \subset \Gamma(n_j, n_{j+1}-1)$. We remark $supp\,\hat{g_j} \subset \Gamma(n_j, n_{j+1}-1)$. Also we define

$$h = \sum_{k=1}^{\infty} \varepsilon^k f_k\,, \quad \text{and} \quad P_n = (1-\varepsilon)\varepsilon^{-1} h * f_n \quad (n \geq 1)\,.$$

Then by the choice of $\{n_j\}$ and $\{f_j\}$, $\mu = \prod_{j=1}^{\infty} P_j$ is a generalized Riesz product. Moreover, by the choice of $\{g_j\}$ similar to Case 1, μ is not L^p-improving and $\sup_{\gamma \neq 0} |\hat{\mu}(\gamma)| \leq 1 - \varepsilon$. We omit the details.

Case 3: Let Γ be the unbounded ordered group. Also let $\{P_n\}_n$ be the trigonometric polynomials of Case 1. Then there exist $\{\gamma_s\}_{s=1}^{\infty} \subset \Gamma$ (order of $\gamma_s \geq 3$, $s \geq 1$) and $\mu \in M(G)$ a probability measure such that

(1) $\Psi_j(x) = \sum_{|k| \leq m_j} \hat{P}_j(k)(x, k\gamma_j) \geq 0$, where m_j is the degree of P_j with the order of $\gamma_j \geq m_j$ ($j \geq 1$),

(2) $d\mu_n = \Psi_1 \Psi_2 \cdots \Psi_n dm_G$ (m_G is the Haar measure on G), which has the weak*-limit μ,

(3) when $I_j = \{k_j\gamma_j \,|\, |k_j| \leq m_j\}$, $\{I_j\}$ is a dissociate set,

(4) $\hat{\mu}(k_1\gamma_1 + \cdots + k_n\gamma_n) = \prod_{j=1}^{n} \hat{P_j}(k_j)$, where $|k_j| \leq m_j$ ($j \geq 1$), and

(5) $\hat{\mu}(\gamma) = 0$ on $\Gamma \setminus \bigcup_{n=1}^{\infty} \{k_1\gamma_1 + \cdots + k_n\gamma_n \,|\, |k_j| \leq N_j,\ 1 \leq j \leq n\}$.

Then μ is a generalized Riesz product with $\sup_{\gamma \neq 0} |\hat{\mu}(\gamma)| \leq \delta$, and μ is not L^p-improving (cf. [15]).

Case 4: Let Γ be an infinite group. Then Γ contains Γ_0, which is the group in Case 1, Case 2, or Case 3. Let μ be the generalized Riesz product on G in Case 1, Case 2, or Case 3. Then μ is considered as a generalized Riesz product on G, which is not an L^p-improving measure with $\sup_{\gamma \neq 0} |\hat{\mu}(\gamma)| \leq \delta$. We omit the details. q.e.d. of proof of Theorem 5.2.

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