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Kobayashi-Hitchin Correspondence for Perturbed Seiberg-Witten Equations

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1. Introduction.

By theorem of Donaldson-Uhlenbeck-Yau [D2], [UY] there exists a unique irreducible Hermitian-Einstein connection on any stable vector bundle over a compact Kähler manifold. This, together with the result of Kobayashi-Lübke [Ko], [Lü], implies that there is a one-to-one correspondence between the differential geometric and algebro-geometric objects. This correspondence is called Kobayashi-Hitchin correspondence, and can be realized as the relation between the symplectic quotient and stable orbits via the moment map (cf [DK]). The purpose of this paper is to establish a correspondence of Kobayashi-Hitchin type.

Recently Seiberg and Witten introduced new invariants for smooth 4-manifolds. These invariants are defined to be the number of the solutions of the Seiberg-Witten equations. For a closed Kähler surface, there is a correspondence of Kobayashi-Hitchin type between the gauge equivalence classes of irreducible solutions of these equations and a certain type of divisors [W], [FM]. However the moduli spaces of solutions of the unperturbed equations may not be useful to compute the invariants because Kähler metrics are not generic. In [W], Witten introduced a certain perturbation of the Seiberg-Witten equations for Kähler surfaces with $b_2^+ > 1$ to compute the invariants, and he used there the fact that there is also a correspondence of Kobayashi-Hitchin type between the gauge equivalence classes of solutions of the perturbed equations and pairs of divisors. It turns out that the set of these equivalence classes is finite. However there he did not give a proof of this fact. In this paper we shall establish the correspondence for the perturbed equations.

We shall explain the contents of this paper. In section 2, we observe how the unperturbed Seiberg-Witten equations are described for Kähler surfaces. In this situation we find that to each gauge equivalence class of solutions, we can associate an effective divisor. The proof of the bijectivity of this correspondence, which can be considered as

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a variant of "Kobayashi-Hitchin correspondence", can be reduced to the existence and the uniqueness of the solution of a certain nonlinear elliptic differential equation. This equation is identical with that of Kazdan and Warner [KW], and we can prove the bijectivity mentioned above by a straightforward application of their results. The results in this section are not due to the author, but have been already stated in [W] or [FM]. In section 3, we consider the perturbed equations of Witten on Kähler surfaces satisfying $b_2^+ > 1$. We will find that this time there corresponds a pair of effective divisors to each equivalence class of solutions. As in the unperturbed case, we can reduce the bijectivity of this "Kobayashi-Hitchin correspondence" to the existence and the uniqueness of the solution of a certain elliptic equation. This equation differs from that of Kazdan-Warner, although similar to theirs, and we cannot apply their results directly. In section 4 we shall prove the existence and the uniqueness of the solution of this equation by a modification of the iteration method which has been exploited by Kazdan-Warner.

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2. The Seiberg-Witten equations for Kähler surfaces.

In this section we describe the moduli spaces of solutions of the unperturbed Seiberg-Witten equations for Kähler surfaces in terms of holomorphic geometry. All of the results in this section are described in [W], [FM]. However we shall sketch some of proofs to consider the moduli spaces of solutions of the perturbed Seiberg-Witten equations.

Let X be a closed Kähler surface with Kähler metric g. K_X will denote the canonical bundle of X, and ω will be the Kähler form. In this case, the structure group of the tangent bundle reduces to $U(2)$, and the lifting

$$\begin{tikzcd} & \mathrm{Spin}^c(4) \arrow[d] \\ U(2) \arrow[ur, "l"] \arrow[r, hook] & SO(4) \end{tikzcd}$$

defines the *canonical* Spin^c structure, whose associated spinor bundles $\mathbf{S} = \mathbf{S}^+ \oplus \mathbf{S}^-$ are given by

$$\mathbf{S}^+ = \underline{\mathbf{C}} \oplus K_X^{-1} = \Lambda^{0,0} \oplus \Lambda^{0,2} \qquad \mathbf{S}^- = T'X = \Lambda^{0,1} .$$

Here $\underline{\mathbf{C}}$ is the trivial line bundle over X, $T'X$ the holomorphic tangent bundle of X and $\Lambda^{p,q}$ the bundle of (p, q)-forms. The spinor bundles associated to a general Spin^c structure ξ are given by

$$\mathbf{S}^+ = \mathscr{L} \oplus K_X^{-1}\mathscr{L} = \Lambda^{0,0} \otimes \mathscr{L} \oplus \Lambda^{0,2} \otimes \mathscr{L}$$

$$\mathbf{S}^- = T'X \otimes \mathscr{L} = \Lambda^{0,1} \otimes \mathscr{L} ,$$

where $\mathscr{L}$ is a $U(1)$ line bundle. Thus the choice of ξ is equivalent to that of $\mathscr{L}$. For Spin^c structure ξ, we write L for the determinant of $\mathbf{S}^{\pm}$. Then $L = K_X^{-1}\mathscr{L}^2$ satisfies $c_1(L) \equiv c_1(K_X) \equiv w_2(X) \bmod 2$. In other words, L is characteristic, i.e. $c_1(L)$ is an integral lift of $w_2(X)$. Thus a Spin^c structure determines the associated characteristic line bundle. Conversely for any line bundle L with $c_1(L) \equiv w_2(X)$, the Spin^c structures with L as the associated characteristic line bundle are in one-to-one correspondence with the elements of the 2-torsion part of $H^2(X; \mathbf{Z})$.

Now we shall introduce *Clifford multiplication.* For any point x in X, T_x^*X acts on $\mathbf{S}_x = \sum_{q=0}^{2} (\Lambda^{0,q} \otimes \mathscr{L})_x$ by

$$\rho : v \mapsto \sqrt{2}\,(\mathrm{ext}(v^{(0,1)}) - \mathrm{int}(\overline{v^{(0,1)}}))\,,$$

where $v^{(0,1)}$ is the $(0, 1)$-part of $v \in T_x^*X$, ext is left exterior multiplication, and int is contraction. It follows immediately from the definition that non-zero v maps $\mathbf{S}_x^{\pm}$ onto $\mathbf{S}_x^{\mp}$. Since this action satisfies $\rho(v) \circ \rho(v) = -|v|^2$, ρ is extended to the isomorphism

$$\rho : \Lambda T^*X \otimes \mathbf{C} \simeq Cl(X) \otimes \mathbf{C} \to \mathrm{End}(\mathbf{S})\,,$$

where $Cl(X)$ is the Clifford bundle of (T^*X, g). This isomorphism is called *Clifford multiplication.* For convenience, we write c for the associated pairing

$$c : \sum_{p=0}^{4} \Omega_{\mathbf{C}}^p(X) \otimes C^{\infty}(\mathbf{S}^{\pm}) \to C^{\infty}(\mathbf{S}^{\mp})\,,$$

where $\Omega_{\mathbf{C}}^p(X)$ is the space of $\mathbf{C}$-valued C^{∞} p-forms.

Next we shall introduce *Dirac operator* on the spinor bundles. Let $\nabla^{\mathrm{L.C.}}$ be the Levi-Civita connection of (X, g). Since $\nabla^{\mathrm{L.C.}}$ is compatible with the holomorphic structure of $T'X$, it induces a unitary connection $A^{\mathrm{L.C.}}$ on K_X that is compatible with the holomorphic structure of K_X. Now suppose that a unitary connection A on L is given. Since Lie group Spin$^c(4)$ is realized as a central extension of $SO(4)$, $\nabla^{\mathrm{L.C.}}$ and A induce a unitary connection ∇_A on $\mathbf{S}^{\pm}$. For unitary connection A on L, the *Dirac operator*

$$D_A : C^{\infty}(\mathbf{S}^{\pm}) \to C^{\infty}(\mathbf{S}^{\mp})$$

is defined by

$$D_A : \Phi \mapsto c(\nabla_A \Phi)\,.$$

In view of our definition of Clifford multiplication, D_A can be written as

$$D_A = \sqrt{2}\,(\bar{\partial}_A + \bar{\partial}_A^*)\,,$$

where $\bar{\partial}_A$ is a $\bar{\partial}$ operator on $\mathscr{L}$ defined by $A^{\mathrm{L.C.}}$ and A. We note

$$\bar{\partial}_A \bar{\partial}_A = \tfrac{1}{2} F_A^{0,2}\,,$$

where $F_A^{0,2}$ is the $(0, 2)$-part of the curvature of A.

For a Kähler surface X with a Spin^c structure ξ, the Seiberg-Witten equations are equations for a pair $(A, \Phi) \in \mathscr{A}(L) \times C^{\infty}(S^+)$, where $\mathscr{A}(L)$ is the space of unitary connections of L. The equations are

$$D_A\Phi = 0\,, \qquad \frac{1}{2}F_A^+ = \rho^{-1}\left(\Phi \otimes \Phi^* - \frac{|\Phi|^2}{2}\mathrm{Id}\right),$$

where F_A^+ is the self-dual part of the curvature of A. We note that the gauge transformations are given by

$$g(A, \Phi) = (A - 2g^{-1}dg, g\Phi) \qquad \text{for } g \in \mathscr{G} = C^{\infty}(X, U(1))\,,$$

and the solution space of these equations is $\mathscr{G}$-invariant. A solution (A, Φ) is called *reducible* if $\Phi = 0$, and *irreducible* otherwise. The irreducible solutions form free orbits of $\mathscr{G}$-action, while the reducible solutions have stabilizer $U(1)$.

Now we denote the component of Φ in $\mathscr{L}$ and in $K_X^{-1}\mathscr{L}$ as α and $-i\bar{\beta}$, respectively. The curvature part of the Seiberg-Witten equations says that

$$\begin{aligned} iF_A^{2,0} &= \alpha\beta \\ iF_A^{\omega} &= -\frac{\omega}{2}(|\alpha|^2 - |\beta|^2) \qquad (2.1) \\ iF_A^{0,2} &= \bar{\alpha}\bar{\beta}\,, \end{aligned}$$

where F_A^{ω} is the $(1, 1)$-part of F_A^+. On the other hand the Dirac equation becomes

$$\bar{\partial}_A\alpha - i\bar{\partial}_A^*\bar{\beta} = 0\,. \qquad (2.2)$$

Suppose that (A, α, β) is a solution of these equations. Then we have

$$0 = \bar{\partial}_A(\bar{\partial}_A\alpha - i\bar{\partial}_A^*\bar{\beta}) = \frac{1}{2}F_A^{0,2}\alpha - i\bar{\partial}_A\bar{\partial}_A^*\bar{\beta} = -\frac{i}{2}|\alpha|^2\bar{\beta} - i\bar{\partial}_A\bar{\partial}_A^*\bar{\beta}\,.$$

Taking the L^2-inner product with $-i\bar{\beta}$, we obtain

$$0 = \tfrac{1}{2}\|\alpha\beta\|_{L^2}^2 + \|\bar{\partial}_A^*\bar{\beta}\|_{L^2}^2\,.$$

It follows that $\alpha\beta$ and $\bar{\partial}_A^*\bar{\beta}$ are zero. Thus we have $F_A^{2,0} = -\overline{F_A^{0,2}} = -i\alpha\beta = 0$. In other words F_A is a $(1, 1)$-form, and so A defines a holomorphic structure on L, and A also determines a holomorphic structure on $\mathscr{L}$. Moreover $\bar{\partial}_A^*\bar{\beta} = 0$ implies $\bar{\partial}_A\alpha = 0$ and $\bar{\partial}_{-A}\beta = 0$, where $\bar{\partial}_{-A}$ is a $\bar{\partial}$ operator on $\mathscr{L}^{-1}$ defined by $A^{\mathrm{L.C.}}$ and A. Consequently α is a holomorphic section of $\mathscr{L}$ with respect to $\bar{\partial}_A$, and β is a holomorphic section of $K_X\mathscr{L}^{-1}$ with respect to $\bar{\partial}_{-A}$. Hence α and β do not vanish on any open subset unless they vanish identically. Furthermore,

$$\omega \cdot c_1(L) = \int_X \omega \wedge \frac{i}{2\pi}F_A^+ = -\frac{1}{2\pi}(\|\alpha\|_{L^2}^2 - \|\beta\|_{L^2}^2)\,,$$

and so α is not zero iff $\omega \cdot c_1(L) < 0$ and β is not zero iff $\omega \cdot c_1(L) > 0$. In case $\omega \cdot c_1(L) = 0$, any solution (A, α, β) is reducible, i.e. $\Phi=\alpha-i\beta$ vanishes identically, and hence we do not consider this case any more. In case $\omega\cdot c_1(L)<0$, any solution (A, α, β) satisfies $\alpha\neq 0$ and $\beta=0$, and α defines effective divisor (α), whose homology class is the Poincaré dual of $c_1(\mathscr{L})$. In case $\omega\cdot c_1(L)>0$, any solution (A, α, β) satisfies $\beta\neq 0$ and $\alpha=0$, and β defines effective divisor (β), whose homology class is the Poincaré dual of $c_1(K_X\mathscr{L}^{-1})$.

For simplicity, we assume that $\omega\cdot c_1(L)<0$. We have seen that each solution $(A, \alpha, 0)$ of the unperturbed equations defines effective divisor (α). This correspondence can be interpreted as follows. We write $\mathscr{A}^{1,1}(L)$ for the space of unitary connections whose curvature are (1,1)-forms, and V for the subspace

$$\{(A, \alpha)\in\mathscr{A}^{1,1}(L)\times C^\infty(\mathscr{L})\mid \bar\partial_A\alpha=0\}\subset\mathscr{A}^{1,1}(L)\times C^\infty(\mathscr{L})\,.$$

As described in [W], $\mathscr{A}^{1,1}(L)\times C^\infty(\mathscr{L})$ can be interpreted as a symplectic manifold and the moduli space to the unperturbed equations can be realized as the moduli space of the intersection of V and the zero set of the moment map

$$\mu:\mathscr{A}^{1,1}(L)\times C^\infty(\mathscr{L})\to C^\infty(X, \mathrm{u}(1))$$

given by

$$\mu:(A, \alpha)\mapsto\Lambda\left(F_A^+-\frac{i}{2}|\alpha|^2\omega\right), \tag{2.3}$$

where Λ denote the contraction with ω. The complex gauge group $\mathscr{G}^{\mathbf{C}}=C^\infty(X, \mathbf{C}^*)$ acts on $\mathscr{A}^{1,1}(L)\times C^\infty(\mathscr{L})$ by

$$g(A, \alpha)=(A-2g^{-1}\bar\partial g+2\overline{g^{-1}\bar\partial g}, g\alpha)$$

for $g\in\mathscr{G}^{\mathbf{C}}$. Note that the connection part of this action means

$$\bar\partial_{g(A)}=g\circ\bar\partial_A\circ g^{-1}\,.$$

We write $\mathrm{Div}^+(X)$ for the space of effective divisors of X, and V^{st} for the $\mathscr{G}^{\mathbf{C}}$-invariant subset $\{(A, \alpha)\in V\,|\,\alpha\neq 0\}\subset V$. The correspondence described above can be interpreted as the map

$$\mu^{-1}(0)\cap V/\mathscr{G}\to V^{st}/\mathscr{G}^{\mathbf{C}}=\{D\in\mathrm{Div}^+(X)\,|\,c_1([D])=c_1(\mathscr{L})\}\,. \tag{2.4}$$

Here we shall prove that correspondence (2.4) is bijective. Suppose that $(A, \alpha)\in V^{st}$ is given, and set $\mu(e^{\lambda/2}(A, \alpha))=0$ where $\lambda\in C^\infty(X)$. Then we obtain

$$F_A^+ +2(\bar\partial\partial\lambda)^+=\frac{i}{2}e^\lambda|\alpha|^2\omega\,.$$

Taking the contraction with $-(i/2)\omega$ yields

$$\Delta\lambda-\frac{|\alpha|^2}{2}e^\lambda-\frac{1}{2}*(iF_A^+\wedge\omega)=0\,. \tag{2.5}$$

Here Δ is the negative Laplacian on $C^\infty(X)$. According to results of Kazdan and Warner [KW], there is a unique solution $\lambda \in C^\infty(X)$ to equation (2.5) if $\int_X iF_A^+ \wedge \omega < 0$, which is just the condition that $\omega \cdot c_1(L) < 0$. Thus we have proved that correspondence (2.4) is bijective.

3. The perturbed equations.

Let X be a closed Kähler surface with $b_2^+ > 1$. Suppose that a Spin^c structure ξ is given. Since the condition $b_2^+ > 1$ is equivalent to $H^0(X, \mathcal{O}(K_X)) \neq 0$, we can pick a non-zero holomorphic 2-form η. The perturbed equations introduced by Witten [W] are

$$
\begin{aligned}
&iF_A^{2,0} = \alpha\beta - \eta \\
(3.1) \qquad &iF_A^{\omega} = -\frac{\omega}{2}(|\alpha|^2 - |\beta|^2) \\
&iF_A^{0,2} = \bar{\alpha}\bar{\beta} - \bar{\eta}
\end{aligned}
$$

and

$$(3.2) \qquad \bar{\partial}_A \alpha - i\bar{\partial}_A^* \beta = 0$$

where $(A, \alpha, \beta) \in \mathcal{A}(L) \times C^\infty(\mathcal{L}) \times C^\infty(K_X \mathcal{L}^{-1})$.

Now we assume that L admits a holomorphic structure, since the moduli space to the unperturbed equations is empty otherwise. Under this assumption we have

$$(F_A^{2,0}, \eta)_{L^2} = (F_A^{0,2}, \bar{\eta})_{L^2} = 0 .$$

Suppose that (A, α, β) is a solution of the perturbed equations. Then we have

$$0 = \bar{\partial}_A(\bar{\partial}_A \alpha - i\bar{\partial}_A^* \beta) = \tfrac{1}{2} F_A^{0,2} \alpha - i\bar{\partial}_A \bar{\partial}_A^* \beta .$$

Taking the L^2-inner product with $-i\beta$, we obtain

$$
\begin{aligned}
0 &= \tfrac{1}{2}(iF_A^{0,2}, \bar{\alpha}\bar{\beta})_{L^2} + \|\bar{\partial}_A^* \beta\|_{L^2}^2 \\
&= \tfrac{1}{2}(iF_A^{0,2}, \bar{\alpha}\bar{\beta} - \bar{\eta})_{L^2} + \|\bar{\partial}_A^* \beta\|_{L^2}^2 \\
&= \tfrac{1}{2}\|\alpha\beta - \eta\|_{L^2}^2 + \|\bar{\partial}_A^* \beta\|_{L^2}^2 .
\end{aligned}
$$

It follows that $(\alpha\beta - \eta)$ and $\bar{\partial}_A^* \beta$ are zero. Thus we have $F_A^{2,0} = -\overline{F_A^{0,2}} = -i(\alpha\beta - \eta) = 0$, and hence A defines a holomorphic structure on L and determines a holomorphic structure on $\mathcal{L}$. Moreover $\bar{\partial}_A^* \beta = 0$ implies $\bar{\partial}_A \alpha = 0$ and $\bar{\partial}_{-A} \beta = 0$. Note that α and β are non-zero because $\alpha\beta = \eta \neq 0$. Consequently α is a non-zero holomorphic section of $\mathcal{L}$ with respect to $\bar{\partial}_A$ and β is a non-zero holomorphic section of $K_X \mathcal{L}^{-1}$ with respect to $\bar{\partial}_{-A}$. Therefore (α, β) defines pairs of effective divisors $((\alpha), (\beta))$. This correspondence can be interpreted as follows. We write

$$W = \{(A, \alpha, \beta) \in \mathcal{A}^{1,1}(L) \times C^\infty(\mathcal{L}) \times C^\infty(K_X \mathcal{L}^{-1}) \mid \alpha\beta = \eta, \bar{\partial}_A \alpha = \bar{\partial}_{-A} \beta = 0\} .$$

We can interpret $\mathscr{A}^{1,1}(L)\times C^{\infty}(\mathscr{L})\times C^{\infty}(K_X\mathscr{L}^{-1})$ as a symplectic manifold and realize the moduli space to the perturbed equations as the moduli space of the intersection of W and the zero set of the moment map

$$\mu : \mathscr{A}^{1,1}(L)\times C^{\infty}(\mathscr{L})\times C^{\infty}(K_X\mathscr{L}^{-1}) \to C^{\infty}(X, \mathfrak{u}(1))$$

given by

$$\mu : (A, \alpha, \beta)\mapsto \Lambda\left(F_A^+ - \frac{i}{2}(|\alpha|^2-|\beta|^2)\omega\right). \tag{3.3}$$

The complex gauge group $\mathscr{G}^{\mathbf{C}}$ acts on $\mathscr{A}^{1,1}(L)\times C^{\infty}(\mathscr{L})\times C^{\infty}(K_X\mathscr{L}^{-1})$ by

$$g(A, \alpha, \beta)=(A-2g^{-1}\bar{\partial}g+2\overline{g^{-1}\bar{\partial}g}, g\alpha, g^{-1}\beta)$$

for $g\in\mathscr{G}^{\mathbf{C}}$. The correspondence described above can be interpreted as the map

$$\mu^{-1}(0)\cap W/\mathscr{G} \to W/\mathscr{G}^{\mathbf{C}}. \tag{3.4}$$

The purpose of this paper is to prove that (3.4) is bijective, which now implies the following theorem.

THEOREM 3.5 (Main Theorem). *Let X be a closed Kähler surface with $b_2^+>1$, and ξ be a Spin^c structure on X whose associated line bundle admits a holomorphic structure. Then for any non-zero holomorphic 2-form η, the gauge equivalence classes of solutions of the perturbed equations are in one-to-one correspondence with the pairs of effective divisors (D_1, D_2) which satisfy*
(i) $(\eta)=D_1+D_2$
(ii) $c_1([D_1])=c_1(\mathscr{L})$ *and* $c_1([D_2])=c_1(K_X\mathscr{L}^{-1})$.

As mentioned in section 1, Theorem 3.5 is used in [W] without establishing bijectivity of (3.4).

4. The proof of the main theorem.

In this section we shall prove Theorem 3.5. By the argument in section 3, it remains only to show that (3.4) is bijective.

Suppose that $(A, \alpha, \beta)\in W$ is given, and set $\mu(e^{\lambda/2}(A, \alpha, \beta))=0$ where $\lambda\in C^{\infty}(X)$. Then we obtain

$$F_A^+ + 2(\bar{\partial}\partial\lambda)^+ = \frac{i}{2}(e^{\lambda}|\alpha|^2-e^{-\lambda}|\beta|^2)\omega .$$

Taking the contraction with $-(i/2)\omega$ yields

$$\Delta\lambda-\frac{|\alpha|^2}{2}e^{\lambda}+\frac{|\beta|^2}{2}e^{-\lambda}-\frac{1}{2}*(iF_A^+\wedge\omega)=0 . \tag{4.1}$$

Let

$$c=\frac{\int_X iF_A^+ \wedge \omega}{\int_X \omega \wedge \omega}.$$

Then we can choose a C^∞ function v such that

$$\Delta v=\tfrac{1}{2}*(iF_A^+ \wedge \omega)-c.$$

For simplicity, write

$$u=\lambda+v\,,\quad a=\frac{|\alpha|^2}{2}e^{-v}\quad\text{and}\quad b=\frac{|\beta|^2}{2}e^{v}.$$

Equation (4.1) then becomes

$$\Delta u-ae^{u}+be^{-u}-c=0. \tag{4.2}$$

To establish bijectivity of (3.4), it is sufficient to prove the following theorem.

THEOREM 4.3. *Let M be an oriented, closed n-dimensional Riemannian manifold, and let a, b be C^∞ functions on M which satisfy $a, b \geq 0$ and a, $b \not\equiv 0$. Then, for any $c \in \mathbf{R}$, there exists a unique solution $u \in C^\infty(M)$ of* (4.2).

Following Kazdan and Warner [KW §9, 10], we shall use the method of upper and lower solutions. Let p be a constant with $p>n$. We shall call $u_+ \in L_2^p(M)$ an upper solution of (4.2) if

$$\Delta u_+ - ae^{u_+}+be^{-u_+}-c\leq 0\,,$$

and $u_- \in L_2^p(M)$ a lower solution of (4.2) if

$$\Delta u_- - ae^{u_-}+be^{-u_-}-c\geq 0\,.$$

LEMMA 4.4. *Let p be a constant with $p>n$. If there exist upper and lower solutions, u_+, $u_- \in L_2^p(M)$ of* (4.2) *and if $u_- \leq u_+$, then there exists a unique solution $u \in C^\infty(M)$ of* (4.2).

PROOF. By the Sobolev embedding theorem, u_+ and u_- are continuous. Let

$$k(x)=a(x)e^{u_+(x)}+b(x)e^{-u_-(x)}.$$

Then k is continuous and $k(x)\geq 0$ for all $x\in M$ but $k\not\equiv 0$. We shall find the desired solution of (4.2) by iterations. Let $L=\Delta-k$. Since L is a self-adjoint, elliptic and negative definite operator, there exist positive constants c_1, c_2 such that for any $u \in L_2^p(M)$,

$$\|u\|_{L_2^p}\leq c_1\|Lu\|_{L^p}\,, \tag{4.5}$$

$$\|u\|_\infty+\|du\|_\infty\leq c_2\|Lu\|_{L^p} \tag{4.6}$$

where $\|\ \ \|_\infty$ denotes the uniform norm. Moreover $L : L_2^p(M) \to L^p(M)$ is a continuous bijection.

Let $u_0=u_+$, and we define inductively $u_{j+1}\in L^p_2(M)$ as the unique solution of

$$Lu_{j+1}=ae^{u_j}-be^{-u_j}+c-ku_j\,.$$

Then $\{u_j\}$ has the following property.

CLAIM. The functions u_j are upper solutions of (4.2), and satisfy

$$u_-\le\cdots\le u_{j+1}\le u_j\le\cdots\le u_+\,. \tag{4.7}$$

We shall show this claim inductively. First we assume that u_j is an upper solution with $u_-\le u_j\le u_+$. Then we have

$$L(u_{j+1}-u_j)=-(\Delta u_j-ae^{u_j}+be^{-u_j}-c)\ge 0\,,$$

$$\begin{aligned}L(u_--u_{j+1})&=\Delta u_--ku_--(ae^{u_j}-be^{-u_j}+c-ku_j)\\&=(\Delta u_--ae^{u_-}+be^{-u_-}-c)+a(e^{u_-}-e^{u_j})+b(e^{-u_j}-e^{-u_-})+k(u_j-u_-)\\&\ge a(e^{u_-}-e^{u_j})+b(e^{-u_j}-e^{-u_-})+(ae^{u_j}+be^{-u_-})(u_j-u_-)\\&=ae^{u_j}\{e^{u_--u_j}-1-(u_--u_j)\}+be^{-u_-}\{e^{u_--u_j}-1-(u_--u_j)\}\ge 0\,.\end{aligned}$$

So the maximum principle says $u_-\le u_{j+1}\le u_j$. Moreover

$$\begin{aligned}\Delta u_{j+1}-ae^{u_{j+1}}+be^{-u_{j+1}}-c&=Lu_{j+1}+ku_{j+1}-ae^{u_{j+1}}+be^{-u_{j+1}}-c\\&=(ae^{u_j}-be^{-u_j}+c-ku_j)+ku_{j+1}-ae^{u_{j+1}}+be^{-u_{j+1}}-c\\&\le a(e^{u_j}-e^{u_{j+1}})+b(e^{-u_{j+1}}-e^{-u_j})+(ae^{u_j}+be^{-u_{j+1}})(u_{j+1}-u_j)\\&=ae^{u_j}\{1+(u_{j+1}-u_j)-e^{u_{j+1}-u_j}\}+be^{-u_{j+1}}\{1+(u_{j+1}-u_j)-e^{u_{j+1}-u_j}\}\le 0\,.\end{aligned}$$

This completes the proof of the claim.

Since u_+, u_- and u_j are continuous, inequality (4.7) shows that $\{u_j\}$ is uniformly bounded. Then by (4.6),

$$\begin{aligned}\|u_{j+1}\|_\infty+\|du_{j+1}\|_\infty&\le c_2\|Lu_{j+1}\|_{L^p}\\&\le c_2\|ae^{u_j}-be^{-u_j}+c-ku_j\|_{L^p}\le\text{const}\,,\end{aligned}$$

so the Arzela-Ascoli theorem implies that a subsequence of $\{u_j\}$ converges uniformly to some continuous function u. It follows, from the monotonicity (4.7), that $\{u_j\}$ itself converges uniformly to u. Then by (4.5),

$$\begin{aligned}\|u_{i+1}-u_{j+1}\|_{L^p_2}&\le\text{const}\|L(u_{i+1}-u_{j+1})\|_{L^p}\\&\le\text{const}(\|a\|_{L^p}\|e^{u_i}-e^{u_j}\|_\infty+\|b\|_{L^p}\|e^{-u_i}-e^{-u_j}\|_\infty+\|k\|_{L^p}\|u_i-u_j\|_\infty)\,.\end{aligned}$$

Thus $\{u_j\}$ converges in $L^p_2(M)$, and hence $u\in L^p_2(M)$. Consequently u satisfies

$$Lu=ae^u-be^{-u}+c-ku\,,$$

so $u\in L^p_2(M)$ is a solution of (4.2) with $u_-\le u\le u_+$. The Sobolev embedding theorem now implies $u\in C^1(M)$. Therefore by a bootstrapping argument, one can prove that

$u \in C^\infty(M)$.

To prove the uniqueness, we suppose that u and v are C^∞ solutions of (4.2). Then we obtain

$$\Delta(u-v)-a(e^u-e^v)+b(e^{-u}-e^{-v})=0\,.$$

Taking the L^2 inner product with $(u-v)$ yields

$$-\|d(u-v)\|_{L^2}^2-\int_M \{a(e^u-e^v)+b(e^{-v}-e^{-u})\}(u-v)*1=0\,.$$

Here we note that $\{a(e^u-e^v)+b(e^{-v}-e^{-u})\}(u-v)\geq 0$. Hence we have

$$d(u-v)=0 \tag{4.8}$$

$$\{a(e^u-e^v)+b(e^{-v}-e^{-u})\}(u-v)=0\,. \tag{4.9}$$

Equality (4.8) shows $u-v=\text{const}$. Write C for the constant $u-v$, then equality (4.9) shows

$$(ae^v+be^{-u})(e^C-1)C=0\,.$$

Thus we conclude that $u=v$. $\square$

To complete the proof of Theorem 4.3, it suffices to show that there exist upper and lower solutions u_+, $u_- \in C^\infty(M)$ which satisfy $u_- \leq u_+$. Firstly, we shall construct an upper solution $u_+ \in C^\infty(M)$. According to [KW, Theorem 10.5(a)], there exists a C^∞ function u_0 which satisfies $\Delta u_0 - ae^{u_0}-c<0$. Let

$$\gamma=-\max_{x\in M}\{\Delta u_0(x)-a(x)e^{u_0(x)}-c\}>0\,,$$

and choose a constant $\delta\geq 0$ such that $e^\delta \geq (b/\gamma)e^{-u_0}$. Then $u_1:=u_0+\delta$ satisfies

$$\begin{aligned}\Delta u_1-ae^{u_1}+be^{-u_1}-c&\leq(\Delta u_0-ae^{u_0}-c)+be^{-u_0}e^{-\delta}\\ &\leq e^{-\delta}\gamma\left(-e^\delta+\frac{b}{\gamma}e^{-u_0}\right)\leq 0\,,\end{aligned}$$

so u_1 is an upper solution of (4.2).

Secondly, we shall construct a lower solution $u_- \in C^\infty(M)$ with $u_- \leq u_+$. Let $u_2 \in C^\infty(M)$ be an upper solution of $\Delta u - be^u + ae^{-u}-(-c)=0$. Then $u_3:=-u_2$ is a lower solution of (4.2). Choose a constant $\varepsilon>0$ such that $u_3-\varepsilon\leq u_+$, and let $u_-=u_3-\varepsilon$. Then u_- is a lower solution of (4.2) with $u_-\leq u_+$. This completes the proof of Theorem 4.3.

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