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On Presheaves Associated to Modules

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Introduction.

Let A be a commutative ring with unity. For a subset E of $\mathrm{Spec}A$, we put

$$S_E = \bigcap_{\mathfrak{p}\in E}(A\backslash\mathfrak{p}) \qquad (S_\varnothing = A). \tag{1}$$

Then S_E is a saturated multiplicatively closed set.

To an A-module M, we associate a presheaf $\bar{M}$ in the following way. By putting

$$\bar{M}(U) = S_U^{-1}M \tag{2}$$

for an open subset U of $\mathrm{Spec}A$, we define a presheaf $\bar{M}$ of modules on $\mathrm{Spec}A$. Then

$$\bar{M}(D(f)) = M_f \qquad \text{for} \quad f \in A\,, \tag{3}$$

$$\bar{M}_\mathfrak{p} = M_\mathfrak{p} \qquad \text{for} \quad \mathfrak{p}\in\mathrm{Spec}A\,, \tag{4}$$

where $D(f) = \{\mathfrak{p}\in\mathrm{Spec}A \mid f\notin\mathfrak{p}\}$. Here $\bar{M}$ is not a sheaf in general. But the sheafification of $\bar{M}$ turns out to be the quasi-coherent $\tilde{A}$-module $\tilde{M}$. Then we ask the question: When is the presheaf $\bar{M}$ actually a sheaf?

Noting that $\bar{M}$ is a sheaf if and only if $\bar{M}=\tilde{M}$, we introduce the following three conditions for a ring A:

(S.1) $\bar{M}=\tilde{M}$ for any A-module M.

(S.2) $\bar{\mathfrak{a}}=\tilde{\mathfrak{a}}$ for any ideal $\mathfrak{a}$ of A.

(S.3) $\bar{A}=\tilde{A}$.

Then it is obvious that (S.1) $\Rightarrow$ (S.2) $\Rightarrow$ (S.3).

The main results of this paper are as follows.

THEOREM 1. *Suppose that A is a valuation ring. Then*

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(i) *A satisfies the condition* (S.3).
(ii) (S.1) $\Leftrightarrow$ (S.2) $\Leftrightarrow$ Spec$A$ *is a noetherian topological space.*

COROLLARY. *A valuation ring of finite dimension satisfies the condition* (S.1).

THEOREM 2. *Let A be a Dedekind domain. Then*
(S.1) $\Leftrightarrow$ (S.2) $\Leftrightarrow$ (S.3) $\Leftrightarrow$ *the ideal class group of A is torsion.*

COROLLARY. (i) *The ring of integers of an algebraic number field of finite degree satisfies the condition* (S.1).

(ii) *Let A be a coordinate ring of a non-singular affine algebraic curve over* **C**. *Then A is a Dedekind domain, and A satisfies the condition* (S.1) *if and only if the curve is rational.*

THEOREM 3. *Suppose that A is a unique factorization domain* (*UFD*). *Then*
(i) *A satisfies the condition* (S.3).
(ii) (S.1) $\Leftrightarrow$ (S.2) $\Leftrightarrow$ *A is a principal ideal domain* (*PID*).

For integral rings which are not integrally closed, we obtain;

EXAMPLE 1. Let $A=\mathbf{Z}[\sqrt{m}]$. If $m=-3, 5$, then A satisfies the condition (S.1). Moreover, if m is a square free integer such that $m\equiv 1 \pmod 8$ and $\mathbf{Z}[(1+\sqrt{m})/2]$ is a PID, then A satisfies the condition (S.1).

EXAMPLE 2. Let $A=\mathbf{C}[X, Y]/(Y^2-X^3-aX-b)$, where $a, b\in\mathbf{C}$, $4a^3+27b^2=0$. Then A does not satisfy the condition (S.3).

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1. Here we introduce the topological conditions (T.1), (T.2) and (T.3).

Let A be a ring and M an A-module. For any subset E of Spec$A$, we obtain

$$S_E^{-1}M \cong \text{ind.}\lim \tilde{M}(U)\,, \tag{5}$$

where U runs over all open sets of Spec$A$ which contain E. Therefore we can write

$$\tilde{M}(E)=S_E^{-1}M \qquad \text{for} \quad E\subset \operatorname{Spec}A\,. \tag{2'}$$

Let A be an integral ring and $\mathfrak{a}$ an ideal of A. Since $\tilde{\mathfrak{a}}$ satisfies the condition that $\tilde{\mathfrak{a}}(U)=\bigcap_{\mathfrak{p}\in U}\mathfrak{a}_{\mathfrak{p}}$ for any non-empty open sets U of Spec$A$, we obtain

$$\bigcap_{\mathfrak{p}\in E}\mathfrak{a}_{\mathfrak{p}} \cong \text{ind.}\lim \tilde{\mathfrak{a}}(U) \tag{6}$$

for any non-empty subset E of Spec$A$. Here U runs over all open sets of Spec$A$ which contain E (see [6], Lemma 1). Therefore we can write

(7) $$\tilde{\mathfrak{a}}(E)=\bigcap_{\mathfrak{p}\in E}\mathfrak{a}_{\mathfrak{p}} \quad \text{for} \quad E\subset \operatorname{Spec}A,\ E\neq\varnothing .$$

For a ring A, we put

$$\Sigma=\{D(f)\mid f\in A\}, \qquad \Sigma_1=\{D(\mathfrak{a}_\alpha)\mid \alpha\in QA\}\cup\{\varnothing\} .$$

Here QA is the total quotient ring of A and $\mathfrak{a}_\alpha=\{b\in A\mid b\alpha\in A\}$. Then

(8) $$\Sigma\subset\Sigma_1 \quad \text{if} \quad A \text{ is integral} .$$

For any subset E of $\operatorname{Spec}A$, we put

(9) $$\tilde{E}=\bigcap_{\substack{U\in\Sigma\\ U\supset E}} U ,$$

(10) $$\tilde{E}^1=\bigcap_{\substack{V\in\Sigma_1\\ V\supset E}} V ,$$

(11) $$E^*=\{\mathfrak{p}\in\operatorname{Spec}A\mid {}^{\exists}\mathfrak{p}'\in E \text{ such that } \mathfrak{p}\subset\mathfrak{p}'\} ,$$

(12) $$E^0=\text{m-}\operatorname{Spec}\bar{A}(E)\subset\operatorname{Spec}A .$$

Then

(9′) $$\tilde{E}=\bigcap_{f\in S_E} D(f)=\{\mathfrak{p}\in\operatorname{Spec}A\mid \mathfrak{p}\subset\bigcup_{\mathfrak{p}'\in E}\mathfrak{p}'\}=\operatorname{Spec}\bar{A}(E) .$$

Moreover if A is integral and $E\neq\varnothing$, then

(9″) $$\tilde{E}=\{\mathfrak{p}\in\operatorname{Spec}A\mid \bar{A}(E)\subset A_{\mathfrak{p}}\} ,$$

(10′) $$\tilde{E}^1=\{\mathfrak{p}\in\operatorname{Spec}A\mid \tilde{A}(E)\subset A_{\mathfrak{p}}\} .$$

LEMMA 1. *Let A be a ring and E a subset of* $\operatorname{Spec}A$. *Then*
(i) $E\subset E^*\subset\tilde{E}^1$, $E^*\subset\tilde{E}$, $E^o\subset\tilde{E}$.
(ii) *If E is open, then $E=E^*$.*
(iii) *If A is integral, then $\tilde{E}^1\subset\tilde{E}$.*

The proof is easy.

Let A be a ring. We introduce the following conditions for the topology of $\operatorname{Spec}A$.

(T.1) For any open set U of $\operatorname{Spec}A$, there exists $f\in A$ such that $U=D(f)$.

(T.2) For any open set U of $\operatorname{Spec}A$, $U=\tilde{U}$.

(T.3) For any open set U of $\operatorname{Spec}A$, $\tilde{U}^1=\tilde{U}$.

Then the following lemma is easy to prove.

LEMMA 2. *Let A be a ring.*

(i) (T.1) $\Leftrightarrow$ *For any ideal* $\mathfrak{a}$ of A, *there exists* $f \in A$ *such that* $\sqrt{\mathfrak{a}} = \sqrt{(f)}$.

(ii) (T.1) $\Rightarrow$ *For any* $\mathfrak{p} \in \operatorname{Spec} A$, *there exists* $f \in A$ *such that* $\mathfrak{p} = \sqrt{(f)}$
$\Rightarrow$ *For any subset E of* $\operatorname{spec} A$, $E^* = \tilde{E}$
$\Rightarrow$ (T.2).

(iii) *If A is integral, then* (T.2) $\Rightarrow$ (T.3).

Moreover, we consider the condition:

(T.1′) For any compact open set U of $\operatorname{Spec} A$, there exists $f \in A$ such that $U = \mathrm{D}(f)$.

Then we have

(13) (T.1) $\Leftrightarrow$ (T.1′) and $\operatorname{Spec} A$ is a noetherian topological space.

2. Here we consider the relationship between the conditions (S.1), (S.2), (S.3) and the conditions (T.1), (T.2), (T.3).

Let A be an integral ring. For any intermediate ring B of QA/A, we put

$$B_* = \{a/b \in QA \mid a \in A, b \in A \cap B^\times\}.$$

Then $A \subset B_* \subset B$.

Let us fix an integral ring A, and consider the correspondence between non-empty subsets E of $\operatorname{Spec} A$ and intermediate rings B of QA/A defined by

(14) $$E \mapsto B = \bar{A}(E), \quad B \mapsto E = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap B^\times = \varnothing\}.$$

LEMMA 3. *Let A be an integral ring. Then*

(i) $\bar{A}(E) \mapsto \tilde{E}$ *by* (14) *for any non-empty subsets E of* $\operatorname{Spec} A$.

(ii) $\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap B^\times = \varnothing\} \mapsto B_*$ *by* (14) *for any intermediate rings B of QA/A.*

(iii) $E = \tilde{E} \neq \varnothing \Leftrightarrow$ *there exists B such that* $E = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap B^\times = \varnothing\}$.

(iv) $B_* = B \Leftrightarrow$ *there exists E such that* $B = \bar{A}(E)$.

The proof is easy.

COROLLARY. *The mapping* (14) *is a bijection between the set of subsets E of* $\operatorname{Spec} A$ *such that $E = \tilde{E} \neq \varnothing$ and the set of intermediate rings B of QA/A such that $B_* = B$.*

Let K be a fied and A a subring of K. We denote by $\mathrm{Loc}(K|A)$ the set of local subrings of K which contain A. We consider the mapping Ψ_A: $\operatorname{Spec} A \to \mathrm{Loc}(QA|A)$ defined by $\mathfrak{p} \mapsto A_\mathfrak{p}$. Then Ψ_A is an into-homeomorphism.

For any intermediate ring B of QA/A, we put

$$B^* = \bigcap_{R \in \operatorname{Im} \Psi_A \cap \mathrm{Loc}(QA|B)} R.$$

Then $B \subset B^* \subset QA$.

We consider the correspondence between E and B defined by

(15) $$E \mapsto B = \tilde{A}(E), \qquad B \mapsto E = \bigcap_{\alpha \in B} \mathrm{D}(\mathfrak{a}_\alpha).$$

Note that

(16) $$\bigcap_{\alpha \in B} \mathrm{D}(\mathfrak{a}_\alpha) = \{\mathfrak{p} \in \operatorname{Spec} A \mid B \subset A_\mathfrak{p}\},$$

for any intermediate ring B of QA/A.

LEMMA 4. *Let A be an integral ring. Then*
(i) $\tilde{A}(E) \mapsto \tilde{E}^1$ *by* (15) *for any non-empty subsets E of* Spec A.
(ii) $\bigcap_{\alpha \in B} \mathrm{D}(\mathfrak{a}_\alpha) \mapsto B^*$ *by* (15) *for any intermediate rings B of QA/A.*
(iii) $E = \tilde{E}^1 \neq \varnothing \Leftrightarrow$ *there exists B such that* $E = \bigcap_{\alpha \in B} \mathrm{D}(\mathfrak{a}_\alpha)$.
(iv) $B = B^* \Leftrightarrow$ *there exists E such that* $B = \tilde{A}(E)$.

The proof is easy.

COROLLARY. *The mapping* (15) *is a bijection between the set of subsets E of* Spec A *such that $E = \tilde{E}^1 \neq \varnothing$ and the set of intermediate rings B of QA/A such that $B = B^*$.*

LEMMA 5. *Let A be an integral ring and E a subset of* Spec A. *Then*
(i) $\bar{A}(E) = \tilde{A}(\tilde{E})$.
(ii) $\bar{A}(E) = \tilde{A}(E) \Leftrightarrow \tilde{E}^1 = \tilde{E}$.

PROOF. We may assume that $E \neq \varnothing$.
(i) $\bar{A}(E) = \bigcap_{\mathfrak{q} \in \operatorname{Spec} \tilde{A}(E)} \tilde{A}(E)_\mathfrak{q} = \bigcap_{\mathfrak{p} \in \tilde{E}} A_\mathfrak{p} = \tilde{A}(\tilde{E})$ by (9').
(ii) If part: By Lemma 4 and (i), we have $\tilde{A}(E) = \tilde{A}(\tilde{E}^1) = \tilde{A}(\tilde{E}) = \bar{A}(E)$. Only if part: By (9'') and (10'), we have $\tilde{E}^1 = \{\mathfrak{p} \in \operatorname{Spec} A \mid \tilde{A}(E) \subset A_\mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} A \mid \bar{A}(E) \subset A_\mathfrak{p}\} = \tilde{E}$. Q.E.D.

COROLLARY. (S.3) $\Leftrightarrow$ (T.3) $\Leftrightarrow \tilde{E}^1 = \tilde{E}$ *for any* $E \subset \operatorname{Spec} A$.

LEMMA 6. *Let A be an integral ring. For a subset E of* Spec A, *the following five conditions are equivalent*:
(a) $\bar{\mathfrak{a}}(E) = \tilde{\mathfrak{a}}(E)$ *for any ideal $\mathfrak{a}$ of A.*
(a') $\bar{\mathfrak{p}}(E) = \tilde{\mathfrak{p}}(E)$ *for any* $\mathfrak{p} \in \operatorname{Spec} A$.
(b) $\mathfrak{b} \subset \bigcup_{\mathfrak{p}' \in E} \mathfrak{p}' \Rightarrow$ *there exists $\mathfrak{p}' \in E$ such that* $\mathfrak{b} \subset \mathfrak{p}'$, *for any ideal $\mathfrak{b}$ of A.*
(b') $E^* = \tilde{E}$.
(c) $E^o \subset E$.

PROOF. We may assume that $E \neq \varnothing$.
(a) $\Rightarrow$ (b): Take an ideal $\mathfrak{b} \subset \bigcup_{\mathfrak{p}' \in E} \mathfrak{p}'$. Then $S_E^{-1}\mathfrak{b} = \bigcap_{\mathfrak{p} \in E} \mathfrak{b}_\mathfrak{p}$ by (a). Since $\mathfrak{b} \cap S_E = \varnothing$, we have $S_E^{-1}\mathfrak{b} \subsetneqq S_E^{-1}A$. Therefore there exists $\mathfrak{p}' \in E$ such that $\mathfrak{b}_{\mathfrak{p}'} \subsetneqq A_{\mathfrak{p}'}$. Then $b \subset \mathfrak{p}'$.
(b) $\Rightarrow$ (b'): Obvious.
(b') $\Rightarrow$ (c): For any $\mathfrak{p} \in E^o \subset \tilde{E} = E^*$, there exists $\mathfrak{p}' \in E$ such that $\mathfrak{p} \subset \mathfrak{p}'$. Then

$S_E^{-1}\mathfrak{p}\subset S_E^{-1}\mathfrak{p}'\neq S_E^{-1}A$ and so $\mathfrak{p}=\mathfrak{p}'\in E$.

(c) $\Rightarrow$ (a): It is sufficient to prove $\bigcap_{\mathfrak{p}\in E}\mathfrak{a}_{\mathfrak{p}}\subset S_E^{-1}\mathfrak{a}$. For any $\alpha\in\bigcap_{\mathfrak{p}\in E}\mathfrak{a}_{\mathfrak{p}}$, we put $\mathfrak{b}=\{b\in A \mid b\alpha\in\mathfrak{a}\}$. If we assume that $\mathfrak{b}\cap S_E=\emptyset$, then $S_E^{-1}\mathfrak{b}\subsetneqq S_E^{-1}A$. Thus there exists $\mathfrak{p}\in E^o\subset E$ such that $S_E^{-1}\mathfrak{b}\subset S_E^{-1}\mathfrak{p}$. Then $\mathfrak{b}\subset\mathfrak{p}$. Since $\alpha\in\mathfrak{a}_{\mathfrak{p}}$, we can write $\alpha=a/b$, $a\in\mathfrak{a}$, $b\in A\backslash\mathfrak{p}$. Then $b\alpha=a\in\mathfrak{a}$ and hence $b\in\mathfrak{b}\subset\mathfrak{p}$. This is a contradiction. Therefore $\mathfrak{b}\cap S_E\neq\emptyset$. If $b\in\mathfrak{b}\cap S_E$, then $a=b\alpha\in\mathfrak{a}$. Thus $\alpha=a/b\in S_E^{-1}\mathfrak{a}$.

(a) $\Rightarrow$ (a$'$): Obvious.

(a$'$) $\Rightarrow$ (c): For any $\mathfrak{p}\in E^o$, we have $S_E^{-1}\mathfrak{p}=\bigcap_{\mathfrak{p}'\in E}\mathfrak{p}_{\mathfrak{p}'}$ by (a$'$). Since $\mathfrak{p}\in E^o\subset\tilde{E}$, we obtain $S_E^{-1}\mathfrak{p}\subsetneqq S_E^{-1}A$. There exists $\mathfrak{p}'\in E$ such that $\mathfrak{p}_{\mathfrak{p}'}\subsetneqq A_{\mathfrak{p}'}$. Then $\mathfrak{p}\subset\mathfrak{p}'$ and hence $S_E^{-1}\mathfrak{p}\subset S_E^{-1}\mathfrak{p}'$. Since $S_E^{-1}\mathfrak{p}$ is maximal, we obtain $\mathfrak{p}=\mathfrak{p}'\in E$. Q.E.D.

COROLLARY.
$$\begin{aligned}(\mathrm{S}.2) &\Leftrightarrow (\mathrm{T}.2)\\ &\Leftrightarrow \bar{\mathfrak{p}}=\tilde{\mathfrak{p}} \text{ for any } \mathfrak{p}\in\operatorname{Spec}A\\ &\Leftrightarrow E^*=\tilde{E} \text{ for any } E\subset\operatorname{Spec}A\\ &\Leftrightarrow E^o\subset E \text{ for any } E\subset\operatorname{Spec}A\,.\end{aligned}$$

REMARK. The above conditions are not equivalent to the following one: $\bar{\mathfrak{m}}=\tilde{\mathfrak{m}}$ for any $\mathfrak{m}\in\text{m-Spec}A$. See Example 3 in §4.

LEMMA 7. *Let A be an integral ring satisfying the condition* (S.3). *Then*
(i) $\bar{\mathfrak{p}}=\tilde{\mathfrak{p}} \Leftrightarrow \mathfrak{p}\notin\widetilde{\mathrm{D}(\mathfrak{p})}$, *for any* $\mathfrak{p}\in\operatorname{Spec}A$.
(ii) *If $\mathfrak{a}$ is a principal ideal of A, then* $\bar{\mathfrak{a}}=\tilde{\mathfrak{a}}$.

PROOF. (i) Note that $\tilde{E}^1=\tilde{E}$ for any subset E of Spec$A$. Thus

$$\begin{aligned}\bar{\mathfrak{p}}=\tilde{\mathfrak{p}} &\Leftrightarrow \text{if } \mathfrak{p}\in\tilde{E}, \text{ then } \mathfrak{p}\in E^* \text{ for any } E\subset\operatorname{Spec}A\\ &\Leftrightarrow \text{if } \mathfrak{p}\notin E^*, \text{ then } \mathfrak{p}\notin\tilde{E} \text{ for any } E\subset\operatorname{Spec}A\\ &\Leftrightarrow \text{if } E\cap\mathrm{V}(\mathfrak{p})=\emptyset, \text{ then } \mathfrak{p}\notin\tilde{E} \text{ for any } E\subset\operatorname{Spec}A\\ &\Leftrightarrow \mathfrak{p}\notin\widetilde{\mathrm{D}(\mathfrak{p})}\,.\end{aligned}$$

(ii) Let $\mathfrak{a}=aA$. We may assume $a\neq0$. It is sufficient to prove $\bigcap_{\mathfrak{p}\in E}\mathfrak{a}_{\mathfrak{p}}\subset S_E^{-1}\mathfrak{a}$ for any non-empty subset E of Spec$A$. For any $\alpha\in\bigcap_{\mathfrak{p}\in E}\mathfrak{a}_{\mathfrak{p}}$, we obtain $\alpha/a\in\bigcap_{\mathfrak{p}\in E}A_{\mathfrak{p}}=S_E^{-1}A$. Thus $\alpha\in S_E^{-1}\mathfrak{a}$. Q.E.D.

LEMMA 8. *For an integral ring A, we obtain*

$$\begin{array}{ccccccc} & & (\mathrm{S}.1) & \Rightarrow & (\mathrm{S}.2) & \Rightarrow & (\mathrm{S}.3)\\ & & \Uparrow & & \Updownarrow & & \Updownarrow\\ \mathrm{PID} & \Rightarrow & (\mathrm{T}.1) & \Rightarrow & (\mathrm{T}.2) & \Rightarrow & (\mathrm{T}.3)\end{array}$$

The proof is obvious from (3), Lemma 2 and Corollaries of Lemmas 5 and 6.

Next we consider several integral rings which are not integrally closed.

EXAMPLE 1. Let $A=\mathbf{Z}[\sqrt{m}]$. If $m=-3$, 5, then A satisfies the condition (T.1). Moreover if m is a square free integer such that $m\equiv1(\mathrm{mod}\,8)$ and $\mathbf{Z}[(1+\sqrt{m})/2]$ is a

PID, then A satisfies the condition (T.1).

In fact, since any prime ideal of A is the radical of a principal ideal and any closed subset of Spec$A$ is either Spec$A$ itself or a finite set, A satisfies the condition (T.1).

EXAMPLE 2′. Let $A=\mathbf{C}[X, Y]/(Y^2-X^3)$. If we put $\mathfrak{m}_a=(X-a^2, Y-a^3)$ for $a\in\mathbf{C}$, then m-Spec$A=\{\mathfrak{m}_a \mid a\in\mathbf{C}\}$ and $\mathfrak{m}_0$ is the unique singular point of Spec$A$. Moreover,

(i) If a subset E of Spec$A$ does not contain $\mathfrak{m}_0$, the $\bar{A}(E)=\tilde{A}(E)$.

(ii) If we put $U=\mathrm{Spec}A\backslash\{\mathfrak{m}_a\}$, then $\bar{A}(U)=\tilde{A}(U)\Leftrightarrow a=0$.

Thus A does not satisfy the condition (S.3).

EXAMPLE 2″. Let $A=\mathbf{C}[X, Y]/(Y^2-X^3-X^2)$. If we put $\mathfrak{m}_a=(X-a^2+1, Y-a^3+a)$ for $a\in\mathbf{C}$, then m-Spec $A=\{\mathfrak{m}_a \mid a\in\mathbf{C}\}$ and $\mathfrak{m}_{-1}=\mathfrak{m}_1$ is the unique singular point of Spec$A$. Moreover,

(i) If a subset E of Spec$A$ does not contain $\mathfrak{m}_{-1}=\mathfrak{m}_1$, then $\bar{A}(E)=\tilde{A}(E)$.

(ii) If we put $U=\mathrm{Spec}A\backslash\{\mathfrak{m}_a\}$, where $a\neq\pm1$, then $\bar{A}(U)=\tilde{A}(U)\Leftrightarrow(a+1)/(a-1)$ is a root of unity.

Thus A does not satisfy the condition (S.3)

3. Here we collect some properties of Prüfer rings and Krull rings.

First we recall the definition of Prüfer rings. An integral ring A is said to be *Prüfer* if $A_\mathfrak{p}$ is a valuation ring for any $\mathfrak{p}\in\mathrm{Spec}\,A$. Let A be a Prüfer ring. Then

$$\text{(16')}\qquad \bigcap_{\alpha\in B} \mathrm{D}(\mathfrak{a}_\alpha)=\{\mathfrak{p}\in\mathrm{Spec}A \mid B\subset A_\mathfrak{p}\}=\{\mathfrak{p}\in\mathrm{Spec}A \mid \mathfrak{p}B\subsetneqq B\}$$
$$=\{\mathfrak{p}\in\mathrm{Spec}A \mid {}^{\exists}\mathfrak{q}\in\mathrm{Spec}B \text{ such that } \mathfrak{p}=A\cap\mathfrak{q}\}\cong\mathrm{Spec}B\,,$$

for any intermediate ring B of QA/A (see [1], (26.1)).

LEMMA 9. *For an integrally closed integral ring A, the following two conditions are equivalent*:

(a) *A is a Prüfer ring.*

(b) *For any intermediate ring B of QA/A, there exists a subset E of* Spec$A$ *such that $B=\tilde{A}(E)$.*

For a proof, see [1], (26.2).

THEOREM 4. *For an integral ring A, the following three conditions are equivalent*:

(a) *A is a Prüfer ring which satisfies the condition* (S.3).

(a′) *A is a Prüfer ring which satisfies the condition* (T.1′).

(b) *For any intermediate ring B of QA/A, there exists a subset E of* Spec$A$ *such that $B=\bar{A}(E)$.*

The proof is easy from Lemma 9 and [1], (27.5).

COROLLARY. *Suppose that A is a Prüfer ring such that* $\mathrm{Spec}\,A$ *is a noetherian topological space. Then the conditions* (S.1), (S.2), (S.3), (T.1), (T.2) *and* (T.3) *are all equivalent.*

Next we recall the definition of Krull rings. An integral ring A is said to be *Krull* if there exists a subset W of $\mathrm{Zar}(QA|A)$ such that

(17) $$\text{If } R\in W, \text{ then } R \text{ is a discrete valuation ring}.$$

(18) $$\text{For any } \alpha\in QA, \text{ the set } \{R\in W \mid \alpha\notin R\} \text{ is finite}.$$

(19) $$A=\bigcap_{R\in W} R.$$

Here we denote by $\mathrm{Zar}(QA|A)$ the set of valuation rings of QA which contain A. In this case we say that W defines A. If A is a Krull ring, then $W_0=\{A_{\mathfrak{p}} \mid \mathfrak{p}\in\mathrm{Spec}\,A, \dim A_{\mathfrak{p}}=1\}$ is the smallest subset of $\mathrm{Zar}(QA|A)$ which defines A (see [4], Theorem 12.3).

LEMMA 10. *Let A be a Krull ring and E a non-empty subset of* $\mathrm{Spec}\,A$. *Then $\bar{A}(E)$ is also a Krull ring. Moreover,*

(i) *If W defines A, then $W\cap\Phi_A^{-1}(\tilde{E})$ defines $\bar{A}(E)$. Here Φ_A is the mapping from* $\mathrm{Zar}(QA|A)$ *to* $\mathrm{Spec}\,A$ *defined by $\Phi_A(R)=A\cap m(R)$ for any $R\in\mathrm{Zar}(QA|A)$.*

(ii) *$\Phi_A^{-1}(\mathfrak{P}\cap\tilde{E})$ is the smallest subset of* $\mathrm{Zar}(QA|A)$ *which defines $\bar{A}(E)$, where* $\mathfrak{P}=\{\mathfrak{p}\in\mathrm{Spec}\,A \mid \dim A_{\mathfrak{p}}=1\}$.

PROOF. (i) It is clear that $W\cap\Phi_A^{-1}(\tilde{E})$ satisfies (17) and (18). By [4], Theorem 12.1 and $\Phi_A^{-1}(\tilde{E})=\mathrm{Zar}(QA|\bar{A}(E))$, we obtain $\bar{A}(E)=\bigcap_{R\in W\cap\Phi_A^{-1}(\tilde{E})} R$.
(ii) is easy from (i) and $\mathfrak{P}\cap\tilde{E}=\{\mathfrak{p}'\in\mathrm{Spec}\,\bar{A}(E) \mid \mathrm{ht}\,\mathfrak{p}'=1\}$. Q.E.D.

LEMMA 11. *Let A be a Krull ring and E a non-empty subset of* $\mathrm{Spec}\,A$. *Then $\tilde{A}(E)$ is also a Krull ring. Moreover if W defines A, then all the sets $W\cap\Phi_A^{-1}(E^*)$, $W\cap\Phi_A^{-1}(\tilde{E}^1)$ and $W\cap\mathrm{Zar}(QA|\tilde{A}(E))$ define $\tilde{A}(E)$.*

PROOF. Since $W\cap\Phi_A^{-1}(E^*)\subset W\cap\Phi_A^{-1}(\tilde{E}^1)\subset W\cap\mathrm{Zar}(QA|\tilde{A}(E))\subset W$, these sets satisfy (17) and (18). By the similar method to the proof of Lemma 10, we obtain $\tilde{A}(E)=\bigcap_{R\in W\cap\Phi_A^{-1}(E^*)} R$. Q.E.D.

THEOREM 5. *Suppose that A is a Krull ring. If we put $\mathfrak{P}=\{\mathfrak{p}\in\mathrm{Spec}\,A \mid \dim A_{\mathfrak{p}}=1\}$, then*

$$\bar{A}(E)=\tilde{A}(E) \quad\Leftrightarrow\quad \mathfrak{P}\cap\tilde{E}\subset E^*$$

for any subset E of $\mathrm{Spec}\,A$.

The proof is easy from Lemmas 10 and 11.

COROLLARY. *For a Krull ring A, we obtain*
(S.3) $\Leftrightarrow$ $\mathfrak{P}\cap\tilde{U}\subset U$ *for any open subsets U of* $\mathrm{Spec}\,A$
$\Leftrightarrow$ $\mathfrak{P}\cap\tilde{E}\subset E^*$ *for any subsets E of* $\mathrm{Spec}\,A$.

4. Here we prove Theorems 1, 2, 3 and their corollaries.
First we shall prove Theorem 1.

LEMMA 12. *If A is a valuation ring, then A satisfies the condition* (S.3).

PROOF. Since any finitely generated ideal of A is principal, A satisfies (T.1′). From Theorem 4, the proof is complete. Q.E.D.

LEMMA 13. *Let A be a valuation ring and $\mathfrak{p} \in \operatorname{Spec} A$. Then*

$$\bar{\mathfrak{p}} = \tilde{\mathfrak{p}} \iff \bigcup_{\mathfrak{p}' \subsetneqq \mathfrak{p}} \mathfrak{p}' \subsetneqq \mathfrak{p}$$

$$\iff \mathrm{D}(\mathfrak{p}) \text{ has the maximal element, if } \mathrm{D}(\mathfrak{p}) \neq \varnothing.$$

The proof is easy from Lemma 7.

LEMMA 14. *For a valuation ring A, the conditions* (S.1), (T.1), (S.2) *and* (T.2) *are equivalent to each other. Moreover, each condition is also equivalent to the condition that* $\operatorname{Spec} A$ *is a noetherian topological space.*

PROOF. From (13) and Lemma 12, we obtain that "$\operatorname{Spec} A$ is a noetherian topological space $\Rightarrow$ (T.1)". By Lemma 8, we have that (T.1) $\Rightarrow$ (T.2) $\Leftrightarrow$ (S.2). Thereofre, it is sufficient to prove that "(S.2) $\Rightarrow$ $\operatorname{Spec} A$ is a noetherian topological space." We assume that $\operatorname{Spec} A$ is not noetherian. Then there exists a chain of open subsets of $\operatorname{Spec} A$ such that

$$U_0 \subsetneqq U_1 \subsetneqq U_2 \subsetneqq \cdots.$$

Since any non-empty closed subsets of $\operatorname{Spec} A$ are irreducible, we obtain a sequence of $\operatorname{Spec} A$ such that

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \mathfrak{p}_2 \subsetneqq \cdots.$$

If we put $\mathfrak{p}_\infty = \bigcup_{i=0}^{\infty} \mathfrak{p}_i$, then $\mathfrak{p}_\infty \in \operatorname{Spec} A$ and $\bar{\mathfrak{p}}_\infty \neq \tilde{\mathfrak{p}}_\infty$. Q.E.D.

Then the proof of Theorem 1 is complete from Lemmas 12 and 14.

EXAMPLE 3. Let k be a field and K the quotient field of a polynomial ring over k of countable indeterminates. Then there exists $A \in \operatorname{Zar}(K|k)$ such that

$$\operatorname{Zar}(K|A) = \{R_0, R_1, R_2, \cdots, B, A\},$$

where $K = R_0 \supsetneqq R_1 \supsetneqq R_2 \supsetneqq \cdots$, $B = \bigcap_{i=0}^{\infty} R_i$, $B \supsetneqq A$. In this case, $\overline{\mathrm{m}(A)} = \widetilde{\mathrm{m}(A)}$ but A does not satisfy the condition (S.2).

Next we shall prove Theorem 2.

LEMMA 15. *Let A be a Dedekind domain. Then*
(i) *the conditions* (S.1), (T.1), (S.2), (T.2), (S.3) *and* (T.3) *are all equivalent.*

(ii) (T.1) *is equivalent to the condition that the ideal class group of A is a torsion group.*

PROOF. (i) is obvious from the corollary to Theorem 4.
(ii) is easy from Lemma 2.

Therefore the proof of Theorem 2 is over. The proof of Corollary (i) to Theorem 2 is easy from Theorem 2 and the fact that the ideal class group of the ring of integers of an algebraic number field of finite degree is finite. Moreover, Corollary (ii) to Theorem 2 is induced from Theorem 2, the fact that any coordinate ring of a non-singular affine rational curve over $\mathbf{C}$ is a PID and the following lemma:

LEMMA 16. *Let V be an open set of a complete algebraic curve X over $\mathbf{C}$ and $A=\mathcal{O}_X(V)$. If V is non-singular and $\emptyset \neq V \subsetneqq X$, then*
(i) *A is a Dedekind domain and $V \cong \operatorname{Spec} A$.*
(ii) *The ideal class group of A is isomorphic to $\mathbf{Z} \oplus (\mathbf{R}/\mathbf{Z})^{2g}/M$, where M is a finitely generated submodule of $\mathbf{Z} \oplus (\mathbf{R}/\mathbf{Z})^{2g}$ and g is the genus of X.*

Finally we shall prove Theorem 3.

LEMMA 17. *If A is a UFD, then A satisfies the condition* (S.3).

PROOF. Since any prime ideal of height one is principal, we obtain $\mathfrak{P} \cap \tilde{E} \subset E^*$ for any $E \subset \operatorname{Spec} A$. From the corollary of Theorem 5, the proof is complete. Q.E.D.

LEMMA 18. *Let A be a UFD and $\mathfrak{p} \in \operatorname{Spec} A$. Then $\bar{\mathfrak{p}} = \tilde{\mathfrak{p}} \Leftrightarrow \dim A_{\mathfrak{p}} \leqq 1 \Leftrightarrow \mathfrak{p}$ is a principal ideal.*

PROOF. We shall prove in the following three steps:
(i) $\dim A_{\mathfrak{p}} \leqq 1 \Rightarrow \mathfrak{p}$ is principal: This step is clear.
(ii) $\mathfrak{p}$ is principal $\Rightarrow \bar{\mathfrak{p}} = \tilde{\mathfrak{p}}$: This is verified from Lemmas 7 and 17.
(iii) $\bar{\mathfrak{p}} = \tilde{\mathfrak{p}} \Rightarrow \dim A_{\mathfrak{p}} \leqq 1$: If we assume that $\dim A_{\mathfrak{p}} \geqq 2$, then $\mathfrak{P} \subset \mathrm{D}(\mathfrak{p})$. Since $S_{\mathfrak{P}} = A^{\times}$, we also have $S_{\mathrm{D}(\mathfrak{p})} = A^{\times}$ and hence $\widetilde{\mathrm{D}(\mathfrak{p})} = \operatorname{Spec} A$. Thus $\mathfrak{p} \in \widetilde{\mathrm{D}(\mathfrak{p})}$. By Lemma 7, we obtain $\bar{\mathfrak{p}} \neq \tilde{\mathfrak{p}}$. Q.E.D.

LEMMA 19. *For a UFD A, the conditions* (S.1), (T.1), (S.2) *and* (T.2) *are equivalent to each other. Moreover, each condition is also equivalent to the condition that A is a PID.*

PROOF. From Lemma 8, it is sufficient to prove that (S.2) implies PID. Take any $\mathfrak{p} \in \operatorname{Spec} A$. Then $\bar{\mathfrak{p}} = \tilde{\mathfrak{p}}$ by assumption. From Lemma 18, we have $\dim A_{\mathfrak{p}} \leqq 1$ and that $\mathfrak{p}$ is principal. Thus A is noetherian and $\dim A \leqq 1$. Therefore A is a PID. Q.E.D.

Then the proof of Theorem 3 is complete from Lemmas 17 and 19.

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