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On the Existence of a Conjugacy between Weakly Multimodal Maps

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1. Introduction.

In this paper we study the existence of a conjugacy between weakly multimodal maps, which are defined in Definition 1.2, and Hölder continuity of the conjugacy. Throughout this paper let I be a closed interval $[0, 1]$ except §3.1, and n be an integer. We first recall the notion of topological conjugacy.

DEFINITION 1.1. Let $f, g : I \to I$ be two maps. f and g are *topologically conjugate* if there exists a homeomorphism $\varphi : I \to I$ such that

$$\varphi \circ f = g \circ \varphi . \tag{1.1}$$

The map φ is called the *conjugacy* between f and g.

If the map $\varphi : I \to I$ satisfying (1.1) is continuous monotone surjection, f and g are *semi-conjugate*. The map φ is called the *semi-conjugacy*.

The key idea is the following. If we can define the "inverse" g^{-1} in some sense, we have the operator $\mathcal{T}\alpha = g^{-1} \circ \alpha \circ f$ whose fixed point φ, if it exists, should have the equality $g \circ \varphi = \varphi \circ f$. This idea is found in [1] and [2]. We treat with the following class of transformations.

DEFINITION 1.2. The map $f : I \to I$ is *weakly multimodal* if it is continuous and there are points $0 = a_0 \leq b_0 < a_1 \leq b_1 < \cdots < a_{l+1} \leq b_{l+1} = 1$ such that $f|[b_i, a_{i+1}]$ is strictly monotone and $f|[a_i, b_i]$ is a constant function. Assume that the set $\{a_0 = 0, b_0, a_1, b_1, \cdots, a_{l+1}, b_{l+1} = 1\}$ is chosen as small as possible. Let $J_i = [a_i, b_i]$. We say that the J_i's are *flat intervals*.

The following class of maps are known as the l-modal maps, if $a_i = b_i$ for all i. This map is a special case of weakly multimodal maps (cf. [3]).

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DEFINITION 1.3. The map $f: I \to I$ is *l-modal* if it is continuous and there are $0=c_0<c_1<\cdots<c_l<c_{l+1}=1$ such that $f|[c_i, c_{i+1}]$ is strictly monotone. Assume that the set $\{c_i \mid i=0, 1, \cdots, l+1\}$ is chosen as small as possible. We say that $c_1, c_2, \cdots, c_l$ are *turning points*. In particular, if $l=1$ we say that f is *unimodal*.

Our main results are stated in §2. The concrete examples are found is §3. More precisely, we state four examples. Tchebycheff polynomials, the modifications of unimodal maps, and the Cantor function are discussed. In §4 we describe the proofs of main results.

2. Main results.

Our main results are as follows.

THEOREM 2.1. *Suppose that $f, g: I \to I$ are two weakly multimodal maps with flat intervals $0 \in J_0, J_1, \cdots, J_{l+1} \ni 1$, $J_i=[a_i, b_i]$ respectively $0 \in \tilde{J}_0, \tilde{J}_1, \cdots, \tilde{J}_{l+1} \ni 1$, $\tilde{J}_i=[\tilde{a}_i, \tilde{b}_i]$.*

Assume that the map

$$h: \bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} f^n(J_i) \to \bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} g^n(\tilde{J}_i)$$

satisfies the following conditions:

(1) $h(f^n(J_i))=g^n(\tilde{J}_i)$ $(n \geq 1, 1 \leq i \leq l+1)$,

(2) *h is an order preserving homeomorphism (i.e. $x<y \Rightarrow h(x)<h(y)$).*

If both $\bigcup_{i=0}^{l+1} \bigcup_{n \in \mathbf{Z}} f^n(J_i)$ and $\bigcup_{i=0}^{l+1} \bigcup_{n \in \mathbf{Z}} g^n(\tilde{J}_i)$ are dense in I, then f and g are topologically conjugate. That is, there exists a homeomorphism $\varphi: I \to I$ such that $\varphi \circ f = g \circ \varphi$ and $\varphi = h$ on $\bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} f^n(J_i)$.

Corollary 1 is straightforward from the definition of the l-modal map. This result is already known (cf. [3]).

COROLLARY 1. *Suppose that $f, g: I \to I$ are two l-modal maps with turning points $0=c_0<c_1<\cdots<c_{l+1}=1$ respectively $0=\tilde{c}_0<\tilde{c}_1<\cdots<\tilde{c}_{l+1}=1$.*

Assume that the map

$$h: \bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} f^n(c_i) \to \bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} g^n(\tilde{c}_i)$$

defined by $h(f^n(c_i))=g^n(\tilde{c}_i)$ is an order preserving bijection.

If both $\bigcup_{i=0}^{l+1} \bigcup_{n \in \mathbf{Z}} f^n(c_i)$ and $\bigcup_{i=0}^{l+1} \bigcup_{n \in \mathbf{Z}} g^n(\tilde{c}_i)$ are dense in I, then f and g are topologically conjugate.

If the map h in Theorem 2.1 is merely assumed to be an order preserving continuous surjection, then f and g are semi-conjugate.

COROLLARY 2. *Let $f, g: I \to I$ be as in Theorem 2.1.*
Assume that the map

$$h: \bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} f^n(J_i) \to \bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} g^n(\tilde{J}_i)$$

satisfies the following conditions:

(1) $h(f^n(J_i)) = g^n(\tilde{J}_i)$ $(n \geq 1, 1 \leq i \leq l+1)$,

(2) *h is an order preserving continuous surjection.*

If both $\bigcup_{i=0}^{l+1} \bigcup_{n \in \mathbf{Z}} f^n(J_i)$ *and* $\bigcup_{i=0}^{l+1} \bigcup_{n \in \mathbf{Z}} g^n(\tilde{J}_i)$ *are dense in I, then f and g are semi-conjugate. That is, there exists a continuous monotone surjection* $\varphi: I \to I$ *such that* $\varphi \circ f = g \circ \varphi$ *and* $\varphi = h$ *on* $\bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} f^n(J_i)$.

If only $\bigcup_{i=0}^{l+1} \bigcup_{n \in \mathbf{Z}} g^n(\tilde{J}_i)$ is dense in I, then f and g are semi-conjugate. This is clear from the proof of Theorem 2.1.

COROLLARY 3. *Let* $f, g: I \to I$ *and* $h: \bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} f^n(J_i) \to \bigcup_{i=0}^{l+1} \bigcup_{n \geq 0} g^n(\tilde{J}_i)$ *be as in Theorem* 2.1.

If $\bigcup_{i=0}^{l+1} \bigcup_{n \in \mathbf{Z}} g^n(\tilde{J}_i)$ *is dense in I, then f and g are semi-conjugate.*

We obtain the following result for Hölder continuity of the conjugacy. We denote the closure of a set X by $\mathrm{Cl}\, X$.

THEOREM 2.2. *Besides the assumptions of Theorem* 2.1, *suppose further that both* $\bigcup_{i=0}^{l+1} \bigcup_{n > 0} f^n(J_i)$ *and* $\bigcup_{i=0}^{l+1} \bigcup_{n > 0} g^n(\tilde{J}_i)$ *are finite sets, and that the map h is Hölder continuous on* J_i, $i = 0, 1, \cdots, l+1$. *Let* $I_i = (b_i, a_{i+1})$, $\tilde{I}_i = (\tilde{b}_i, \tilde{a}_{i+1})$, $i = 0, 1, \cdots, l$, *and let* $g_i = g | \mathrm{Cl}\, \tilde{I}_i$, $i = 0, 1, \cdots, l$. *Moreover, suppose that for each* $i = 0, 1, \cdots, l$ *there exists* $K_i > 0$ *such that*

$$(2.1) \qquad |g_i^{-1}(x) - g_i^{-1}(y)| \leq K_i^{-1} |x - y|$$

for any $x, y \in g_i(\mathrm{Cl}\, \tilde{I}_i)$, *and that there exists* σ *with* $0 < \sigma \leq 1$ *such that*

$$(2.2) \qquad |f(x) - f(y)| \leq K_i^{1/\sigma} |x - y|$$

for any $x, y \in \mathrm{Cl}\, I_i$. *Then the conjugacy* φ *in Theorem* 2.1 *is Hölder continuous.*

We remark that Theorem 2.2 can be applied to the conjugacy in Corollary 1 and the semi-conjugacy in Corollaries 2, 3.

3. Examples.

3.1. The conjugacy between Tchebycheff polynomials and piecewise linear maps. Let $I = [-1, 1]$. Tchebycheff polynomials $T_n: I \to I$ are defined by $T_n(x) = \cos(n \cos^{-1} x)$, where $0 \leq \cos^{-1} x \leq \pi$. The case when $n = 0, 1$ is trivial. So we deal the case when $n \geq 2$. We remark that $T_n(x)$ are polynomials of degree n and that $T_n(x)$ are $(n-1)$-modal maps (cf. [4]).

LEMMA 3.1. *Let $T_n : I \to I$ be Tchebycheff polynomials for $n \geq 2$. Then the union of the backward orbits of the turning points is dense.*

PROOF. Since

$$T_n^2(x) = \cos(n\cos^{-1}(\cos(n\cos^{-1}x))) = \cos(n^2\cos^{-1}x),$$

we have inductively

$$T_n^k(x) = \cos(n^{k-1}\cos^{-1}(\cos(n\cos^{-1}x))) = \cos(n^k\cos^{-1}x). \tag{3.1}$$

We show that the union of the backward orbits of 1, -1 is dense in I. $f(x) = \cos x$ has local extrema at $x = \pi m$, $m \in \mathbf{Z}$. By (3.1)

$$\{x \in I \mid n^k\cos^{-1}x = \pi m \text{ for some } k > 0,\ m \in \mathbf{Z}\} \tag{3.2}$$

is in the union of the backward orbits of 1, -1. Let $X = \{\pi m/n^k \in [0, \pi] \mid k, m > 0\}$. It follows that X is dense in $[0, \pi]$. The map $f(x) = \cos x$ is a homeomorphism from $[0, \pi]$ to I. Hence $f(X)$ is dense in I. This completes the proof. □

LEMMA 3.2. *Let $f: I \to I$ be a piecewise linear l-modal map with $|f'| > 1$ a.e. Then the union of the backward orbits of the turning points is dense.*

PROOF. Let $-1 = c_0 < c_1 < \cdots < c_l < c_{l+1} = 1$ be turning points of f. We show that for any $x, y \in I$ there exist $n \geq 0$ and c_i such that

$$f^n(x) < c_i \leq f^n(y), \quad \text{or} \quad f^n(y) < c_i \leq f^n(x). \tag{3.3}$$

Suppose that $c_j \leq x < y \leq c_{j+1}$. Since $|f'| > 1$ a.e., $|f^n(x) - f^n(y)|$ must be increasing until we get (3.3). □

By Lemmas 3.1 and 3.2, the maps F_n in the following result and the maps T_n satisfy the assumptions of Corollary 1. This shows the existence of a conjugacy. In fact we have, using the result in [4],

EXAMPLE 3.1. Let $n \geq 2$. Let $F_n : I \to I$ be piecewise linear $(n-1)$-modal maps with slopes $\pm n$ and $F_n(1) = 1$. Then $\varphi(x) = \frac{2}{\pi}\sin^{-1}x$ is a topological conjugacy between T_n and F_n such that $\varphi \circ T_n = F_n \circ \varphi$.

PROOF. Let $R : I \to [0, \pi]$ be the map defined by $R : x \to x' = \cos^{-1}x$. Put

$$V_n = R \circ T_n \circ R^{-1}. \tag{3.4}$$

If

$$\frac{k\pi}{n} \leq x' \leq \frac{(k+1)\pi}{n}, \qquad k = 0, 1, \cdots, n-1,$$

we see that

$$V_n(x')=\begin{cases} nx'-k\pi & \text{if } k \text{ even} \\ -nx'+(k+1)\pi & \text{if } k \text{ odd}. \end{cases}$$

It is easy to check that V_n are piecewise linear $(n-1)$-modal maps with slopes $\pm n$ and $V_n(0)=0$.

Let $L: [0, \pi] \to I$ be the linear map defined by $L: x' \to x = -\frac{2}{\pi}x'+1$. We have

$$V_n = L^{-1} \circ F_n \circ L. \tag{3.5}$$

From (3.4) and (3.5), $F_n \circ (L \circ R) = (L \circ R) \circ T_n$. It follows that the conjugacy $\varphi(x) = L \circ R(x) = \frac{2}{\pi} \sin^{-1} x$. □

3.2. The modifications of the quadratic map and the tent map. Let $G: I \to I$ be

$$G(x)=\begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2(1-x) & \text{if } 1/2 < x \le 1. \end{cases}$$

The map G is called the tent map.

We define the map G_a by

$$G_a(x)=\begin{cases} 2x & \text{if } 0 \le x < a \\ 2a & \text{if } a \le x \le 1-a \\ 2(1-x) & \text{if } 1-a < x \le 1, \end{cases}$$

where $0 < a < 1/2$. Note that the backward orbit of the interval $[a, 1-a]$ is dense in I by the proof of Lemma 3.2.

EXAMPLE 3.2. For any $a, b \in (1/3, 2/5]$, G_a and G_b are topologically conjugate. Let $J=[a, 1-a]$. We have $G_a(J)=2a$, $G_a^2(J)=2(1-2a) \in J$. Hence all of points in J are eventually periodic of period 2. Let $\tilde{J}=[b, 1-b]$. Similarly, we have $G_b^2(\tilde{J}) \in \tilde{J}$. From Theorem 2.1, G_a and G_b are topologically conjugate.

Let $Q: I \to I$ be the map defined by $Q: x \to 4x(1-x)$. We define the map Q_b by

$$Q_b(x)=\begin{cases} 4x(1-x) & \text{if } x \in I \setminus [b, 1-b] \\ 4b(1-b) & \text{if } x \in [b, 1-b]. \end{cases}$$

It is well known that the map $\varphi(x) = \sin^2 \frac{\pi}{2} x$ is a conjugacy between Q and G with $Q \circ \varphi = \varphi \circ G$. From this, we have the following.

EXAMPLE 3.3. G_a are topologically conjugate to Q_b, where $b = \sin^2 \frac{\pi}{2} a$, $\varphi(x) = \sin^2 \frac{\pi}{2} x$ is a conjugacy such that

$$Q_b \circ \varphi = \varphi \circ G_a. \tag{3.6}$$

PROOF. If $x \in I \setminus [a, 1-a]$, then $\varphi(x) \in I \setminus [\sin^2 \frac{\pi}{2} a, \sin^2 \frac{\pi}{2}(1-a)]$. Since $Q \circ \varphi = \varphi \circ G$, (3.6) holds.

If $x \in [a, 1-a]$, then $\varphi(x) \in [\sin^2 \frac{\pi}{2} a, \sin^2 \frac{\pi}{2}(1-a)]$. Hence we have

$$Q_b \circ \varphi(x) = 4\left(\sin^2 \frac{\pi}{2} a\right)\left(1 - \sin^2 \frac{\pi}{2} a\right) = \left(2\left(\sin \frac{\pi}{2} a\right)\left(\cos \frac{\pi}{2} a\right)\right)^2 = \sin^2 \pi a ,$$

$$\varphi \circ G_a(x) = \sin^2\left(\frac{\pi}{2} \cdot 2a\right) = \sin^2 \pi a . \qquad \square$$

3.3. The Cantor function as a semi-conjugacy. We recall the Cantor Middle-Thirds set. Put $E_0 = I = [0, 1]$. Suppose $n \geq 0$ and E_n is constructed so that E_n is the union of 2^n disjoint closed intervals, each of length 3^{-n}. Delete a segment in the center of each of these 2^n intervals, so that each of the remaining 2^{n+1} intervals has length 3^{-n-1}, and let E_{n+1} be the union of these 2^{n+1} intervals. Put

$$E = \bigcap_{n=1}^{\infty} E_n .$$

We say that the set E is the *Cantor Middle-Thirds set*. It is well known that E consists precisely of those numbers in I whose base-3 expansion does not contain the digit 1.

The *Cantor function* $C: I \to I$ is the following. For every $x \in E$ let $x = \alpha_1 3^{-1} + \alpha_2 3^{-2} + \cdots$ with $\alpha_i = 0$ or 2 for each i. We define

$$\beta_n = \begin{cases} 0 & \text{if } \alpha_n = 0 \\ 1 & \text{if } \alpha_n = 2 , \end{cases}$$

and $C(x) = \beta_1 2^{-1} + \beta_2 2^{-2} + \cdots$. If $x \in I \backslash E$, we define $C(x) = \sup\{C(y) \mid y < x, y \in E\}$. Note that C is continuous.

Now let the maps $f, g : I \to I$ be

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x < 1/3 \\ 1 & \text{if } 1/3 \leq x \leq 2/3 \\ 3(1-x) & \text{if } 2/3 < x \leq 1 , \end{cases}$$

$$g(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2(1-x) & \text{if } 1/2 < x \leq 1 . \end{cases}$$

It is easy to check that we may use Corollary 2 for f and g. Furthermore, we have

EXAMPLE 3.4. The Cantor function C is a semi-conjugacy between f and g such that $C \circ f = g \circ C$.

PROOF. Let E be the Cantor Middle-Thirds set. For $x \in E$ we write

$$x = \alpha_1 3^{-1} + \alpha_2 3^{-2} + \cdots ,$$

where $\alpha_i = 0, 2$, and

$$C(x)=\beta_1 2^{-1}+\beta_2 2^{-2}+\cdots,$$

where $\beta_i=0, 1$.

We show that $C\circ f=g\circ C$. For $x\in E$, if $0\le x\le 1/3$ then $\alpha_1=0$ and $\beta_1=0$. Hence we have

$$C\circ f(x)=C(3x)=C(\alpha_2 3^{-1}+\alpha_3 3^{-2}+\cdots)=\beta_2 2^{-1}+\beta_3 2^{-2}+\cdots,$$

$$g\circ C(x)=2C(x)=2(\beta_2 2^{-2}+\beta_3 2^{-3}+\cdots)=\beta_2 2^{-1}+\beta_3 2^{-2}+\cdots.$$

If $2/3\le x\le 1$ then $\alpha_1=2$ and $\beta_1=1$. We define $\bar{\alpha}_i=2-\alpha_i$, and $\bar{\beta}_i=1-\beta_i$. We have

$$C\circ f(x)=C(3(1-x))=C(\bar{\alpha}_2 3^{-1}+\bar{\alpha}_3 3^{-2}+\cdots)=\bar{\beta}_2 2^{-1}+\bar{\beta}_3 2^{-2}+\cdots,$$

$$g\circ C(x)=2(1-C(x))=2(\bar{\beta}_2 2^{-2}+\bar{\beta}_3 2^{-3}+\cdots)=\bar{\beta}_2 2^{-1}+\bar{\beta}_3 2^{-2}+\cdots.$$

If $x\in I\backslash E$ then there exist $u, v\in E$, $u<x<v$ such that $C(u)=C(x)=C(v)$. Hence we have

$$C\circ f(u)=g\circ C(u)=g\circ C(x),$$

$$C\circ f(v)=g\circ C(v)=g\circ C(x).$$

Since f is monotone on $[u, v]$, $C\circ f$ is monotone. Hence we have $C\circ f(x)=g\circ C(x)$. $\square$

Since we can apply Theorem 2.2 to the previous example, it follows that the Cantor function C is Hölder continuous with exponent $\log 2/\log 3$. In fact, from (2.1) we have $K_i=2$ for $i=1, 2$. Since it follows that $|f(x)-f(y)|\le 3|x-y|$ for $x, y\in[0, 1/3]$ or $x, y\in[1/3, 1]$, by (2.2) we have $2^{1/\sigma}=3$. Hence $\sigma=\log 2/\log 3$.

4. Proofs of main results.

4.1. Proof of Theorem 2.1. We consider a complete metric space

$$M=\left\{\alpha: I\to I \,\middle|\, \alpha \text{ non-decreasing, } \alpha=h \text{ on } \bigcup_{i=0}^{l+1}\bigcup_{n\ge 0} f^n(J_i)\right\}$$

with a metric $\|\alpha-\beta\|=\sup_{x\in I}|\alpha(x)-\beta(x)|$. Note that $M\neq\emptyset$.

Let $I_i=(b_i, a_{i+1})$, $\tilde{I}_i=(\tilde{b}_i, \tilde{a}_{i+1})$, and $g_i=g|\mathrm{Cl}\,\tilde{I}_i$, $i=0, 1, \cdots, l$. We define an operator $\mathscr{T}$ on M by

$$\mathscr{T}(\alpha)(x)=\begin{cases} g_i^{-1}(\alpha(f(x))) & \text{if } x\in I_i,\ i=0, 1, \cdots, l \\ h(x) & \text{if } x\in\bigcup_{i=0}^{l+1} J_i. \end{cases} \tag{4.1}$$

It is easy to check that $\mathscr{T}$ is well-defined.

LEMMA 4.1. $\mathscr{T}M\subset M$.

PROOF. We first show that $\mathscr{T}(\alpha)=h$ on $\bigcup_{i=0}^{l+1}\bigcup_{n\ge 0} f^n(J_i)$. For $n=0$ it follows that $\mathcal{T}(\alpha)=h$ on $\bigcup_{i=0}^{l+1} J_i$ by the definition of $\mathcal{T}$. Let $n>0$. If $f^n(J_i)\in J_j$ then $\mathcal{T}(\alpha)(f^n(J_i))=h(f^n(J_i))$. If $f^n(J_i)\in I_j$ then we have $g^n(\tilde{J}_i)\in \tilde{I}_j$ since $h(f^n(J_i))\in h(I_j)$. Hence we have

$$\mathcal{T}(\alpha)(f^n(J_i))=g_j^{-1}(\alpha(f^{n+1}(J_i)))=g_j^{-1}(g_j(g^n(\tilde{J}_i)))=h(f^n(J_i)) .$$

Therefore $\mathcal{T}(\alpha)=h$ on $\bigcup_{i=0}^{l+1}\bigcup_{n\geq 0} f^n(J_i)$.

To show that $\mathcal{T}(\alpha)$ is non-decreasing, suppose $x<y$. If $x, y\in J_i$ then $h(x)<h(y)$. Hence $\mathcal{T}(\alpha)(x)<\mathcal{T}(\alpha)(y)$. Let $x, y\in I_i$. If f is increasing on I_i then g is increasing on $\tilde{I}_i$. Hence, it follows that g_i^{-1} is also increasing, so that we have $\mathcal{T}(\alpha)(x)\leq\mathcal{T}(\alpha)(y)$. Similarly, if f is decreasing on I_i then it follows that $\mathcal{T}(\alpha)(x)\leq\mathcal{T}(\alpha)(y)$. Since $\mathcal{T}(\alpha)(I_j)\subset \mathrm{Cl}(\tilde{I}_j)$ and $\mathcal{T}(\alpha)(J_j)\subset\tilde{J}_j$, if $x\in I_i$, $y\in J_j$ for $i<j$ or $x\in J_j$, $y\in I_i$ for $j\leq i$ then we have $\mathcal{T}(\alpha)(x)\leq \mathcal{T}(\alpha)(y)$. □

By Lemma 4.1, we have $M\supset\mathcal{T}M\supset\mathcal{T}^2M\supset\cdots$. We denote the boundary of X by ∂X.

LEMMA 4.2. *Let $\alpha\in M$. The restriction of the map $\mathcal{T}^n(\alpha)$*

$$\mathcal{T}^n(\alpha): \bigcup_{i=0}^{l+1}\bigcup_{k\geq -n} f^k(J_i)\to\bigcup_{i=0}^{l+1}\bigcup_{k\geq -n} g^k(\tilde{J}_i)$$

is an order preserving homeomorphism with

$$\mathcal{T}^n(\alpha)(f^k(J_i))=g^k(\tilde{J}_i)\,,\quad k\geq -n\,,\qquad i=0, 1, \cdots, l+1 .$$

PROOF. We use induction on n. If $n=0$, this is straightforward from the definition of $\mathcal{T}(\alpha)$. So we may assume that the statement holds for $n-1$.

We note that the restriction of $\mathcal{T}^n(\alpha)$ is an order preserving homeomorphism with $\mathcal{T}^n(\alpha)(f^k(J_i))=g^k(\tilde{J}_i)$ for $k\geq -n+1$ since $\mathcal{T}^nM\subset\mathcal{T}^{n-1}M$. We first show that $\mathcal{T}^n(\alpha)(f^{-n}(J_i))\subset g^{-n}(\tilde{J}_i)$ for all i. Let $x\in f^{-n}(J_i)$. Suppose $x\in J_j$ for some j. We claim that if $f^{-n}(J_i)\cap J_j\neq\varnothing$ then $f^{-n}(J_i)\supset J_j$. In fact, assume that there exists $x\in f^{-n}(J_i)\cap J_j$. Since f is constant on J_j, we have $f^n(x)=f^n(y)$ for any $y\in J_j$. From this and $f^n(x)\in J_i$, we have $f^n(J_j)\in J_i$. Therefore $J_j\subset f^{-n}(J_i)$.

Since $f^n(J_j)\in J_i$, we have

$$g^n(\tilde{J}_j)=h(f^n(J_j))\in h(J_i)=\tilde{J}_i .$$

This shows that $\tilde{J}_j\subset g^{-n}(\tilde{J}_i)$. Therefore $\mathcal{T}^n(\alpha)(x)=h(x)\in\tilde{J}_j\subset g^{-n}(\tilde{J}_j)$. If $x\in I_j$ for some j then we have

$$\mathcal{T}^n(\alpha)(x)=g_j^{-1}(\mathcal{T}^{n-1}(\alpha)(f(x)))\in g_j^{-1}(\mathcal{T}^{n-1}(\alpha)(f^{-n+1}(J_i)))=g^{-n}(\tilde{J}_i)$$

by induction.

We show that the restriction of $\mathcal{T}^n(\alpha)$ is onto. Since $\mathcal{T}^n(\alpha)\in\mathcal{T}^{n-1}M$, for $y\in \bigcup_{i=0}^{l+1}\bigcup_{k\geq -n+1} g^k(\tilde{J}_i)$ there exists $x\in\bigcup_{i=0}^{l+1}\bigcup_{k\geq -n+1} f^k(J_i)$ sucht that $\mathcal{T}^n(\alpha)(x)=y$. So let $y\in\bigcup_{i=0}^{l+1} g^{-n}(\tilde{J}_i)$. If $y\in\tilde{J}_j$, then it is clear. Assume that $y\in\tilde{I}_i$, and $g^n(y)\in\tilde{J}_j$. By induction and $g(y)\in g^{-n+1}(\tilde{J}_j)$, there exists $\xi\in f^{-n+1}(J_j)$ such that $T^{n-1}(\alpha)(\xi)=g(y)$. Hence there exists $x\in I_i$, such that $f(x)=\xi$. It follows that $x\in f^{-1}(\xi)\subset f^{-n}(J_j)$, so that we have

$$\mathscr{T}^n(\alpha)(x)=g_i^{-1}(\mathscr{T}^{n-1}(\alpha)(f(x)))=g_i^{-1}(\mathscr{T}^{n-1}(\alpha)(\xi))=y\,.$$

We now prove that the restriction of $\mathscr{T}^n(\alpha)$ is one-to-one. Suppose that $\mathscr{T}^n(\alpha)(x)=\mathscr{T}^n(\alpha)(y)$ for any $x, y\in\bigcup_{i=0}^{l+1}\bigcup_{k\geq -n} f^k(J_i)$, $x<y$. If $x\in I_i$, $y\in I_j$ then we have that $i=j$ and $\mathscr{T}^{n-1}(\alpha)(f(x))=\mathscr{T}^{n-1}(\alpha)(f(y))$ since $g_i^{-1}(\mathscr{T}^{n-1}(\alpha)(f(x)))=g_j^{-1}(\mathscr{T}^{n-1}(\alpha)(f(y)))$. By induction, it follows that $f(x)=f(y)$. Hence $x=y$. If $x\in I_i$, $y\in J_j$ then we must have $j=i+1$ and $y=\min J_{i+1}$ since $g_i^{-1}(\mathscr{T}^{n-1}(\alpha)(f(x)))=h(y)$ and $x<y$. Hence $h(y)=\min\tilde{J}_{i+1}$, so that

$$\mathscr{T}^{n-1}(\alpha)(f(x))=g_i(\min\tilde{J}_{i+1})=g(\tilde{J}_{i+1})\,.$$

By induction, it follows that $f(x)=f(J_{i+1})$. Hence $x\in J_{i+1}$. This contradicts our assumption.

If $x\in J_i$, $y\in I_j$ then the proof is similar. If $x, y\in\bigcup_{i=0}^{l+1} J_i$ then $h(x)=h(y)$. Hence $x=y$.

$\mathscr{T}^n(\alpha)$ is non-decreasing on I. This implies that the restriction of $\mathscr{T}^n(\alpha)$ is order preserving.

Finally, we prove that $\mathscr{T}^n(\alpha)$ is continuous on $\bigcup_{i=0}^{l+1}\bigcup_{k\geq -n} f^k(J_i)$. By induction, it follows that $\mathscr{T}^n(\alpha)=g_i^{-1}(\mathscr{T}^{n-1}(\alpha)(f(x)))$ is continuous on $I_i\cap(\bigcup_{i=0}^{l+1} f^{-n}(J_i))$ for $0\leq i\leq l$. $\mathscr{T}^n(\alpha)$ is also continuous on all of J_i. We show that $\mathscr{T}^n(\alpha)$ is continuous on ∂J_i. Let $c=\min J_i$. We have

$$\lim_{x\downarrow c}\mathscr{T}^n(\alpha)(x)=h(c)\,,$$

$$\begin{aligned}\lim_{x\uparrow c}\mathscr{T}^n(\alpha)(x)&=\lim_{x\uparrow c} g_{i-1}^{-1}(\mathscr{T}^{n-1}(\alpha)(f(x)))\\ &=g_{i-1}^{-1}(\mathscr{T}^{n-1}(\alpha)(f(c)))=g_{i-1}^{-1}(g(\tilde{J}_i))=h(c)\,.\end{aligned}$$

If $c=\max J_i$ then the proof is similar. □

Let Lemma 4.3 be ready to prove Lemma 4.4.

LEMMA 4.3. *For any* $n\geq 0$ *and any* $x\in\bigcup_{i=0}^{l+1}\bigcup_{k\geq -n} f^k(J_i)$, *we have*

$$\mathscr{T}^{n+1}(\alpha)(x)=\mathscr{T}^n(\alpha)(x)\,.$$

PROOF. We use induction on n. Suppose $n=0$, then $x\in\bigcup_{i=0}^{l+1}\bigcup_{k\geq 0} f^k(J_i)$. If $x\in J_i$, we have $T(\alpha)(x)=h(x)=\alpha(x)$. Now we may write $x=f^k(J_j)$. If $x\in I_i$ then we have

$$\mathscr{T}(\alpha)(x)=g_i^{-1}(\alpha(f^{k+1}(J_j)))=g_i^{-1}(g^{k+1}(\tilde{J}_j))=g^k(\tilde{J}_j)=\alpha(f^k(J_j))=\alpha(x)\,.$$

So we assume that the result is true for $n-1$ and prove it for n.

If $x\in J_i$, we have $\mathscr{T}^{n+1}(\alpha)(x)=\mathscr{T}^n(\alpha)(x)=h(x)$. If $x\in I_i$, we have

$$\mathscr{T}^{n+1}(\alpha)(x)=g_i^{-1}(\mathscr{T}^n(\alpha)(f(x)))=g_i^{-1}(\mathscr{T}^{n-1}(\alpha)(f(x)))=\mathscr{T}^n(\alpha)(x)\,.$$ □

Using Lemma 4.2 and Lemma 4.3, we will show that $\mathscr{T}^n(\alpha)$ converges in M.

LEMMA 4.4. *For any $\alpha \in M$, there exists $\varphi \in M$ such that*

$$\lim_{n\to\infty} \|\mathscr{T}^n(\alpha)-\varphi\|=0 .$$

PROOF. Let $\varepsilon>0$ and $B_\varepsilon(x)=(x-\varepsilon, x+\varepsilon)$. Since $\bigcup_{i=0}^{l+1}\bigcup_{n\in\mathbf{Z}} g^n(\tilde{J}_i)$ is dense in I, for any $\varepsilon>0$ there exists a number $N>0$ such that $\{B_\varepsilon(x) \mid x\in\bigcup_{i=0}^{l}\bigcup_{k\geq -N} g^k(\tilde{J}_i)\}$ is covering of I. By Lemma 4.2, for any $x\in I$ for $m, n\geq N$ there exists $y\in\bigcup_{i=0}^{l}\bigcup_{k\geq -N} f^k(J_i)$ such that $\mathscr{T}^n(\alpha)(x)\in B_\varepsilon(\mathscr{T}^m(\alpha)(y))$ and $\mathscr{T}^n(\alpha)(x)\in B_\varepsilon(\mathscr{T}^n(\alpha)(y))$. By Lemma 4.3, we have

$$\begin{aligned}|\mathscr{T}^m(\alpha)(x)-\mathscr{T}^n(\alpha)(x)|&\leq|\mathscr{T}^m(\alpha)(x)-\mathscr{T}^m(\alpha)(y)|+|\mathscr{T}^m(\alpha)(y)-\mathscr{T}^n(\alpha)(y)|\\&\quad+|\mathscr{T}^n(\alpha)(y)-\mathscr{T}^n(\alpha)(x)|\leq 2\varepsilon .\end{aligned}$$

Hence,

$$\|\mathscr{T}^m(\alpha)-\mathscr{T}^n(\alpha)\|=\sup_{x\in I}|\mathscr{T}^m(\alpha)(x)-\mathscr{T}^n(\alpha)(x)|\leq 2\varepsilon .$$

Thus, it follows that $\{\mathscr{T}^n(\alpha)\}_{n=0}^{\infty}$ is a Cauchy sequence in M. This shows that $\{\mathscr{T}^n(\alpha)\}_{n=0}^{\infty}$ has a limit point φ in M. □

By Lemma 4.4, it follows that φ is an order preserving homeomorphism from $\bigcup_{i=0}^{l+1}\bigcup_{n\in\mathbf{Z}} f^n(J_i)$ to $\bigcup_{i=0}^{l+1}\bigcup_{n\in\mathbf{Z}} g^n(\tilde{J}_i)$. Hence $\varphi: I\to I$ is a homeomorphism with $\mathscr{T}\varphi=\varphi$. This completes the proof of Theorem 2.1.

4.2. Proof of Theorem 2.2. The complete metric space M and the operator $\mathscr{T}$ are those defined in the proof of Theorem 2.1. We consider

$$H_A=\left\{\alpha\in M \;\middle|\; \sup_{0\leq i\leq l}\ \sup_{\substack{x,y\in \mathrm{Cl}\, I_i\\ x\neq y}} \frac{|\alpha(x)-\alpha(y)|}{|x-y|^\sigma}\leq A\right\}, \qquad A>0 ,$$

which is a subset of M.

Since both $\bigcup_{i=0}^{l}\bigcup_{n>0} f^n(J_i)$ and $\bigcup_{i=0}^{l}\bigcup_{n>0} g^n(\tilde{J}_i)$ are finite sets, we can choose a number A sufficiently large and a number σ sufficiently small such that $H_A\neq\varnothing$. It is clear that H_A is a closed set in M. Hence H_A is complete. Let $\alpha\in H_A$ and $x, y\in \mathrm{Cl}\, I_i$. Using (2.1) and (2.2), we have

$$\begin{aligned}|\mathscr{T}(\alpha)(x)-\mathscr{T}(\alpha)(y)|&=|g_i^{-1}(\alpha(f(x)))-g_i^{-1}(\alpha(f(y)))|\\&\leq K_i^{-1}|\alpha(f(x))-\alpha(f(y))|\leq AK_i^{-1}|f(x)-f(y)|^\sigma\leq A|x-y|^\sigma .\end{aligned}$$

Thus, we have $\mathscr{T}H_A\subset H_A$. Therefore φ is Hölder continuous on each of closed intervals $\mathrm{Cl}\, I_i$ and J_i. This shows that φ in the proof of Theorem 2.1 is Hölder continuous on I.

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