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On an Inverse Problem for Quasilinear Parabolic Equations

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1. Introduction.

Let us consider the initial boundary value problem:

$$u_t - a(x, u)u_{xx} = 0 \quad \text{in} \quad Q_T, \tag{1.1}$$

$$u(x, 0) = 0 \quad \text{for} \quad 0 < x < 1, \tag{1.2}$$

$$u(0, t) = f(t) \quad \text{and} \quad u(1, t) = 0 \quad \text{for} \quad 0 < t < T, \tag{1.3}$$

where $Q_T \equiv \{(x, t) : 0 < x < 1, 0 < t < T\}$.

Assume that the following conditions for a and f are satisfied:

(i) for any finite $M > 0$, $a(x, z) \in C^1([0, 1] \times [-M, M])$,
(ii) for fixed, positive constants ν and μ, $0 < \nu \leq a(x, z) \leq \mu$ on $[0, 1] \times [-M, M]$,
(iii) for fixed, positive constant C, $|\partial_x a(x, z)| + |\partial_z a(x, z)| \leq C$ on $[0, 1] \times [-M, M]$,
(iv) $f \in H^{1+\beta/2}([0, T])$ $(0 < \beta < 1)$ with $f'(t) > 0$ for $0 < t < T$,
(v) $f(0) = 0 = f'(0)$.

From the conditions (i)–(v) applying Theorem 5.2 and Remark 5.1 in [5], we see that there exists a unique solution $u(x, t) \in H^{2+\beta, 1+\beta}([0, 1] \times [0, T])$ to the initial boundary value problem (1.1)–(1.3). So we may define D-N map as follows:

$$\Lambda(a, f) : u(0, t) = f(t) \mapsto u_x(0, t) \quad \text{on} \quad [0, T].$$

We are interested in uniqueness results for $a(x, u)$ of the equation (1.1) from $\Lambda(a, f)$. Isakov [4] proved the uniqueness for $a(x, u)$ in the case that the spatial dimension is greater than or equal to 2 by using the completeness of products of solutions for linear parabolic equations. But in the case that the spatial dimension is one, the completeness of products of solutions has not been proved yet. So we need another method for proving the uniqueness for $a(x, u)$. In [1] it was shown that the coefficient κ of the equation $a(u)u_t = \kappa(a(u)u_x)_x$ was uniquely determined from overspecified boundary data

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by transforming the original equation to the linear one $v_t = \kappa v_{xx}$. In [3] DuChateau studied the monotonicity and the uniqueness of the coefficient $a(u)$ of the equation $u_t - (a(u)u_x)_x = 0$ by using the methods in Muzylev [6]. In this paper we will prove the uniqueness for $a(x, u)$ in an admissible class A by modifying and extending the methods in [6] for the equation $u_t - a(x, u)u_{xx} = 0$.

DEFINITION. Two functions $a(x, z)$ and $b(x, z)$ satisfying the conditions (i)–(iii) will be said to belong to the admissible class $A[0, 1] \times [0, d]$ provided that, if $a(x, 0) = b(x, 0)$ on $[0, 1]$, then there exists $d > 0$ such that $a(x, z) = b(x, z)$ on $[0, 1] \times [0, d]$.

Our theorem is as follows:

THEOREM. *Let u^j be a solution to the problem* (1.1)–(1.3) *with* $a = a^j \in A[0, 1] \times [0, f(T)]$, $j = 1, 2$. *If* $A(a^1, f) = A(a^2, f)$, *then* $a^1 = a^2$ *on* $[0, 1] \times [0, f(T)]$.

We will prove our theorem by using an integral identity (Lemma 2.1). This paper is organized as follows. Some lemmas are proved in Section 2. Section 3 is devoted to the proof of our theorem.

2. Lemmas.

From (ii) in Introduction, we may define

$$c(x, u) = \int_0^u \frac{1}{a(x, z)} dz .$$

Then the solution $u(x, t)$ to the original problem (1.1)–(1.3) satisfies the equation $(c(x, u))_t - u_{xx} = 0$. It is easily seen that for any $\phi(x, t) \in C^{2,1}(\bar{Q}_T)$, we have

$$\begin{aligned} 0 &= \int_{\bar{Q}_T} \{(c(x, u))_t - u_{xx}\}\phi dxdt \\ &= -\int_{\bar{Q}_T} \{c(x, u)\phi_t + u\phi_{xx}\}dxdt + \int_0^1 [c(x, u)\phi]_0^T dx + \int_0^T [u\phi_x - u_x\phi]_0^1 dt . \end{aligned}$$

This implies

$$\begin{aligned} &\int_{\bar{Q}_T} \{c(x, u)\phi_t + u\phi_{xx}\}dxdt \\ &\quad = \int_0^1 \{c(x, u(x, T))\phi(x, T) - c(x, u(x, 0))\phi(x, 0)\}dx \\ &\quad + \int_0^T \{u(1, t)\phi_x(1, t) - u_x(1, t)\phi(1, t) - u(0, t)\phi_x(0, t) + u_x(0, t)\phi(0, t)\}dt . \end{aligned} \tag{2.1}$$

Using (2.1), we prove the following Lemma 2.1.

LEMMA 2.1. *Let u^j be a solution to the mixed problem* (1.1)–(1.3) *with* $a=a^j$, $j=1,2$. *If* $\Lambda(a^1,f)=\Lambda(a^2,f)$, *then we have*

$$I_T \equiv \int_{\bar{Q}_T} \left(\frac{1}{a^2(x,u^2)} - \frac{1}{a^1(x,u^2)} \right) u_t^2 \phi \, dx dt = 0 , \tag{2.2}$$

where $\phi(x,t)$ *is a solution to the following mixed problem*:

$$\phi_t + p(x,t)\phi_{xx} = 0 \quad \text{in} \quad Q_T , \tag{2.3}$$

$$\phi(x,T) = 0 \quad \text{for} \quad 0<x<1 , \tag{2.4}$$

$$\phi(1,t) = 0 \quad \text{for} \quad 0<t<T , \tag{2.5}$$

$$\phi(0,t) = \chi(t) \quad \text{for} \quad 0<t<T , \tag{2.6}$$

here $p(x,t)>0$ *defined by*

$$p(x,t) = \begin{cases} \dfrac{u^1 - u^2}{c^1(x,u^1) - c^1(x,u^2)} & \text{for} \quad u^1(x,t) \neq u^2(x,t) , \\ \dfrac{1}{\partial_u c^1(x,u^1)} & \text{for} \quad u^1(x,t) = u^2(x,t) , \end{cases}$$

and $\chi(t)$ *is an infinitely differentiable function satisfying* $\chi(t)>0$ *for* $0<t<T$ *and* $\chi(T)=0$.

PROOF. If $p(x,t)$ is Lipschitz with respect to x and t, then it is known that there exists a unique classical solution for the mixed problem (2.3)–(2.6). First we prove $p(x,t)$ is Lipschietz with respect to x. From the definition of c, if $u^1(x,t)=u^2(x,t)$, then we see that $p(x,t)=a^1(x,u^1(x,t))$. Hence we obtain

$$|p_x(x,t)| \le \max_{x,z} |a_x^1(x,z)| + \max_{x,z} |a_z^1(x,z)| \cdot \max_{\bar{Q}_T} |u_x^1| .$$

Therefore, using (ii) in Introduction and $u^1 \in H^{2+\beta,1+\beta}(\bar{Q}_T)$, there exists a positive constant $C_1>0$ such that

$$|p_x(x,t)| \le C_1 \qquad \text{if} \quad u^1(x,t) = u^2(x,t) .$$

If $u^1(x,t) \neq u^2(x,t)$, by the mean value theorem, we have

$$\begin{aligned} |p(x,t)-p(y,t)| &= \left| \frac{u^1(x,t)-u^2(x,t)}{\int_{u^2(x,t)}^{u^1(x,t)} dz/a^1(x,z)} - \frac{u^1(y,t)-u^2(y,t)}{\int_{u^2(y,t)}^{u^1(y,t)} dz/a^1(y,z)} \right| = |a^1(x,\xi_1) - a^1(y,\xi_2)| \\ &\le |a^1(x,\xi_1) - a^1(y,\xi_1)| + |a^1(y,\xi_1) - a^1(y,\xi_2)| \\ &\le \max_{x,z} |a_x^1(x,z)| \cdot |x-y| + \max_{x,z} |a_z^1(x,z)| \cdot |\xi_1 - \xi_2| , \end{aligned}$$

where

$$\xi_1=\theta u^1(x,t)+(1-\theta)u^2(x,t)\,,\qquad 0<\theta<1\,,$$
$$\xi_2=\eta u^1(y,t)+(1-\eta)u^2(y,t)\,,\qquad 0<\eta<1\,.$$

We define an interval such as

$$M(z)\equiv(\min\{u^1(z,t),u^2(z,t)\},\quad \max\{u^1(z,t),u^2(z,t)\})\,.$$

If $M(x)\cap M(y)=\varnothing$, then $p(x,t)=a^1(x,\xi_1)$ is differentiable with respect to x and we obtain

$$p_x(x,t)=a^1_x(x,\xi_1)+a^1_z(x,\xi_1)(\theta u^1_z(x,t)+(1-\theta)u^2_x(x,t))\,.$$

Hence there exists a positive constant $C_2>0$ such that

$$|p_x(x,t)|\le\max_{x,z}|a^1_x(x,z)|+\max_{x,z}|a^1_z(x,z)|\left(\max_{\bar{Q}_T}|u^1_x|+\max_{\bar{Q}_T}|u^2_x|\right)\le C_2\,.$$

If $M(x)\cap M(y)\neq\varnothing$, then there exists $z\in(x,y)$ such that $u^1(z,t)=u^2(z,t)$. By the mean value theorem, there exist γ_j, $0<\gamma_j<1$ $(j=1,2,3,4)$ such that

$$u^1(x,t)=u^1(z,t)+u^1_x(\gamma_1,t)(x-z)\,,$$
$$u^2(x,t)=u^2(z,t)+u^2_x(\gamma_2,t)(x-z)\,,$$
$$u^1(y,t)=u^1(z,t)+u^1_x(\gamma_3,t)(y-z)\,,$$
$$u^2(y,t)=u^2(z,t)+u^2_x(\gamma_4,t)(y-z)\,.$$

Thus there exists a positive constant $C_4>0$ such that

$$\begin{aligned}|\xi_1-\xi_2|=|&\theta u^1(z,t)+\theta u^1_x(\gamma_1,t)(x-z)+(1-\theta)u^2(z,t)+(1-\theta)u^2_x(\gamma_2,t)(x-z)\\&-\eta u^1(z,t)-\eta u^1_x(\gamma_3,t)(y-z)-(1-\eta)u^2(z,t)-(1-\eta)u^2_x(\gamma_4,t)(y-z)|\\ \le{}& C_4|x-y|\,,\end{aligned}$$

here we have used $u^1(z,t)=u^2(z,t)$. Therefore we get

$$|p(x,t)-p(y,t)|\le(C+CC_4)|x-y|\,.$$

The proof is similar in the case of variable t. Hence $p(x,t)$ is Lipschitz with respect to x and t.

Now we are going to derive (2.2) in Lemma 2.1. From (2.1), (2.4), (2.5), (2.6) and noting that $c^j(x,u^j(x,0))=c^j(x,0)$, $j=1,2$, we have

$$\int_{\bar{Q}_T}\{c^j(x,u^j)\phi_t+u^j\phi_{xx}\}dxdt=\int_0^T\{\chi(t)u^j_x(0,t)-f(t)\phi_x(0,t)\}dt\,,\qquad j=1,2\,.$$

This and the assumption $u^1_x(0,t)=u^2_x(0,t)$ implies

$$\int_{\bar{Q}_T}\{(c^1(x,u^1)-c^2(x,u^2))\phi_t+(u^1-u^2)\phi_{xx}\}dxdt$$

$$=\int_0^T \chi(t)(u_x^1(0,t)-u_x^2(0,t))dt=0\,.$$

Combining this equality with (2.3), (2.4), and $c^1(x,0)=c^2(x,0)=0$, we obtain

$$\begin{aligned}
0&=\int_{\bar{Q}_T}\{(c^1(x,u^1)-c^2(x,u^2))\phi_t+(u^1-u^2)\phi_{xx}\}dxdt\\
&=\int_{\bar{Q}_T}\{(c^1(x,u^1)-c^1(x,u^2))\phi_t+(u^1-u^2)\phi_{xx}\}dxdt\\
&\quad+\int_{\bar{Q}_T}\{c^1(x,u^2)-c^2(x,u^2)\}\phi_t dxdt\\
&=\int_{\bar{Q}_T}\{c^1(x,u^2)-c^2(x,u^2)\}\phi_t dxdt\\
&=\int_0^1[(c^1(x,u^2)-c^2(x,u^2))\phi]_0^T dt-\int_{\bar{Q}_T}\left(\frac{1}{a^2(x,u^2)}-\frac{1}{a^1(x,u^2)}\right)u_t^2\phi dxdt\\
&=\int_{\bar{Q}_T}\left(\frac{1}{a^2(x,u^2)}-\frac{1}{a^1(x,u^2)}\right)u_t^2\phi dxdt\,.
\end{aligned}$$

Therefore we get the desired equality (2.2). The proof is complete.

To prove our theorem, we need some lemmas related to the positivity of u_t and ϕ.

LEMMA 2.2. *Let u be a solution to the problem* (1.1)–(1.3) *with* (i)–(v). *Then the following inequality holds*:

$$0\le z(x,t)\le u(x,t)\le v(x,t)\le f(t)\qquad in\quad \bar{Q}_T\,,$$

where z and v are unique solutions to the following problems:

$$\begin{cases} v_t-\mu v_{xx}=0 & in\quad Q_T\,,\\ v(x,0)=0 & for\quad 0<x<1\,,\\ v(0,t)=f(t) & for\quad 0<t<T\,,\\ v(1,t)=0 & for\quad 0<t<T\,,\end{cases}$$

$$\begin{cases} z_t-\nu z_{xx}=0 & in\quad Q_T\,,\\ z(x,0)=0 & for\quad 0<x<1\,,\\ z(0,t)=f(t) & for\quad 0<t<T\,,\\ z(1,t)=0 & for\quad 0<t<T\,.\end{cases}$$

PROOF. From a maximum principle, we have

$$0\le v(x,t)\le f(t)\quad\text{and}\quad 0\le z(x,t)\le f(t)\,.$$

$v-u$ satisfies the following mixed problem:

$$\begin{cases} (v-u)_t - a(x, u)(v-u)_{xx} = (\mu - a(x, u))v_{xx} & \text{in } Q_T, \\ (v-u)(x, 0) = 0 & \text{for } 0<x<1, \\ (v-u)(0, t) = 0 & \text{for } 0<t<T, \\ (v-u)(1, t) = 0 & \text{for } 0<t<T. \end{cases}$$

Noting that $\mu \geq a$, if $v_{xx} \geq 0$ in $\bar{Q}_T$, then a maximum principle yields $v-u \geq 0$ in $\bar{Q}_T$. So we are going to prove $v_{xx} \geq 0$ in $\bar{Q}_T$. Set $w = v_t$. Then w satisfies the following:

$$\begin{cases} w_t - \mu w_{xx} = 0 & \text{in } Q_T, \\ w(x, 0) \geq 0 & \text{for } 0<x<1, \\ w(0, t) = f'(t) > 0 & \text{for } 0<t<T, \\ w(1, t) = 0 & \text{for } 0<r<T. \end{cases}$$

A maximum principle implies that $w = v_t \geq 0$ in $\bar{Q}_T$. Combining this with $\mu > 0$ and $v_t = \mu v_{xx}$, we have $v_{xx} \geq 0$ in $\bar{Q}_T$. By the same way, we get $u - z \geq 0$ in $\bar{Q}_T$.

LEMMA 2.3. *Let u be a solution to the problem* (1.1)–(1.3) *with* (i)–(v), *then it holds that*

$$u_t(x, t) > 0 \quad \textit{in} \quad Q_T.$$

PROOF. Set $w = u_t$, then w satisfies the following:

$$\begin{cases} w_t - a_z(x, u)w_{xx} - a(x, u)u_{xx}w = 0 & \text{in } Q_T, \\ w(x, 0) \geq 0, & \text{for } 0<x<1, \\ w(0, t) = f'(t) > 0 & \text{for } 0<t<T, \\ w(1, t) = 0 & \text{for } 0<t<T. \end{cases}$$

From Lemma 2.2 and $u \in H^{2+\beta, 1+\beta}(\bar{Q}_T)$, $a(x, u)u_{xx}$ is bounded. Hence a maximum principle yields $u_t(x, t) > 0$ in Q_T.

3. Proof of Theorem.

A maximum principle implies that $\phi > 0$ in Q_T, here ϕ is a solution to the mixed problem (2.3)–(2.6). Combining this with Lemma 2.3, we have $u_t^2 \phi > 0$ in Q_T. If $a^1(x, 0) \neq a^2(x, 0)$ on $[0, 1]$, then there exist ε_0, $\varepsilon_1 > 0$ such that

$$\frac{1}{a^2(x, u^2)} - \frac{1}{a^1(x, u^2)} > \varepsilon_0 \quad \text{or} \quad \frac{1}{a^1(x, u^2)} - \frac{1}{a^2(x, u^2)} > \varepsilon_0$$

for $0 < x < 1$ and $0 \leq u^2 \leq \varepsilon_1$. Taking $f(t)$ such that $\max_{\bar{Q}_T} u = f(T) = \varepsilon_1$, then by Lemma 2.1 and $u_t^2 \phi > 0$ in Q_T, we obtain $I_T > 0$. This contradicts (2.2). Since a^1 and a^2 belong to the admissible class stated in Introduction, then we get $a^1(x, z) = a^2(x, z)$ on $[0, 1] \times [0, f(T)]$.

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