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On Deformations of Einstein-Weyl Structures

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1. Introduction.

Let M be an n-dimensional manifold with a conformal class C. A conformal connection on M is an affine connection D preserving the conformal class C. We also assume D is torsion-free. The triple (M, C, D) is called a *Weyl manifold* or (C, D) is called a *Weyl structure* on M. A Weyl manifold admits an *Einstein-Weyl structure* if the symmetric part of the Ricci curvature of the conformal connection is proportional to a conformal metric which belongs to C. The Einstein-Weyl equations on the metric and affine connection are conformally invariant nonlinear partial differential equations. If (M, g) is an Einstein manifold, then this conformal class C and the Levi-Civita connection defines an Einstein-Weyl structure. So the notion of the Einstein-Weyl manifolds is a generalization of an Einstein metric to conformal structures.

In this paper we consider infinitesimal deformations of an Einstein metric as an Einstein-Weyl structure, and we prove any such deformation comes from conformal Killing vector fields provided certain conditions of curvatures are satisfied.

2. Preliminaries.

Let (M, C, D) be a Weyl manifold. We assume $n = \dim M \geq 3$. This implies the existence of a 1-form ω_g such that $Dg = \omega_g \otimes g$. Let Ric^D denote the Ricci curvature of D. In general, Ricci curvature of conformal connection is not symmetric, so we denote by $\mathrm{Sym}(\mathrm{Ric}^D)$ its symmetric part. The scalar curvature R_g^D of D with respect to $g \in C$ is defined by

$$R_g^D = \mathrm{tr}_g \mathrm{Ric}^D . \tag{1}$$

A Weyl manifold (M, C, D) is said to be *Einstein-Weyl manifold* if the symmetric part of the Ricci curvature Ric^D is proportional to the metric g in C. So the

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Einstein-Weyl equations are

$$\mathrm{Sym}(\mathrm{Ric}^D)=\frac{R_g^D}{n}g\,. \tag{2}$$

Note that $R_g^D g$ is conformally invariant quantity. In terms of the Ricci curvature and the scalar curvature of the metric $g\in C$, the Einstein-Weyl equations can be written by

$$\mathrm{Ric}_g+\frac{n-2}{4}(\mathscr{L}_{\omega_g^\sharp}g+\omega_g\otimes\omega_g)=\frac{1}{n}\left\{R_g+\frac{n-2}{4}|\omega_g|^2-\frac{n-2}{2}\delta_g\omega_g\right\}g \tag{3}$$

where $\mathscr{L}$ is the Lie derivative, δ_g is the codifferential of g, and the vector field $\omega_g^\sharp$ is defined as $\omega_g(X)=g(X,\omega_g^\sharp)$ for all vector fields X.

LEMMA 1. *Let (M,g) be an Einstein manifold, then the conformal class $[g]$ of g, and the Levi-Civita connection ∇_g of g defines an Einstein-Weyl structure on M.*

PROOF. Obvious from the definition. □

3. Deformations of Einstein-Weyl structures.

In this section, we consider deformations of Einstein-Weyl structures at Einstein metrics. Let (M,g) be an Einstein n-manifold. Consider a 1-parameter family of Riemannian metrics g_t with $g_0=g$, and 1-forms ω_t with $\omega_0=0$. These define Weyl structures (C_t, D_t) on M by $D_t g_t=\omega_t\otimes g_t$. Set $h:=dg_t/dt|_{t=0}$, and $\alpha:=d\omega_t/dt|_{t=0}$. Without loss of generality, we may assume that $\mathrm{tr}_g h=0$ and $\delta_g h=0$.

DEFINITION 2. The Einstein-Weyl structure $(M,[g],\nabla_g)$ on an Einstein manifold is *conformally rigid* if $h=0$ and $\alpha^\sharp$ is a conformal Killing vector of g for all deformations $(g(t),\omega(t))$ as above.

Define the curvature operator $\mathrm{Rm}_g\colon \Gamma(S^2(T^*M))\to\Gamma(S^2(T^*M))$ by

$$\mathrm{Rm}_g(h)_{ij}=-R_{ikjl}h^{kl}\,. \tag{4}$$

Its first eigenvalue $\lambda_1(\mathrm{Rm}_g)$ is given by

$$\lambda_1(\mathrm{Rm}_g)=\inf_{h\neq 0}\left(\int_M(\mathrm{Rm}_g(h),h)d\mu_g\right)\Big/\left(\int_M|h|^2d\mu_g\right). \tag{5}$$

THEOREM 3. *Let (M,g) be a closed Einstein n-manifold. Assume that first eigenvalue $\lambda_1(\mathrm{Rm}_g)$ of the curvature operator Rm_g satisfies*

$$\lambda_1(\mathrm{Rm}_g)>\min\left\{\frac{R_g}{n},\,-\frac{R_g}{2n}\right\}. \tag{6}$$

Then the Einstein-Weyl structure $(M,[g],\nabla_g)$ is conformally rigid.

PROOF. A direct calculation shows

$$\frac{d}{dt}\Gamma^i_{jk}\Big|_{t=0}=\frac{1}{2}(\nabla_j h^i_k+\nabla_k h^i_j-\nabla^i h_{jk}),$$
$$\frac{d}{dt}R^k_{lij}\Big|_{t=0}=\nabla_i\left(\frac{d}{dt}\Gamma^k_{lj}\Big|_{t=0}\right)-\nabla_j\left(\frac{d}{dt}\Gamma^k_{li}\Big|_{t=0}\right)$$
$$=\frac{1}{2}(\nabla_i\nabla_l h^k_j+\nabla_i\nabla_j h^k_l-\nabla_i\nabla^k h_{lj}-\nabla_j\nabla_l h^k_i-\nabla_j\nabla_i h^k_l+\nabla_j\nabla^k h_{li}). \tag{7}$$

Because h is traceless and divergence-free, we get, from the second Bianchi identity,

$$\frac{d}{dt}\mathrm{Ric}_g\Big|_{t=0}=-\frac{1}{2}\bar{\Delta}_g h+\frac{R_g}{n}h+\mathrm{Rm}_g(h), \tag{8}$$

where $\bar{\Delta}_g$ is the rough Laplacian acting on symmetric 2-tensor defined by

$$(\bar{\Delta}_g h)_{ij}=\nabla^k\nabla_k h_{ij}. \tag{9}$$

The Einstein-Weyl equation is

$$\mathrm{Ric}_g+\frac{n-2}{4}\mathscr{L}_{\omega_g^\sharp}g+\frac{n-2}{4}\omega_g\otimes\omega_g=A_g g, \tag{10}$$

where

$$A_g:=\frac{1}{n}\left\{R_g+\frac{n-2}{4}|\omega_g|^2-\frac{n-2}{2}\delta_g\omega_g\right\}. \tag{11}$$

Differentiating this, we get

$$\frac{d}{dt}\mathrm{Ric}_g\Big|_{t=0}+\frac{n-2}{4}\mathscr{L}_{\alpha^\sharp}g=A_g h+\frac{d}{dt}A_g\Big|_{t=0}g. \tag{12}$$

Therefore, we have

$$-\frac{1}{2}\bar{\Delta}_g h+\mathrm{Rm}_g(h)+\frac{n-2}{4}\mathscr{L}_{\alpha^\sharp}g=\frac{d}{dt}A_g\Big|_{t=0}g. \tag{13}$$

Define the operators $T\colon \Gamma(S^2(T^*M))\to\Gamma(T^0_3(M))$ and $S\colon \Gamma(S^2(T^*M))\to\Gamma(T^0_3(M))$ by

$$(T(h))(X,Y,Z)=\alpha(\nabla_X h)(Y,Z)+\beta(\nabla_Y h)(Z,X)+\gamma(\nabla_Z h)(X,Y),$$
$$(S(h))(X,Y,Z)=(\nabla_Y h)(Z,X),$$

where $\alpha,\beta,\gamma\in\mathbf{R}$, $\alpha^2+\beta^2+\gamma^2=1$. Set $u=\alpha\beta+\beta\gamma+\gamma\alpha$. Then $\max u=1$ and $\min u=-1/2$. By a direct caluculation, we get

$$\int_M |T(h)|^2 d\mu_g = \int_M |\nabla h|^2 d\mu_g + 2u \int_M (S(h), \nabla h) d\mu_g$$
$$= \int_M (-\bar{\Delta}_g h, h) d\mu_g + 2u \int_M (\delta_g S(h), h) d\mu_g .$$

Note that h is divergence-free, and we have, from the second Bianchi identity,

$$(\delta_g S(h))_{ij} = -\nabla^k (S(h))_{kij} = -\nabla^k \nabla_i h_{jk}$$
$$= -\nabla_i \nabla^k h_{jk} + g^{km} R^l_{jmi} h_{lk} + g^{km} R^l_{kmi} h_{jl}$$
$$= -(\mathrm{Rm}_g(h))_{ij} - R^l_i h_{jl} = -(\mathrm{Rm}_g(h))_{ij} - \frac{R_g}{n} h_{ij} .$$

Hence using the tracelessness and divergence-freeness of h, we get

$$0 \leq \int_M \left(-\bar{\Delta}_g h - 2u \mathrm{Rm}_g(h) - 2u \frac{R_g}{n} h, h \right) d\mu_g .$$
$$= \int_M \left(-(1+u) \mathrm{Rm}_g(h) - 2u \frac{R_g}{n} h - \frac{n-2}{2} \mathscr{L}_{\alpha^\sharp} g, h \right) d\mu_g$$
$$= \int_M \left(-2(1+u) \mathrm{Rm}_g(h) - 2u \frac{R_g}{n} h, h \right) d\mu_g .$$

Thus we get

$$u \frac{R_g}{n} \int_M |h|^2 d\mu_g \leq -(1+u) \int_M (\mathrm{Rm}_g(h), h) d\mu_g . \tag{14}$$

Assume $h \neq 0$. If $u = -\frac{1}{2}$, then

$$\frac{R_g}{n} \geq \left(\int_M (\mathrm{Rm}_g(h), h) d\mu_g \right) \Big/ \left(\int_M |h|^2 d\mu_g \right) \geq \lambda_1(\mathrm{Rm}_g) > \frac{R_g}{n} , \tag{15}$$

which is a contradiction. If $u = 1$, then

$$-\frac{R_g}{2n} \geq \left(\int_M (\mathrm{Rm}_g(h), h) d\mu_g \right) \Big/ \left(\int_M |h|^2 d\mu_g \right) \geq \lambda_1(\mathrm{Rm}_g) > -\frac{R_g}{2n} , \tag{16}$$

which leads also a contradiction. Therefore $h = 0$, and $\alpha^\sharp$ is a conformal Killing vector of g. □

REMARK 4. (1) The deformations of Einstein-Weyl structures on odd dimensional sphere includes all the perturbation of the standard Einstein metric.

(2) If (M, g) is not conformal to the standard sphere, then all conformal Killing vector fields are Killing ([4]).

(3) Recently, Pedersen et al. also consider the deformations of Einstein-Weyl

structures at Einstein metrics using the Gauduchon metrics (see [7]).

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