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On the Uniqueness of a Weyl Structure with Prescribed Ricci Curvature

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1. Introduction.

Let M be an n-dimensional manifold with a conformal class C. A conformal connection on M is an affine connection D preserving the conformal class C. We also assume D is torsion-free. The triple (M, C, D) is called a *Weyl manifold* or (C, D) is called a *Weyl structure* on M. In general, the Ricci curvature Ric^D of D is not symmetric, so we denote by $\mathrm{Sym}(\mathrm{Ric}^D)$ its symmetric part.

We consider a problem of a Weyl structure with prescribed Ricci curvature as follows: For a given conformal class C and a $(0, 2)$-tensor H, can we find a conformal connection D such that $\mathrm{Ric}^D = H$? In this paper, we prove the following result on uniqueness for the problem.

THEOREM 1. *Let M be a closed connected n-manifold, $n \geq 3$, with a conformal class C, and let D and $\bar{D}$ be torsion-free conformal connections of (M, C). If* $\mathrm{Sym}(\mathrm{Ric}^D) = \mathrm{Sym}(\mathrm{Ric}^{\bar{D}})$, *then $D = \bar{D}$.*

The result shows for a conformal connection, the symmetric part of the Ricci curvature determines the full Ricci curvature. The following corollary is due to [7].

COROLLARY 2. *Let (M, C, D) be a closed connected Weyl n-manifold, $n \geq 3$. If* $\mathrm{Sym}(\mathrm{Ric}^D) = \mathrm{Ric}_g$ *for some Riemannian metric $g \in C$, then D is the Levi-Civita connection of g, and such a g is unique in C up to a constant multiple.*

2. Preliminaries.

Let (M, C, D) be a Weyl manifold. We assume $n = \dim M \geq 3$. Then there is a unique 1-form ω_g such that $Dg = \omega_g \otimes g$.

We denote by Ric^D the Ricci curvature of D, and by $\mathrm{Sym}(\mathrm{Ric}^D)$ the symmetric part of the Ricci curvature. The scalar curvature R_g^D of D with respect to $g \in C$ is defined

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by $R_g^D := \mathrm{tr}_g \mathrm{Ric}^D$. We denote the Ricci curvature and the scalar curvature of g by Ric_g and R_g respectively.

LEMMA 3. *Let (M, C, D) be a Weyl n-manifold. Then the symmetric part of Ricci curvature* $\mathrm{Sym}(\mathrm{Ric}^D)$ *of D and the scalar curvature R_g^D of D with respect to $g \in C$ are related in terms of* Ric_g *and R_g as follows.*

$$\mathrm{Sym}(\mathrm{Ric}^D) = \mathrm{Ric}_g + \frac{n-2}{4}(\mathscr{L}_{\omega_g^\sharp} g + \omega_g \otimes \omega_g) - \left(\frac{n-2}{4}|\omega_g|^2 + \frac{1}{2}\delta_g \omega_g\right) g \,, \tag{1}$$

$$R_g^D = R_g - \frac{(n-1)(n-2)}{4}|\omega_g|^2 - (n-1)\delta_g \omega_g \,, \tag{2}$$

where the vector field $\omega_g^\sharp$ is defined by $\omega_g(X) = g(X, \omega_g^\sharp)$ for all vector field X, $\mathscr{L}$ is the Lie derivative, and δ_g is the codifferential of d with respect to g.

PROOF. Direct calculations. □

LEMMA 4. *Let (M, C, D) be Weyl n-manifold. Then for $g \in C$, we have*

$$\delta_g \left\{ \mathrm{Sym}(\mathrm{Ric}^D) - \frac{n-2}{4}(\mathscr{L}_{\omega_g^\sharp} g + \omega_g \otimes \omega_g) \right\}$$
$$= -\frac{1}{2} d \left\{ R_g^D + \frac{(n-2)(n-3)}{4}|\omega_g|^2 + (n-2)\delta_g \omega_g \right\}. \tag{3}$$

PROOF. A direct calculation with the second Bianchi identity: $\delta_g \mathrm{Ric}_g + \frac{1}{2} dR_g = 0$. □

LEMMA 5. *Let α be a 1-form on M. If $\delta_g \alpha = 0$ for all $g \in C$, then $\alpha = 0$.*

PROOF. For $h \in C$, define a vector field X_h by $\alpha(X) = h(X, X_h)$. Fix an arbitrary $g \in C$. For a smooth function u on M, set $\bar{g} := e^{2u} g$. Then we have

$$0 = (\mathrm{div}_{\bar{g}} X_{\bar{g}}) d\mu_{\bar{g}} = \mathscr{L}_{X_{\bar{g}}} d\mu_{\bar{g}} = n e^{nu}(X_{\bar{g}} u) d\mu_g + e^{nu} \mathscr{L}_{X_{\bar{g}}} d\mu_g$$
$$= n(e^{-2u} X_g u) d\mu_{\bar{g}} + (\mathrm{div}_g(e^{-2u} X_g)) d\mu_{\bar{g}} = (n-2) e^{-2u}(X_g u) d\mu_{\bar{g}} \,,$$

where $d\mu_g$ denote the volume element of g. Therefore $X_g u = 0$ for all smooth function u, so $X_g = 0$, and $\alpha = 0$. □

3. Proof of Theorem.

Fix an arbitrary $g \in C$, and $Dg = \omega_g \otimes g$, $\bar{D} g = \bar{\omega}_g \otimes g$. Put $\alpha := \bar{\omega}_g - \omega_g$. Note that α is independent of the choice of Riemannian metric g. By our assumption $\mathrm{Sym}(\mathrm{Ric}^{\bar{D}}) = \mathrm{Sym}(\mathrm{Ric}^D)$, we have

$$(n-2)(\mathscr{L}_{\alpha^\sharp} g + \bar{\omega}_g \otimes \bar{\omega}_g - \omega_g \otimes \omega_g) - (n-2)(|\bar{\omega}_g|^2 - |\omega_g|^2) g - 2(\delta_g \alpha) g = 0 \,. \tag{4}$$

From $R_g^{\bar{D}}=R_g^D$, we have

$$\delta_g\alpha=-\frac{n-2}{4}(|\bar{\omega}_g|^2-|\omega_g|^2)\,, \tag{5}$$

so we get

$$2\delta_g(\mathscr{L}_{\alpha^\sharp}g+\bar{\omega}_g\otimes\bar{\omega}_g-\omega_g\otimes\omega_g)=-d(|\bar{\omega}_g|^2-|\omega_g|^2)\,. \tag{6}$$

On the other hand, from the second Bianchi identity,

$$\delta_g(\mathscr{L}_{\alpha^\sharp}g+\bar{\omega}_g\otimes\bar{\omega}_g-\omega_g\otimes\omega_g)=d\left\{\frac{n-3}{2}(|\bar{\omega}_g|^2-|\omega_g|^2)-2\delta_g\alpha\right\}. \tag{7}$$

Combining the above equations, we get

$$(n-2)d(|\bar{\omega}_g|^2-|\omega_g|^2)=0\,, \tag{8}$$

therefore, $|\bar{\omega}_g|^2-|\omega_g|^2=:c=\text{const}$. So

$$0=\int_M\delta_g\alpha d\mu_g=-\frac{n-2}{4}\int_M(|\bar{\omega}_g|^2-|\omega_g|^2)d\mu_g=-\frac{c(n-2)}{4}\mathrm{Vol}(M,g)\,. \tag{9}$$

Therefore for all $g\in C$,

$$\delta_g\alpha=-\frac{n-2}{4}(|\bar{\omega}_g|^2-|\omega_g|^2)=-\frac{n-2}{4}c=0\,,$$

so we get desired result $\bar{\omega}_g=\omega_g$ for all $g\in C$. □

References

[1] G. B. FOLLAND, Weyl manifolds, J. Diff. Geom. **4** (1970), 145–153.
[2] H. PEDERSEN, Y. S. POON and A. SWANN, The Hitchin-Thorpe inequality for Einstein-Weyl manifolds, Bull. London Math. Soc. **26** (1994), 191–194.
[3] H. PEDERSEN and A. SWANN, Riemannian submersions, four-manifolds and Einstein-Weyl geometry, Proc. London Math. Soc. **66** (1993), 381–399.
[4] H. PEDERSEN and A. SWANN, Einstein-Weyl geometry, the Bach tensor and conformal scalar curvature, J. Reine Angew. Math. **441** (1993), 99–133.
[5] H. PEDERSEN and K. P. TOD, Three-dimensional Einstein-Weyl geometry, Adv. in Math. **97** (1993), 74–109.
[6] K. P. TOD, Compact 3-dimensional Einstein-Weyl structures, J. London Math. Soc. **45** (1992), 341–351.
[7] X. XU, Prescribing a Ricci tensor in a conformal class of Riemannian metrics, Proc. Amer. Math. Soc. **115** (1992), 455–459, and **118** (1993), 333.

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