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A Degree Condition for the Existence of $[a, b]$-Factors in $K_{1,n}$-Free Graphs

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Abstract. A graph is called $K_{1,n}$-free if it contains no $K_{1,n}$ as an induced subgraph. Let a, b $(0 \le a < b)$, and n (≥ 3) be integers. Let G be a $K_{1,n}$-free graph. We prove that G has an $[a, b]$-factor if its minimum degree is at least

$$\left(\frac{(a+1)(n-1)}{b}+1\right)\left\lceil \frac{a}{2}+\frac{b}{2(n-1)} \right\rceil - \frac{n-1}{b}\left(\left\lceil \frac{a}{2}+\frac{b}{2(n-1)} \right\rceil\right)^2 - 1 .$$

This degree condition is sharp for any integers a, b, and n with $b \le a(n-1)$. If $b \ge a(n-1)$, it exists if its minimum degree is at least a.

1. Introduction and notation.

We begin with definitions and notation. In this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges, respectively. Let S and T be disjoint subsets of $V(G)$. We denote by $e(S, T)$ the number of edges joining S and T. A vertex x is often identified with $\{x\}$. So, $e(x, T)$ means $e(\{x\}, T)$. For $x \in V(G)$, we denote the degree of x in G by $\deg_G(x)$, the set of vertices adjacent to x in G by $N_G(x)$. If $S \subset V(G)$, $G - S$ is the subgraph of G induced by $V(G) - S$. The minimum degree of G is denoted by $\delta(G)$. We denote by $\omega(G)$ the number of components of a graph G. A spanning subgraph F of a graph G with $\deg_F(v) = r$ for all $v \in V(G)$ is called an r-factor. And a spanning subgraph F' of a graph G with $a \le \deg_{F'}(v) \le b$ for all $v \in V(G)$ is called an $[a, b]$-factor. A graph is called $K_{1,n}$-free if it contains no $K_{1,n}$ as an induced subgraph. The other notation may be found in [1].

Here, we note the following result which presents a degree condition for the existence of an r-factor in a $K_{1,n}$-free graph.

THEOREM A ([2]). *Let n $(n \ge 3)$ and r be positive integers. If r is odd, we assume*

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that $r \geq n-1$. *Let G be a connected* $K_{1,n}$*-free graph with* $r|V(G)|$ *even, and suppose that the minimum degree of G is at least* $(n^2/4(n-1))r+(3n-6)/2+(n-1)/4r$. *Then G has an r-factor.*

It is easy to see that every connected graph G with $r|V(G)|$ odd has no r-factor. And it is described in [2] that the condition "$r \geq n-1$ if r is odd" in Theorem A cannot be dropped. However, the degree condition is not best for some pairs of integers n and r. For that reason, there exists the following theorem in [4], in which the degree condition is sharp.

THEOREM B ([4]). *Let* n (≥ 3) *and r be positive integers. If r is odd, we assume that* $r \geq n-1$. *Let G be a connected* $K_{1,n}$*-free graph with* $r|V(G)|$ *even. If the minimum degree of G is at least*

$$\left(n+\frac{n-1}{r}\right)\left\lceil \frac{n}{2(n-1)}r \right\rceil - \frac{n-1}{r}\left(\left\lceil \frac{n}{2(n-1)}r \right\rceil\right)^2 + n-3,$$

then G has an r-factor.

As mentioned in [4], the degree condition is sharp for every pairs of integers n and r. We obtain the following theorem which is extended Theorem B for $[a, b]$-factors.

THEOREM 1. *Let* a, b $(0 \leq a < b)$ *and* n (≥ 3) *be integers. Let G be a* $K_{1,n}$*-free graph. If the minimum degree of G is at least*

$$\left(\frac{(a+1)(n-1)}{b}+1\right)\left\lceil \frac{a}{2}+\frac{b}{2(n-1)} \right\rceil - \frac{n-1}{b}\left(\left\lceil \frac{a}{2}+\frac{b}{2(n-1)} \right\rceil\right)^2 - 1, \tag{1}$$

then G has an $[a, b]$*-factor.*

This degree condition is sharp for any integers a, b, and n with $b \leq a(n-1)$ (we will show that in Section 2). Here, by using Theorem 1 with $b=a(n-1)$, we know that every $K_{1,n}$-free graph with $\delta(G) \geq a$ has an $[a, a(n-1)]$-factor. Hence we obtain that every $K_{1,n}$-free graph with $\delta(G) \geq a$ has an $[a, b]$-factor with $b \geq a(n-1)$.

2. Proof of theorem.

We use the following theorem for the existence of an $[a, b]$-factor with $a < b$.

THEOREM C (Lovász [3]). *A graph G has an* $[a, b]$*-factor* $(a < b)$, *if and only if*

$$\theta(S, T) = b|S| + \sum_{x \in T} (\deg_{G-S}(x) - a) \geq 0$$

for any disjoint subsets S and T of $V(G)$. □

Let n, a, b, and G be as in Theorem 1. First, we prove the following claim.

CLAIM 1. *$\delta(G) \geq (n-1)y(a-y+1)/b+y-1$ for any integer y.*

PROOF. We fix n, a, and b, and define $f(y)$ to be the RHS (right hand side) of the above inequality. Among all integers y, $f(y)$ is maximum when y is the nearest integer to $a/2+b/(2(n-1))+\frac{1}{2}$, i.e., when $y=\lceil a/2+b/(2(n-1))\rceil$. It is easy to check that $f(\lceil a/2+b/(2(n-1))\rceil)$ is identical to the expression (1). Hence, $f(y) \leq f(\lceil a/2+b/(2(n-1))\rceil) \leq \delta(G)$ for any integer y. □

Let S and T be disjoint subsets of $V(G)$. Here, we want to show that $\theta(S, T) \geq 0$ which implies that G has an $[a, b]$-factor by Theorem C.

We define x_i and N_i $(i \geq 1)$ as follows: If $T \neq \emptyset$, let $x_1 \in T$ be a vertex such that $\deg_{G-S}(x_1)$ is minimum, and $N_1 = (N_G(x_1) \cup \{x_1\}) \cap T$. For $i \geq 2$, if $T - \bigcup_{j<i} N_j \neq \emptyset$, let $x_i \in T - \bigcup_{j<i} N_j$ be a vertex such that $\deg_{G-S}(x_i)$ is as small as possible, and $N_i = (N_G(x_i) \cup \{x_i\}) \cap (T - \bigcup_{j<i} N_j)$.

We suppose $x_1, x_2, \cdots, x_m$ are defined, but x_{m+1} cannot. When $T = \emptyset$, we define $m=0$. By definition, $\{x_1, x_2, \cdots, x_m\}$ is an independent set of G, and T is the disjoint union of $N_1, N_2, \cdots, N_m$.

Under this notation, we show the following claim.

CLAIM 2. *$|S| \geq (1/(n-1)) \sum_{i=1}^{m} e(x_i, S)$.*

PROOF. Let X be the set $\{x_1, x_2, \cdots, x_m\}$. Since X is an independent set of G and G is $K_{1,n}$-free, every vertex $v \in S$ is adjacent to at most $n-1$ vertices of X. Therefore, $(n-1)|S| \geq e(X, S) = \sum_{i=1}^{m} e(x_i, S)$. □

By Claim 2,

$$\begin{aligned}\theta(S, T) &\geq \frac{b}{n-1} \sum_{i=1}^{m} e(x_i, S) + \sum_{x \in T} (\deg_{G-S}(x) - a) \\ &= \sum_{i=1}^{m} \left(\frac{b}{n-1} e(x_i, S) + \sum_{x \in N_i} (\deg_{G-S}(x) - a) \right).\end{aligned}$$

We show the following inequality that implies $\theta(S, T) \geq 0$, and hence the existence of an $[a, b]$-factor in G.

$$\frac{b}{n-1} e(x_i, S) + \sum_{x \in N_i} (\deg_{G-S}(x) - a) \geq 0 \qquad \text{for each } i \quad (1 \leq i \leq m). \tag{2}$$

Here we fix i $(1 \leq i \leq m)$ and define $d = \deg_{G-S}(x_i)$. Since $\deg_{G-S}(x) \geq d$ for all $x \in N_i$,

$$\sum_{x \in N_i} (\deg_{G-S}(x) - a) \geq |N_i|(d-a).$$

If $d-a \geq 0$, then inequality (2) holds. Hence we may assume $d-a<0$. Since $|N_i| \leq d+1$,

$$\sum_{x \in N_i} (\deg_{G-S}(x) - a) \geq (d+1)(d-a).$$

By using Claim 1 with $y=d+1$, we obtain $\delta(G)\geq(n-1)(d+1)(a-d)/b+d$. Hence,

$$\frac{b}{n-1}e(x_i, S)+\sum_{x\in N_i}(\deg_{G-S}(x)-a)\geq\frac{b}{n-1}(\deg_G(x_i)-d)+(d+1)(d-a)$$

$$\geq\frac{b}{n-1}\left(\frac{n-1}{b}(d+1)(a-d)+d-d\right)+(d+1)(d-a)=0\,. \qquad \square$$

Finally we give the following remark.

REMARK 1. In Theorem 1 with $a<b\leq a(n-1)$, the degree condition is sharp.

To show this remark with an example, let

$$y=\left\lceil\frac{a}{2}+\frac{b}{2(n-1)}\right\rceil, \qquad x=\left\lceil\frac{(n-1)y(a-y+1)}{b}\right\rceil-1\,.$$

Let L be the complete graph K_x, and M be $n-1$ disjoint copies of K_y. Here, let G be a graph obtained from the join of L and M. Then G is a $K_{1,n}$-free graph with

$$\delta(G)=\deg_G v\,(v\in V(M))\geq\left(\frac{(a+1)(n-1)}{b}+1\right)\left\lceil\frac{a}{2}+\frac{b}{2(n-1)}\right\rceil$$

$$-\frac{n-1}{b}\left(\left\lceil\frac{a}{2}+\frac{b}{2(n-1)}\right\rceil\right)^2-2\,.$$

Application of Lovász's theorem with $S=V(L)$ and $T=V(M)$ proves that G has no $[a, b]$-factor.

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