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On Pseudoconvex Domains in $\mathbf{P}^n$

Klas DIEDERICH and Takeo OHSAWA

University of Wuppental and Nagoya University

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To the memory of Minoru Tada and Nobuo Sasakura

1. Introduction.

Let $\Omega \subsetneqq \mathbf{P}^n$ be a locally pseudoconvex domain and denote, for $z \in \Omega$, by $\delta_\Omega(z)$ the distance between z and $\partial\Omega$ measured with respect to the Fubini-Study metric. It is known from the work of A. Takeuchi [7], that $-\log\delta_\Omega$ is strictly plurisubharmonic on all of Ω and, hence, Ω is Stein. Therefore, it is reasonable to try to generalize function theory to locally pseudoconvex domains in $\mathbf{P}^n$.

In this article we will consider two questions in this direction, namely:
1) Are there localization principles for the Bergman kernel function and the Bergman metric (with respect to the measure coming from the Fubini-Study metric) on a suitable class of such domains?
2) Does local hyperconvexity of pseudoconvex domains $\Omega \subset \mathbf{P}^n$ imply also their global hyper-convexity?

In order to formulate our results with respect to 1) we denote by $d\lambda_{FS}$ the Fubini-Study volume element on $\mathbf{P}^n$ and put

$$A^2(\Omega) := \left\{ f \in \mathcal{O}(\Omega) : \int_\Omega |f(z)|^2 d\lambda_{FS} < \infty \right\}.$$

This is a Hilbert space with respect to the inner product

$$(f, g) := \int_\Omega f(z)\overline{g(z)} d\lambda_{FS}.$$

Notice, that always $\mathbf{C} \subset A^2(\Omega)$, since the volume of Ω is finite. The space $A^2(\Omega)$ has a (possibly constant) reproducing kernel

$$K_\Omega(\cdot, \cdot) : \Omega \times \Omega \to \mathbf{C}$$

which, in this article, will be called the *Bergman kernel* of Ω. We denote by $K_\Omega(z) := K_\Omega(z, z)$

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its restriction to the diagonal and call it the *Bergman kernel function* of Ω. We will show the following localization principle for $K_\Omega(\cdot)$.

THEOREM 1.1. *Suppose, that the interior of the complement of the pseudoconvex domain $\Omega \subset \mathbf{P}^n$ is non-empty. Then there is, for any point $x \in \bar{\Omega}$, and any pair $V \Subset U$ of arbitrarily small open neighborhoods of x a consant $C > 0$, such that for all $z \in \Omega \cap V$*

$$C^{-1} K_\Omega(z) < K_{\Omega \cap U}(z) < C \cdot K_\Omega(z) .$$

Since always $\mathbf{C} \subset A^2(\Omega)$ we have $K_\Omega(z) > 0$ for any $z \in \Omega$. Hence, the function $\log K_\Omega(\cdot)$ is well-defined on all of Ω, so that we also can consider, for any $z \in \Omega$, the hermitian form $i\partial\bar{\partial} \log K_\Omega(z)$. It is always positive semi-definite, and we call it the *Bergman function pseudometric* of Ω (with respect to the Fubini-Study volume element). Also for it we have an analogous localization principle, namely

THEOREM 1.2. *Suppose, that the domain $\Omega \subset \mathbf{P}^n$ satisfies the same hypothesis as in Theorem 1.1. Then there is, for any point $x \in \bar{\Omega}$ and any pair $U \Supset V \ni x$ of open neighborhoods of x, a constant $C > 0$ such that*

$$C^{-1} i\partial\bar{\partial} \log K_\Omega(z) \leq i\partial\bar{\partial} \log K_{\Omega \cap U}(z) \leq C \cdot i\partial\bar{\partial} \log K_\Omega(z)$$

for all $z \in \Omega \cap V$.

As an immediate consequence of this we obtain the fact, that the Bergman function metric of such domains Ω is (strictly) positive definite everywhere on the domain, namely we have

COROLLARY 1.3. *For pseudoconvex domains $\Omega \subset \mathbf{P}^n$ as in Theorem 1.1 one has for all $z \in \Omega$*

$$i\partial\bar{\partial} \log K_\Omega(z) > 0 .$$

In order to formulate the result concerning 2) we recall, that a complex manifold X is said to be hyperconvex, if there exists a bounded plurisubharmonic exhaustion function on X, i.e. a plurisubharmonic function $\varphi : X \to [-\infty, 0)$ such that, for any $c \in [-\infty, 0)$,

$$X_c := \{x \in X : \varphi(x) < c\} \Subset X .$$

A domain Ω in a complex manifold M is called locally hyperconvex, if, for each boundary point $x \in \partial\Omega$, there is a neighborhood $U \ni x$ such that $U \cap \Omega$ is hyperconvex. It is known from a theorem of Vâjâitu [8], that a relatively compact domain in a Stein manifold is hyperconvex, if and only if it is locally hyperconvex. However, on $\mathbf{P}^n$ we have the following:

THEOREM 1.4. *There exists a locally hyperconvex Stein domain Ω in $\mathbf{P}^5$ which is not hyperconvex.*

REMARK 1.5. A domain as in this theorem necessarily has to have a rather "bad" boundary, since it is known, that all pseudoconvex domains in $\mathbf{P}^n$ with $\mathscr{C}^2$-smooth boundary are hyperconvex (cf. [5]).

2. Proof of Theorem 1.1.

Under the hypothesis of Theorem 1.1 on the pseudoconvex domain $\Omega \subset \mathbf{P}^n$ we fix a point $y \in (\mathbf{P}^n \backslash \bar{\Omega})$ and denote by X the set of complex lines passing through y. We have a canonical identification $X \xrightarrow{\pi} \mathbf{P}^{n-1}$. For any point $x \in \bar{\Omega}$, we take $l_x \in X$ to be the complex line passing through x and y. Furthermore, we choose an arbitrary complex hyperplane H_x that intersects l_x only at y. Let, now, W be an open neighborhood of $l_x \cap \Omega$ of the form $\pi^{-1}(W') \cap \Omega$ such that $W \Subset \mathbf{P}^n \backslash H_x \simeq \mathbf{C}^n$. Such a neighborhood exists, since y is a point in $(\mathbf{P}^n \backslash \bar{\Omega})^\circ$.

In order to prove the statement of Theorem 1.1 it suffices to show it for $U = \pi^{-1}(W')$, since, afterwards, for treating arbitrary small neighborhoods U, we can apply well-known localization principles for bounded domains in $\mathbf{C}^n$. So, let $z \in V \cap \Omega$ and take a function $f \in A^2(U \cap \Omega)$ such that

$$K_{U \cap \Omega}(z) = |f(z)|^2 \quad \text{and} \quad \int_{U \cap \Omega} |f|^2 d\lambda_{FS} = 1 . \tag{2.1}$$

Let $\rho : \mathbf{P}^{n-1} \to [0, 1]$ be a $\mathscr{C}^\infty$-function such that

$$\begin{cases} \operatorname{supp} \rho \subset \pi(U) = W' \\ \rho = 1 \text{ on } \pi(V) . \end{cases}$$

We claim, that we, then, can solve the $\bar{\partial}$-equation

$$\bar{\partial}(\rho f) = \bar{\partial} u \tag{2.2}$$

with the constraints

$$\begin{cases} \|u\| \leq C_1 \\ u \,|\, l_x = 0 . \end{cases} \tag{2.3}$$

In order to see this, let $s_1, \cdots, s_{n-1}$ be sections of the hyperplane bundle $\mathcal{O}(1)$ on $\mathbf{P}^n$, such that $\{s_1 = \cdots = s_{n-1} = 0\} = l_x$ and fix fiber metrics a resp. b on $\mathcal{O}(1)$ and on the canonical bundle $K(\mathbf{P}^n) \simeq \mathcal{O}(-n-1)$ respectively so that their curvature forms $\Theta_{\mathcal{O}(1)}$ and $\Theta_{K(P^n)}$ satisfy $i\Theta_{\mathcal{O}(1)} > 0$ and $\Theta_{K(P^n)} = -(n+1)\Theta_{\mathcal{O}(1)}$. Then we note, that $\bar{\partial}(\rho f)$ can be considered as a $\bar{\partial}$-closed $(n, 1)$-form on Ω with values in $K^{-1}(\mathbf{P}^n)$ which is square integrable with respect to the singular hermitian metric

$$h := \left(\frac{1}{\sum_{j=1}^{n-1} |s_j|^2} \right)^{n-1} b^{-1} \tag{2.4}$$

and that the curvature form of h is equal to

$$-(n-1)\Theta_{\mathcal{O}(1)}-\Theta_{K(P^n)}=2\Theta_{\mathcal{O}(1)} \tag{2.5}$$

on $\Omega\backslash l_x$. Therefore, by applying an L^2-cohomology vanishing theorem on complete Kähler manifolds (cf. [1] or [4]), one obtains a solution of the equation (2.2) with side conditions (2.3). This finishes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2.

We just need to refine the proof of Theorem 1.1 a little bit in order to get a proof of Theorem 1.2. Namely, as usual, we use the fact, that the Bergman function pseudometric $ds^2(z, t) := i\partial\bar{\partial}\log K_\Omega(z)(t)$ is given by the following formula

$$ds^2_\Omega(z, t) = \frac{\sup\{|\langle\partial f(z), t\rangle|^2 : f\in A^2(\Omega),\ \|f\|=1,\ f(z)=0\}}{K_\Omega(z)} \tag{3.1}$$

(and similarly on $U\cap\Omega$). We choose for $z\in V\cap\Omega$ a function $f\in A^2(U\cap\Omega)$ realizing the supremum in the denominator of (3.1) for the case of $U\cap\Omega$ and apply to it the same $\bar{\partial}$-machinery as in the proof of Theorem 1.1 with the only change, that we replace the singular hermitian metric h from (2.4) by

$$h := \left(\frac{1}{\sum_{j=1}^{n-1}|s_j|^2}\right)^n b^{-1} \tag{3.2}$$

in order to assure, that one has for the solution of the equation (2.2) not only the side conditions (2.3), but also $du(z)=0$. This finishes the proof of Theorem 1.2. □

4. Proof of Theorem 1.4.

We show at first

LEMMA 4.1. *Let M be a complex manifold and $N\subset M$ a closed complex submanifold. Furthermore, let $D\subset N$ be a relatively compact locally hyperconvex Stein domain. Then D has a locally hyperconvex open neighborhood $\Omega\subset M$.*

PROOF. For each point $x\in\bar{D}$, take a neighborhood $W_x\ni x$ in M so that a holomorphic retraction $W_x \xrightarrow{\pi_x} W_x\cap N$ exists. Since $\bar{D}$ is compact, it can be covered by finitely many such neighborhoods W_x, say $\{W_{x_i}\}_{i=1,\cdots,m}$. Let $U\supset D$ be a neighborhood satisfying

$$\pi_{x_i}^{-1}(\partial D\cap W_{x_i})\cap\partial U\subset\partial D \qquad \forall i=1,\cdots,m\,.$$

By Siu's theorem [6], there exists a Stein neighborhood $V\supset D$, $V\subset M$ open, such that $V\subset U$. We can choose Stein neighborhoods $\tilde{V}_1\supset\tilde{V}_2\supset D$ with $\tilde{V}_1\subset V$ and such that $\partial\tilde{V}_2\backslash N\subset\tilde{V}_1$, $\partial\tilde{V}_1\backslash N\subset V$. Furthermore, we choose a $\mathscr{C}^\infty$ exhaustion function φ of V and put $V_j := \{z\in V\colon \varphi(z)<j\}$ for all $j\in\mathbf{N}$. The sets $K_j := \{\bar{\tilde{V}}_1\backslash\tilde{V}_2\}\cap\{\bar{V}_j\backslash V_{j-1}\}$ are compact

(here $V_0=\varnothing$), $K_j \cap D=\varnothing$ and

$$\bigcup_{j=1}^{\infty} K_j=(\bar{\tilde{V}}_1\setminus\tilde{V}_2)\cap V.$$

By Runge-Hörmander's approximation theorem, for every $j\in\mathbf{N}$ and every point $z\in K_{j+2}\setminus\mathring{K}_{j+1}$ there is a function $f\in\mathcal{O}(V)$, $f\,|\,D=0$, such that $|f(z)|>1$ and $|f|<\varepsilon$ on V_j for any given $\varepsilon>0$. Since the inequality $|f|>1$ remains true in a neighborhood of z and K_j can be covered by finitely many such neighborhoods, we can, by passing over to suitable powers of the f's, for any $j\in\mathbf{N}$, choose a finite system of holomorphic functions $f_{jk}\in\mathcal{O}(V)$, $k=1,\cdots,N_j$, such that we have

$$\begin{cases} \sum_{k=1}^{N_j}|f_{jk}|^2<2^{-j} & \text{on} \quad V_j \\ \sum_{k=1}^{N_j}|f_{jk}|^2>1 & \text{on} \quad K_{j+2}\,. \end{cases} \tag{4.1}$$

Of course, there are also finitely many $f_{0k}\in\mathcal{O}(V)$ for $k=1,\cdots,N_0$, $f_{0k}\,|\,D=0$, such that

$$\sum_{k=1}^{N_0}|f_{0k}|^2>1 \qquad \text{on} \quad K_1\cup K_2\,. \tag{4.2}$$

From (4.1) and (4.2) we get, that the function

$$\Phi(z):=\sum_{j=0}^{\infty}\sum_{k=1}^{N_j}|f_{jk}|^2$$

is a $\mathscr{C}^\infty$ plurisubharmonic function on V_1 with

$$\delta:=\inf_{V_1\setminus V_2}\Phi(z)>1 \quad \text{and} \quad \Phi\,|\,D=0\,.$$

We put

$$\Omega:=\{z\in V_1:\Phi(z)<\delta/2\}\,. \tag{4.3}$$

Clearly $D\subset\Omega$. The domain Ω is also locally hyperconvex. This follows directly from (4.3) near all points $z\in\partial\Omega\setminus\partial D$. For any $z\in\partial D$ there is a neighborhood W_{x_i} with a local retraction π_{x_i} as chosen above. Furthermore, since D is supposed to be locally hyperconvex, we may assume, that $W_{x_i}\cap D$ has a bounded plurisubharmonic exhaustion function ρ_i. Then

$$\max\{\pi_{x_i}^*\rho_i,\ \Phi-\delta/2\}$$

is a bounded plurisubharmonic exhaustion on $W_{x_i}\cap\Omega$ (in the sense, that it is negative inside and goes to 0 at $\partial\Omega$).

We now come to the

Proof of Theorem 1.4. Let $A:=\mathbf{C}/\mathbf{Z}+i\mathbf{Z}$. As was shown in [3], there exists a worm-like (cf. [2]) Stein open subset $D\subset A\times\mathbf{P}^1$ with $\mathscr{C}^\omega$-smooth boundary and such that D is locally hyperconvex and biholomorphically equivalent to the product of an annulus and $\mathbf{C}^*$.

Notice, that A can be embedded into $\mathbf{P}^2$. Therefore, $A \times \mathbf{P}^1$ can be embedded into $\mathbf{P}^5$ using the Veronese embedding $\mathbf{P}^2 \times \mathbf{P}^1 \to \mathbf{P}^5$. By Lemma 4.1, there is a locally hyperconvex domain $\Omega \subset \mathbf{P}^5$ containing the image D_0 of D under the embedding of $A \times \mathbf{P}^1$ into $\mathbf{P}^5$. However, such a domain Ω cannot be (globally) hyperconvex. Otherwise, there would exist a bounded plurisubharmonic exhaustion function φ on Ω the restriction of which to D_0 would be a bounded plurisubharmonic exhaustion function on D_0. But this is impossible, since D_0 contains a biholomorphic image of $\mathbf{C}^*$ as a closed submanifold. $\square$

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Present Addresses:
Klas Diederich
Mathematik, University of Wuppertal,
Gausstr. 20, D-42095 Wuppertal, Germany.

Takeo Ohsawa
Graduate School of Polymathematics, Nagoya University,
Chikusa-ku, Furocho, Nagoya, 464–8602, Japan.