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Weyl Groups of the Extended Affine Root System $A_1^{(1,1)}$ and the Extended Affine $\mathfrak{sl}(2)$

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1. Introduction.

The notion of the extended affine root systems was introduced by K. Saito ([1]). These root systems relate to the simple elliptic singularity, and by definition, they are extensions of the affine root systems by one dimensional radical. The Weyl groups of the extended affine root systems and their hyperbolic extension groups have been also studied in [1], [2], from a geometric point of view. In this paper, in the case of $A_1^{(1,1)}$, from the view point of representation theory, we study the Weyl group of $A_1^{(1,1)}$ and its hyperbolic extension group, which we denote by $W(A_1^{(1,1)})$ and $\tilde{W}(A_1^{(1,1)})$, respectively. From their constructions, we find that $\tilde{W}(A_1^{(1,1)})$ ([1]) is isomorphic to the Weyl group $W(\hat{\mathfrak{sl}}_2)$ of the extended affine Lie algebra $\hat{\mathfrak{sl}}_2$ (the extended affine $\mathfrak{sl}(2)$). At first, we fix the generators of $W(A_1^{(1,1)})$ and decide their relations by considering $W(A_1^{(1,1)})$ as an extension of the finite Weyl group $W(A_1)$ by translations of two directions, and further using this result, we show that $W(A_1^{(1,1)})$ contains the infinite dihedral group D_∞ as a subgroup and is an extension of the dihedral group D_2. Further we present that $W(A_1^{(1,1)})$ is non-amenable. The extended affine Lie algebra $\hat{\mathfrak{sl}}_2$ is defined as follows. In quantum field theory, the gauge group (the current group) is defined to be the set of smooth functions from the compact manifold M onto the semi-simple compact Lie group G with a pointwise product. When $M=T^\nu=S^1\times\cdots\times S^1$ i.e. ν-dimensional torus, the corresponding Lie algebra is the gauge algebra $P(T^\nu, \mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G and $P(T^\nu, \mathfrak{g})$ means the set of functions from T^ν into $\mathfrak{g}$ with finite fourier series. The central extensions of $P(T^\nu, \mathfrak{g})$ are infinite dimensional Lie algebras, and called quasi-simple Lie algebras ([16]). Especially, when $\nu=1$, $P(S^1, \mathfrak{g})$ is usually written as $\mathbf{C}[t, t^{-1}]\otimes\mathfrak{g}$, where $\mathbf{C}[t, t^{-1}]$ is the ring of Laurent polynomials in t, and its central extension is called affine Lie algebra (Kac-Moody algebra). Then the corresponding Lie group is called loop group ([20]). Further let $M=T^2=S^1\times S^1$, then the Lie algebra

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$\tilde{P}(T^2, \mathfrak{g})$, which is a central extension of $P(T^2, \mathfrak{g})$, has been studied by several authors ([13], [15], [17]). We may consider $P(T^2, \mathfrak{g})$ as $\tilde{\mathfrak{g}}=\mathbf{C}[t, t^{-1}, s, s^{-1}]\otimes\mathfrak{g}$, where $\mathbf{C}[t, t^{-1}, s, s^{-1}]$ is the ring of Laurent polynomials in t and s, then the central extension of $\tilde{\mathfrak{g}}$ is called the extended affine Lie algebra from the view point of its root system ([14], [18], [19]). The Weyl group W of a Lie algebra $\mathfrak{g}$ is defined to be the group generated by the reflections of the root system of $\mathfrak{g}$. In the cases of classical (finite) or affine Lie algebras, as an equivalent definition, W is also the group of those automorphisms of a Cartan subalgebra of $\mathfrak{g}$ which are restrictions of conjugations by elements of G, the Lie group corresponding to $\mathfrak{g}$ ([9], [10]). According to the definition of the latter, we examine the Weyl group $W(\hat{\mathfrak{sl}}_2)$, and as a result, we find that $W(\hat{\mathfrak{sl}}_2)$ is a central extension of $W(A_1^{(1,1)})$ and identified with the hyperbolic extension $\tilde{W}(A_1^{(1,1)})$ ([1]) of $W(A_1^{(1,1)})$. In the hyperbolic extension $\tilde{W}(A_1^{(1,1)})$, the double translation part of $W(A_1^{(1,1)})$ is replaced by a discrete Heisenberg ([1]), and the case of $W(\hat{\mathfrak{sl}}_2)$ is similar. So, using this fact, we describe the generators of $W(\hat{\mathfrak{sl}}_2)$ and their relations.

2. The extended affine root system $A_1^{(1,1)}$ and its Weyl group.

The definition of the extended affine root systems is given as follows ([1]). Let F be a real vector space of finite rank with a symmetric bilinear form $I: F\times F\to\mathbf{R}$, such that I is positive semi-definite and the radical;

$$\mathrm{rad}(I):=\{x\in F: I(x, y)=0 \text{ for } \forall y\in F\},$$

is of rank 2 over $\mathbf{R}$. Then an extended affine root system is defined to be a set Φ of non-isotropic elements $\alpha\in F$ (i.e. $I(\alpha, \alpha)\neq 0$) which satisfy some conditions ([1]). According to them, in the case of the extended affine root system $A_1^{(1,1)}$, F, I and Φ are given as follows;

$$\begin{aligned}
&F=\mathbf{R}(\varepsilon_1-\varepsilon_2)\oplus\mathbf{R}a\oplus\mathbf{R}b,\\
&I(\varepsilon_i, \varepsilon_j)=\delta_{ij}\quad(i, j=1,2), \qquad I(\varepsilon_i, a)=I(\varepsilon_i, b)=0\quad(i=1, 2)\\
&I(a, a)=I(b, b)=I(a, b)=0,\\
&\Phi=\{\pm(\varepsilon_1-\varepsilon_2)+nb+ma\ (n, m\in\mathbf{Z})\}.
\end{aligned}$$

We choose its basis $B\subset\Phi$ such as;

$$B=\{\alpha_0=\varepsilon_2-\varepsilon_1+b,\ \alpha_1=\varepsilon_1-\varepsilon_2,\ \alpha_2=\varepsilon_2-\varepsilon_1+a\}.$$

We note that each root $\beta\in\Phi$ can be written;

$$\beta=\sum_{\alpha\in B} n_\alpha\alpha,$$

where the n_α are integers, but unlike the cases of finite or affine root systems, not necessarily of the same sign. In this basis, we have a direct sum decomposition of the vector space F;

$$F=\mathbf{R}(\varepsilon_1-\varepsilon_2)\oplus\mathbf{R}a\oplus\mathbf{R}b=\mathbf{R}\alpha_0\oplus\mathbf{R}\alpha_1\oplus\mathbf{R}\alpha_2 .$$

The reflection w_α corresponding to the root α is an element of $O(I):=\{g\in GL(F)\mid I(x,y)=I(g(x),g(y))\}$ given by $w_\alpha(u):=u-I(u,\alpha^\vee)\alpha$, $(\forall u\in F)$, where $\alpha^\vee:=2\alpha/I(\alpha,\alpha)\in F$. The group generated by w_α for all $\alpha\in\Phi$ is called the extended affine Weyl group and denoted by $W(A_1^{(1,1)})$. We set $w_i:=w_{\alpha_i}$ $(0\leq i\leq 2)$, then we have the following.

LEMMA 2.1. *$W(A_1^{(1,1)})$ is generated by w_0, w_1 and w_2.*

PROOF. For $\alpha\in\Phi$, w_α acts trivially on a and b, so the action of w_α on F is decided by the action on $\varepsilon_1-\varepsilon_2$. We set $\alpha_1:=\varepsilon_1-\varepsilon_2$, then we see that

$$w_{\alpha_1+nb+ma}(\alpha_1)=-\alpha_1-2nb-2ma , \qquad w_{-\alpha_1+nb+ma}(\alpha_1)=-\alpha_1+2nb+2ma .$$

On the other hand, $w_1(\alpha_1)=-\alpha_1$, $w_0(\alpha_1)=-\alpha_1+2b$, $w_2(\alpha_1)=-\alpha_1+2a$, so we see that

$$(w_1w_0)^n(\alpha_1)=\alpha_1+2nb , \quad (w_2w_0)^n(\alpha_1)=\alpha_1+2na \qquad \text{for} \quad n\geq 1 .$$

From these actions, we find that w_1, w_1w_0 and w_2w_0, so w_1, w_0 and w_2 are generators of $W(A_1^{(1,1)})$. □

The elements w_0 and w_1 are generators of the affine Weyl group of $\tilde{A}_1$ ([3], [4], [8]), and the relations of the generators of the affine Weyl groups are well known ([5], [8]), furthermore each affine Weyl group is the semi-direct extension of finite Weyl group by translations. In the case $\tilde{A}_1$, we set $T:=w_1w_0$, then

$$T\varepsilon_1=\varepsilon_1+b , \qquad T\varepsilon_2=\varepsilon_2-b .$$

The relations of w_0 and w_1 are given by;

$$w_0^2=w_1^2=1 .$$

Rewriting this, we can describe the relations of w_1 and T:

LEMMA 2.2. *The elements w_1 and T are generators of the affine Weyl group of $\tilde{A}_1$, and their relations are*

$$w_1^2=1 , \qquad Tw_1Tw_1=1 .$$

In the case of $A_1^{(1,1)}$, further we set $R:=w_1w_2$, then

$$R\varepsilon_1=\varepsilon_1+a , \qquad R\varepsilon_2=\varepsilon_2-a .$$

To examine the relations of w_i $(0\leq i\leq 2)$, firstly we regard w_1, T and R as the generators of the Weyl group $W(A_1^{(1,1)})$, and examine the relations of them, after that we rewrite these relations by the elements w_i $(0\leq i\leq 2)$. It is clear the R satisfies the same relations as T, and that T and R commute, i.e.

$$Rw_1Rw_1=1 , \qquad TR=RT .$$

Next we must consider about the relations containing all of T, R and w_1. But the relation $Tw_1Tw_1=1$ (resp. $Rw_1Rw_1=1$) is rewritten as $Tw_1=w_1T^{-1}$ (resp. $Rw_1=w_1R^{-1}$), so all commutation relations between each two elements have been already given. Therefore we can obtain all relations of w_1, T and R from only these three relations. Now the relation $TR=RT$ is rewritten as follows;

$$TR=RT \Longleftrightarrow (w_0w_1w_2)^2=1\ ,$$

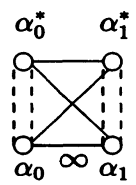
so we have the following;

Proposition 2.3. *The relations of the extended affine Weyl group $W(A_1^{(1,1)})$ are given as follows;*

$$w_i^2=1\ (0\leq i\leq 2)\ , \qquad (w_0w_1w_2)^2=1\ .$$

Proof. We have only to check that it is possible to obtain all relations $Rw_1Rw_1=1$, $Tw_1Tw_1=1$, and $TR=RT$ from the above relations, and it is easy. □

In the above arguments, we have chosen the basis of $A_1^{(1,1)}$ such as $\{\alpha_0, \alpha_1, \alpha_2\}$. But the Dynkin diagram of $A_1^{(1,1)}$ is given by the following ([1]);



where α_0 and α_1 are the previous ones and $\alpha_0^*=\alpha_0+a$, $\alpha_1^*=\alpha_1+a$. The reflections with respect to α_0^* and α_1^* are expressed as follows;

$$w_0^*:=w_{\alpha_0^*}=w_1TR\ , \qquad w_1^*:=w_{\alpha_1^*}=w_1R^{-1}\ .$$

By simple calculations, we see that $w_0^{*2}=w_1^{*2}=1$, and using these relations, $TR=RT$ is rewritten as $w_0w_0^*w_1w_1^*=1$. So we have the following;

Proposition 2.4. *The relations of w_0, w_1, w_0^* and w_1^* are given as follows;*

$$w_0^2=w_1^2=w_0^{*2}=w_1^{*2}=w_0w_0^*w_1w_1^*=1\ .$$

Proof. It is clear from the fact that we can obtain all relations of w_1, T, and R from the above relations. □

3. An extension of dihedral group D_2 and infinite dihedral group D_∞.

The infinite dihedral group D_∞ ([6], [7], [8]) is the multiplicative group generated by the matrices A, B, where

$$A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

This group is also defined by the following generator and relations;

$$\langle A, B; A^2 = B^2 = 1 \rangle ,$$

and isomorphic to the affine Weyl group of $\tilde{A}_1$. The extended affine Weyl group $W(A_1^{(1,1)})$ contains this group as a subgroup, so we imbed A, B into the 3×3 matrix as follows;

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And further add

$$C' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

for symmetry. Then we have;

PROPOSITION 3.1.

$$\mathrm{gp}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\} \cong W(A_1^{(1,1)}) .$$

PROOF. It suffices to show that $\mathrm{gp}\{w_0, w_1, w_2\} \cong \mathrm{gp}\{A', B', C'\}$. By considering the action on the basis α_i $(0 \le i \le 2)$, w_i $(0 \le i \le 2)$ are expressed as

$$w_0 = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & -1 \end{pmatrix}.$$

Using the matrix

$$X = \begin{pmatrix} a & 0 & 0 \\ -2a & 2a & -2a \\ 0 & 0 & a \end{pmatrix}, \qquad a \neq 0 ,$$

we see that

$$Xw_0X^{-1} = A' , \quad Xw_1X^{-1} = B' , \quad Xw_2X^{-1} = C' . \qquad \square$$

From the previous arguments, it turns out that the Weyl group $W(A_1^{(1,1)})$ is isomorphic to

$$\langle a, b, c; a^2 = b^2 = c^2 = (abc)^2 = 1 \rangle \cong \langle a, b, c, d; a^2 = b^2 = c^2 = d^2 = abcd = 1 \rangle .$$

Now the dihedral group D_2, of order 4, is defined by ([7], [8])

$$\langle a, b; a^2 = b^2 = (ab)^2 = 1 \rangle \cong \langle a, b, c; a^2 = b^2 = c^2 = abc = 1 \rangle ,$$

therefore we can also see that the extended affine Weyl group $W(A_1^{(1,1)})$ is an extension of the dihedral group D_2. Further both D_∞ and D_2 are amenable, but we present the following;

PROPOSITION 3.2. *The extended affine Weyl group $W(A_1^{(1,1)})$ is non-amenable.*

PROOF. In the group $\langle a, b, c, d; a^2=b^2=c^2=d^2=abcd=1\rangle$, we can see that ac and bd are generators of the free group F_2 which is non-amenable. Because $W(A_1^{(1,1)})$ has the subgroup F_2 which is non-amenable, $W(A_1^{(1,1)})$ is also non-amenable. $\square$

4. Definition of the extended affine Lie algebra $\hat{\mathfrak{gl}}_n$.

Let $\mathfrak{gl}_n$ denote the Lie algebra of all $n\times n$ matrices with complex entries acting on $\mathbf{C}^n$ and let $\mathbf{C}[t, t^{-1}, s, s^{-1}]$ denote the ring of Laurent polynomials. We define the two loop algebra $\tilde{\mathfrak{gl}}_n$ as $\mathfrak{gl}_n(\mathbf{C}[t, t^{-1}, s, s^{-1}])$ i.e. as the complex Lie algebra of $n\times n$ matrices with Laurent polynomials as entries. An element of $\tilde{\mathfrak{gl}}_n$ has the form

$$a(t,s)=\sum_{k,l} t^k s^l a_{k,l} \qquad (a_{k,l}\in\mathfrak{gl}_n)\,,$$

where k, l run over a finite subset of $\mathbf{Z}$. The central extension of $\hat{\mathfrak{gl}}_n$ is defined as follows ([17], [18]). We set

$$\hat{\mathfrak{gl}}'_n := \tilde{\mathfrak{gl}}_n \oplus \mathbf{C}c \oplus \mathbf{C}c_2\,.$$

For $X(m,n)=t^m s^n\otimes X$, $Y(k,l)=t^k s^l\otimes Y$, the two-cocycle α is defined by

$$\alpha(X(m,n),\ Y(k,l))=(m+n)\delta_{m+k,0}\delta_{n+l,0}\,\mathrm{tr}(XY)\,, \tag{4.1}$$

where tr denotes the trace in $\tilde{\mathfrak{gl}}_n$. For general elements $a(t,s)$, $b(t,s)\in\tilde{\mathfrak{gl}}_n$ the formula (4.1) can be written as follows:

$$\alpha(a(t,s),b(t,s))=\mathrm{Res}_0\mathrm{tr}\left(s^{-1}\tfrac{da(t,s)}{dt}b(t,s)\right)+\mathrm{Res}_0\mathrm{tr}\left(t^{-1}\tfrac{da(t,s)}{dt}b(t,s)\right),$$

where Res_0 is the coefficient of $(st)^{-1}$.

The commutation relations of the Lie algebra $\hat{\mathfrak{gl}}'_n$ are given by:

$$[a(t,s),c]=[a(t,s),c_2]=[c,c_2]=0\,,$$

$$\begin{aligned}[a(t,s),b(t,s)]&=a(t,s)b(t,s)-b(t,s)a(t,s)\\&\quad+\mathrm{Res}_0\mathrm{tr}\left(s^{-1}\tfrac{da(t,s)}{dt}b(t,s)\right)c+\mathrm{Res}_0\mathrm{tr}\left(t^{-1}\tfrac{da(t,s)}{ds}b(t,s)\right)c_2\,.\end{aligned} \tag{4.2}$$

For the elements $X(m,n)=t^m s^n\otimes X$, $Y(k,l)=t^k s^l\otimes Y$, (4.2) is written as follows:

$$[X(m,n),Y(k,l)]=[X,Y](m+k,n+l)+m\delta_{m+k,0}\delta_{n+l,0}\,\mathrm{tr}(XY)c+n\delta_{m+k,0}\delta_{n+l,0}\,\mathrm{tr}(XY)c_2\,.$$

The bilinear form on $\mathfrak{gl}_n$ defined by

$$(X\mid Y)=\mathrm{tr}(XY)\,, \tag{4.3}$$

is symmetric, non-degenerate and invariant. Now $\mathfrak{gl}_n$ is the Lie algebra of the group GL_n and $(\cdot \mid \cdot)$ has the property of being invariant under the adjoint action Ad of this group:

$$(\mathrm{Ad}(A)(X) \mid \mathrm{Ad}(A)(Y)) = (AXA^{-1} \mid AYA^{-1}) = (X \mid Y) \tag{4.4}$$

for all $A \in GL_n$. We can define a bilinear form on $\widetilde{\mathfrak{gl}}_n$ in analogy with (4.3):

$$(X(m, n) \mid Y(k, l)) = \delta_{m+k,0}\delta_{n+l,0}\mathrm{tr}(XY).$$

This definition extends by linearity to general elements $a(t, s)$, $b(t, s)$ of $\widetilde{\mathfrak{gl}}_n$ as follows:

$$(a(t, s) \mid b(t, s)) = \mathrm{Res}_0(ts)^{-1}\mathrm{tr}(a(t, s)b(t, s)).$$

It is easily checked that $(\cdot \mid \cdot)$ is a symmetric, invariant, non-degenerate, bilinear form on $\widetilde{\mathfrak{gl}}_n$. It also has the property, which is analogous to (4.4), of being invariant under the adjoint action Ad of the group $\widetilde{GL}_n$, where

$$\widetilde{GL}_n = GL_n(\mathbf{C}[t, t^{-1}, s, s^{-1}]),$$

is the group of all invertible $n \times n$ matrices over $\mathbf{C}[t, t^{-1}, s, s^{-1}]$, and the adjoint action of $\widetilde{GL}_n$ on $\hat{\mathfrak{gl}}'_n$ are defined by

$$\mathrm{Ad}\,A(t, s)a(t, s) := A(t, s)a(t, s)A(t, s)^{-1}, \quad \mathrm{Ad}\,A(t, s)c := c, \quad \mathrm{Ad}\,A(t, s)c_2 := c_2,$$

for $a(t, s) \in \widetilde{\mathfrak{gl}}_n$ and $A(t, s) \in \widetilde{GL}_n$.

The form $(\cdot \mid \cdot)$ can be extended to $\hat{\mathfrak{gl}}'_n$ by defining $(c \mid \widetilde{\mathfrak{gl}}_n) = (c_2 \mid \widetilde{\mathfrak{gl}}_n) = (c \mid c) = (c_2 \mid c_2) = (c \mid c_2) = 0$. This definition preserves all the previous properties, except that it is degenerate. It is convenient from several points of view to enlarge $\widetilde{\mathfrak{gl}}_n$ by adding two generators d and d_2: we set

$$\hat{\mathfrak{gl}}_n := \hat{\mathfrak{gl}}'_n \oplus \mathbf{C}d \oplus \mathbf{C}d_2,$$

where the commutation relations with d and d_2 are:

$$\begin{aligned} &[d, c] = [d, c_2] = [d_2, c] = [d_2, c_2] = [d, d_2] = 0, \\ &[d, a(t, s)] = t\tfrac{da(t,s)}{dt}, \qquad [d_2, a(t, s)] = s\tfrac{da(t,s)}{ds}. \end{aligned} \tag{4.5}$$

The Lie algebra $\hat{\mathfrak{gl}}_n$ is called the extended affine Lie algebra associated to $\mathfrak{gl}_n$.

LEMMA 4.1. *The extended affine Lie algebra* $\hat{\mathfrak{gl}}_n = \widetilde{\mathfrak{gl}}_n \oplus \mathbf{C}c \oplus \mathbf{C}c_2 \oplus \mathbf{C}d \oplus \mathbf{C}d_2$ *carries a non-degenerate, symmetric, invariant bilinear form* $(\cdot \mid \cdot)$ *defined by*;

$$\begin{aligned} &(a(t, s) \mid b(t, s)) = \mathrm{Res}_0(ts)^{-1}\mathrm{tr}(a(t, s)b(t, s)) \qquad \textit{for } a(t, s),\ b(t, s) \in \widetilde{\mathfrak{gl}}_n, \\ &(d \mid c) = (d_2 \mid c_2) = 1, \\ &(c \mid a(t, s)) = (c_2 \mid a(t, s)) = (c \mid c) = (c \mid c_2) = (c_2 \mid c_2) = 0, \\ &(d \mid a(t, s)) = (d_2 \mid a(t, s)) = (d \mid d) = (d \mid d_2) = (d_2 \mid d_2) = 0. \end{aligned}$$

PROOF. It is clear from the definition. $\square$

REMARK. In the above statement, *invariant* means $([x, y] \mid z)=(x \mid [y, z])$, not the action of $\widetilde{GL}_n$.

From (4.5), we define adjoint action of $\widetilde{GL}_n$ on $\tilde{\mathfrak{gl}}_n \oplus \mathbf{C}d \oplus \mathbf{C}d_2$ by

$$\mathrm{Ad}(A(t, s))(a(t, s)) := A(t, s)a(t, s)A(t, s)^{-1},$$

$$\mathrm{Ad}(A(t, s))(d) := d - t\tfrac{dA(t,s)}{dt}A^{-1}(t, s),$$

$$\mathrm{Ad}(A(t, s))(d_2) := d_2 - s\tfrac{dA(t,s)}{ds}A^{-1}(t, s),$$

for $a(t, s) \in \tilde{\mathfrak{gl}}_n$, $A(t, s) \in \widetilde{GL}_n$.

Further from the commutation relations of $\hat{\mathfrak{gl}}_n$, we define the adjoint action of $\widetilde{GL}_n$ on $\hat{\mathfrak{gl}}_n$ as follows:

$$\mathrm{Ad}(A(t, s))(c) = c, \qquad \mathrm{Ad}(A(t, s))(c_2) = c_2,$$

$$\mathrm{Ad}(A(t, s))(x(t, s)) = Ax(t, s)A^{-1} + \lambda(A, x)c + \lambda_2(A, x)c_2,$$

$$\mathrm{Ad}(A(t, s))(d) = d - t\tfrac{dA}{dt}A^{-1} + \mu(A)c + \mu_2(A)c_2,$$

$$\mathrm{Ad}(A(t, s))(d_2) = d_2 - s\tfrac{dA}{ds}A^{-1} + \nu(A)c + \nu_2(A)c_2,$$

where $x(t, s) \in \tilde{\mathfrak{gl}}_n$ and $\lambda(A, x)$, $\lambda_2(A, x)$, $\mu(A)$, $\mu_2(A)$, $\nu(A)$, $\nu_2(A) \in \mathbf{C}$.

To decide the coefficients $\lambda(A, x)$, $\lambda_2(A, x)$, $\mu(A)$, $\mu_2(A)$, $\nu(A)$ and $\nu_2(A)$, we demand the $\widetilde{GL}_n$-invariants of the form $(\cdot \mid \cdot)$ on $\hat{\mathfrak{gl}}_n$, and the following cocycle condition: $\mathrm{Ad}(A(t, s)) \cdot \mathrm{Ad}(B(t, s)) = \mathrm{Ad}(A(t, s)B(t, s))$. Then we have

$$\begin{aligned} 0 &= (x \mid d) = (\mathrm{Ad}(A)(x) \mid \mathrm{Ad}(A)(d)) \\ &= (AxA^{-1} + \lambda c + \lambda_2 c_2 \mid d - t\tfrac{dA}{dt}A^{-1} + \mu c + \mu_2 c_2), \end{aligned}$$

from this $\lambda = \mathrm{Res}_0\mathrm{tr}(s^{-1}\tfrac{dA}{dt}xA^{-1})$.

Similarly from $0 = (x \mid d_2) = (\mathrm{Ad}(A)(x) \mid \mathrm{Ad}(A)(d_2))$, we get $\lambda_2 = \mathrm{Res}_0\mathrm{tr}(t^{-1}\tfrac{dA}{ds}xA^{-1})$.

Further from $0 = (d \mid d) = (\mathrm{Ad}(A)(d) \mid \mathrm{Ad}(A)(d))$, $0 = (d_2 \mid d_2) = (\mathrm{Ad}(A)(d_2) \mid \mathrm{Ad}(A)(d_2))$ and $0 = (d \mid d_2) = (\mathrm{Ad}(A)(d) \mid \mathrm{Ad}(A)(d_2))$, we get $\mu(A) = -\tfrac{1}{2}\mathrm{Res}_0\mathrm{tr}(s^{-1}t(\tfrac{dA}{dt}A^{-1})^2)$, $\nu_2(A) = -\tfrac{1}{2}\mathrm{Res}_0\mathrm{tr}(t^{-1}s(\tfrac{dA}{ds}A^{-1})^2)$, and

$$\mu_2(A) + \nu(A) = -\mathrm{Res}_0\mathrm{tr}(\tfrac{dA}{dt}A^{-1}\tfrac{dA}{ds}A^{-1}). \tag{4.6}$$

Further from the cocycle condition, we get

$$\mu_2(A) + \mu_2(B) - \mu_2(AB) = \mathrm{Res}_0\mathrm{tr}(\tfrac{dA}{ds}\tfrac{dB}{dt}B^{-1}A^{-1}), \tag{4.7}$$

$$\nu(A) + \nu(B) - \nu(AB) = \mathrm{Res}_0\mathrm{tr}(\tfrac{dA}{dt}\tfrac{dB}{ds}B^{-1}A^{-1}). \tag{4.8}$$

Further we see that (4.6), (4.7), (4.8) $\Leftrightarrow$ (4.6), (4.7). Now we can describe the adjoint action of $\widetilde{GL}_n$ on $\hat{\mathfrak{gl}}_n$.

PROPOSITION 4.2. *In the above notations, we have*:

$$\mathrm{Ad}(A(t,s))(c)=c\,,\qquad \mathrm{Ad}(A(t,s))(c_2)=c_2\,,$$

$$\mathrm{Ad}(A(t,s))(x(t,s))=Ax(t,s)A^{-1}+\mathrm{Res}_0\mathrm{tr}(s^{-1}\tfrac{dA}{dt}xA^{-1})c+\mathrm{Res}_0\mathrm{tr}(t^{-1}\tfrac{dA}{ds}xA^{-1})c_2\,,$$

$$\mathrm{Ad}(A(t,s))(d)=d-t\tfrac{dA}{dt}A^{-1}-\tfrac{1}{2}\mathrm{Res}_0\mathrm{tr}(s^{-1}t(\tfrac{dA}{dt}A^{-1})^2)c+\mu_2(A)c_2\,,$$

$$\mathrm{Ad}(A(t,s))(d_2)=d_2-s\tfrac{dA}{ds}A^{-1}-\tfrac{1}{2}\mathrm{Res}_0\mathrm{tr}(t^{-1}s(\tfrac{dA}{ds}A^{-1})^2)c_2+\nu(A)c\,,$$

where $\mu_2(A)$ *and* $\nu(A)$ *obey the above conditions* (4.6) *and* (4.7).

REMARK. We can check that these actions define an automorphism of the Lie algebra $\hat{\mathfrak{gl}}_n$.

5. Extended affine Lie algebra $\hat{\mathfrak{sl}}_2$ and its Weyl group.

The restriction of the bilinear form (4.3) on $\mathfrak{gl}_n$ to its subalgebra $\mathfrak{sl}_n$ remains non-degenerate and we have the associated extended affine Lie algebra $\hat{\mathfrak{sl}}_n$. Especially in the case of $\hat{\mathfrak{sl}}_2$, we examine its Weyl group. The Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{sl}}_2$ is spanned by h, c, c_2, d and d_2, where $h=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, i.e.

$$\hat{\mathfrak{h}}=\mathbf{C}h\oplus\mathbf{C}c\oplus\mathbf{C}c_2\oplus\mathbf{C}d\oplus\mathbf{C}d_2\,.$$

The bilinear form $(\cdot\mid\cdot)$ on $\hat{\mathfrak{sl}}_2$ is non-degenerate when restricted to $\hat{\mathfrak{h}}$, and we see from Lemma 4.1;

$$\begin{cases}(h\mid h)=2,\ (c\mid d)=(c_2\mid d_2)=1\,,\\ \text{and all other pairs vanish}\,.\end{cases}$$

We shall identify $\hat{\mathfrak{h}}$ with $\hat{\mathfrak{h}}^*$ via this form. We define the Weyl group $W(\widetilde{SL}_2)$ of $\widetilde{SL}_2:=SL_2(\mathbf{C}[t,t^{-1},s,s^{-1}])$ as follows. Let $H=\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\middle|\, a\in\mathbf{C}^*\right\}$ and $\tilde{N}=\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix},\right.$ $\left.\begin{pmatrix}0 & a\\ -a^{-1} & 0\end{pmatrix}\middle|\, a, a^{-1}\in\mathbf{C}[t,t^{-1},s,s^{-1}]\right\}$ be the subgroups of $\widetilde{SL}_2$, and we set $W(\widetilde{SL}_2):=\tilde{N}/H$. Then we easily see the following.

LEMMA 5.1. $W(\widetilde{SL}_2)$ *is generated by the conjugations by* $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}$ *and* $\begin{pmatrix}s & 0\\ 0 & s^{-1}\end{pmatrix}$.

We define $W(\hat{\mathfrak{sl}}_2)$ by the action of $W(\widetilde{SL}_2)$ on $\hat{\mathfrak{h}}$, and we denote that r_α is conjugation by $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, and t_k and s_k $(k\in\mathbf{Z})$ are conjugations by the k-th powers of $\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}$ and

$\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$, respectively. Since the adjoint action of $\widetilde{SL}_2$ on $\hat{\mathfrak{h}}$ can be determined from Proposition 4.2, we can compute the actions of the elements r_α, t_k and s_k on $\hat{\mathfrak{h}}$ as follows:

$$r_\alpha(h) = -\mathrm{h}\,, \quad r_\alpha(c) = c\,, \quad r_\alpha(c_2) = c_2\,, \quad r_\alpha(d) = d\,, \quad r_\alpha(d_2) = d_2\,,$$
$$t_k(h) = h + 2kc\,, \quad t_k(c) = c\,, \quad t_k(c_2) = c_2\,,$$
$$t_k(d) = d - kh - k^2 c + \mu_k c_2\,, \quad t_k(d_2) = d_2 - \mu_k c\,,$$
$$s_k(h) = h + 2kc_2\,, \quad s_k(c) = c\,, \quad s_k(c_2) = c_2\,,$$
$$s_k(d) = d - \nu_k c_2\,, \quad s_k(d_2) = d_2 - kh - k^2 c_2 + \nu_k c\,,$$

where μ_k and ν_k satisfy the conditions;

$$\mu_k + \mu_l = \mu_{k+l} \quad \text{and} \quad \nu_k + \nu_l = \nu_{k+l} \qquad (k, l \in \mathbf{Z})\,, \tag{4.9}$$

which are obtained by (4.7) and (4.8).

Let $\nu: \hat{\mathfrak{h}} \to \hat{\mathfrak{h}}^*$ be a map defined by the bilinear form $(\ |\ )$, and we put

$$\nu(h) := \alpha\,, \quad \nu(c) := \delta\,, \quad \nu(c_2) := \delta_2\,, \quad \nu(d) := \Lambda_0\,, \quad \nu(d_2) := \Lambda_2\,.$$

With this map, identifying $\hat{\mathfrak{h}}$ with $\hat{\mathfrak{h}}^*$, we write the actions of r_α, t_k and s_k on $\hat{\mathfrak{h}}^*$ as follows:

PROPOSITION 5.2. *In the above notations, we have*:

$$r_\alpha(\alpha) = -\alpha\,, \quad r_\alpha(\delta) = \delta\,, \quad r_\alpha(\delta_2) = \delta_2\,, \quad r_\alpha(\Lambda_0) = \Lambda_0\,, \quad r_\alpha(\Lambda_2) = \Lambda_2\,,$$
$$t_k(\alpha) = \alpha + 2k\delta\,, \quad t_k(\delta) = \delta\,, \quad t_k(\delta_2) = \delta_2\,,$$
$$t_k(\Lambda_0) = \Lambda_0 - k\alpha - k^2\delta + \mu_k\delta_2\,, \quad t_k(\Lambda_2) = \Lambda_2 - \mu_k\delta\,,$$
$$s_k(\alpha) = \alpha + 2k\delta_2\,, \quad s_k(\delta) = \delta\,, \quad s_k(\delta_2) = \delta_2\,,$$
$$s_k(\Lambda_0) = \Lambda_0 - \nu_k\delta_2\,, \quad s_k(\Lambda_2) = \Lambda_2 - k\alpha - k^2\delta_2 + \nu_k\delta\,,$$

where μ_k and ν_k satisfy (4.9).

From this proposition, we see that:

COROLLARY 5.3. (i) $t_n \cdot t_m = t_{n+m}$, $s_n \cdot s_m = s_{n+m}$, (ii) $t_n r_\alpha = r_\alpha t_{-n}$, $s_n r_\alpha = r_\alpha s_{-n}$.

From the pairing on $\hat{\mathfrak{h}}$, we get that on $\hat{\mathfrak{h}}^*$:

$$\begin{cases} (\alpha \mid \alpha) = 2,\ (\delta \mid \Lambda_0) = 1,\ (\delta_2 \mid \Lambda_2) = 1\,, \\ \text{and all of other pairs vanish}\,. \end{cases}$$

We introduce the element γ^n ($n \in \mathbf{Z}$) ([19]), which is defined by

$$\gamma^n(\lambda) = \lambda + n\delta_2 \cdot \delta(\lambda) - n\delta \cdot \delta_2(\lambda)\,, \qquad \text{for} \quad \lambda \in \hat{\mathfrak{h}}^*\,,$$

where $\delta(\lambda)=(\delta \mid \lambda)$, $\delta_2(\lambda)=(\delta_2 \mid \lambda)$. Using this, we obtain

PROPOSITION 5.4. *The commutation relation of t_n and s_m is given by*

$$t_n s_m = \gamma^{2nm} s_m t_n \qquad (n, m \in \mathbf{Z})\,.$$

PROOF. It is easily checked by direct calculations. □

We define the reflection w_β for $\beta \in \Phi$ as the element in $GL(\hat{\mathfrak{h}})$, then we see that $W(\hat{\mathfrak{sl}}_2)$ is the group generated by w_β for all $\beta \in \Phi$, and we have the following.

LEMMA 5.5.

$$w_{\alpha+n\delta+m\delta_2} = \gamma^{nm} s_m t_n w_\alpha\,, \qquad w_{-\alpha+n\delta+m\delta_2} = \gamma^{nm} w_\alpha s_m t_n\,.$$

PROOF. It is easily checked by direct calculations. □

REMARK. We can identify $W(\hat{\mathfrak{sl}}_2)$ with $\tilde{W}(A_1^{(1,1)})$ through the reflection w_β (see [1]).

From the above lemma, we have the following.

LEMMA 5.6.

$$w_0^2 = w_1^2 = w_0^{*2} = w_1^{*2} = 1\,, \qquad w_0 w_0^* w_1 w_1^* = \gamma\,,$$

where $w_1 = w_\alpha$, $w_0 = w_{-\alpha+\delta} = w_1 t_1$, $w_1^* = w_{\alpha+\delta_2} = s_1 w_1$, *and* $w_0^* = w_{-\alpha+\delta+\delta_2} = \gamma w_1 s_1 t_1$.

PROOF. It is easily checked by direct calculations. □

From the above arguments, we see that:

PROPOSITION 5.7. (i) *The Weyl group $W(\hat{\mathfrak{sl}}_2)$ is generated by the four elements* γ, t_1, s_1, w_1.

(ii) *The relations of $W(\hat{\mathfrak{sl}}_2)$ are given by*:

$$\begin{gathered}\gamma s_1 = s_1 \gamma\,, \quad \gamma t_1 = t_1 \gamma\,, \quad \gamma w_1 = w_1 \gamma\,, \quad w_1^2 = 1\,, \\ t_1 s_1 = \gamma^2 s_1 t_1\,, \quad t_1 w_1 t_1 w_1 = 1\,, \quad s_1 w_1 s_1 w_1 = 1\,.\end{gathered}$$

PROOF. From Lemma 5.5 and Lemma 5.6, (i) is clear. So we prove (ii). When we restrict the action of $W(\hat{\mathfrak{sl}}_2)$ on α, δ and δ_2, we see that $t_1 = T^{-1}$, $s_1 = R^{-1}$ and $W(\hat{\mathfrak{sl}}_2)|_{\mathbf{R}\alpha \oplus \mathbf{R}\delta \oplus \mathbf{R}\delta_2} = W(A_1^{(1,1)})$. The relations of w_1, T and R are:

$$w_1^2 = 1\,, \quad T w_1 T w_1 = 1\,, \quad R w_1 R w_1 = 1\,, \quad TR = RT\,. \tag{5.1}$$

We see that w_1, t_1 and s_1 satisfy the following relations in (5.1):

$$w_1^2 = 1\,, \quad t_1 w_1 t_1 w_1 = 1\,, \quad s_1 w_1 s_1 w_1 = 1\,.$$

Further it is easily checked that the element γ commutes with all of w_1, t_1 and s_1. We have seen that $W(\hat{\mathfrak{sl}}_2) \cong \tilde{W}(A^{(1,1)})$ before, so using the result [1], we see that $W(\hat{\mathfrak{sl}}_2)$ is the central extension of $W(A_1^{(1,1)})$ and the double translation part is replaced by a central extension, therefore we get (ii). □

REMARK. The element γ is a central element in $W(\widehat{\mathfrak{sl}}_2)$, and corresponds to a 1-cocycle in [11] and the Coxeter element in [1].

COROLLARY 5.8. *The hyperbolic extension group* $\tilde{W}(A^{(1,1)})$ *is generated by the reflections* w_0, w_1, w_0^* *and* w_1^* *and their relations are given by*:

$$w_0^2 = w_1^2 = w_0^{*2} = w_1^{*2} = 1\ , \quad \text{and}$$

$$w_0 w_0^* w_1 w_1^* = w_0^* w_1 w_1^* w_0 = w_1 w_1^* w_0 w_0^* = w_1^* w_0 w_0^* w_1\ .$$

PROOF. It is easily checked by Lemma 5.6 and Proposition 5.7. □

References

[1] K. SAITO, Extended affine root systems I, Publ. RIMS Kyoto Univ. **21** (1985), 75–179.

[2] K. SAITO, Extended affine root systems II, Publ. RIMS Kyoto Univ. **26** (1990), 15–78.

[3] N. BOURBAKI, *Groupes et Algèbres de Lie*, Hermann (1968); Masson (1981), ch. 4–6.

[4] I. G. MACDONALD, Affine root systems and Dedekind's η-functions, Invent. Math. **15** (1972), 91–143.

[5] N. IWAHORI and H. MATSUMOTO, On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, Inst. Hautes Etudes Sci. Publ. Math. **25** (1965), 5–48.

[6] IAN D. MACDONALD, *The Theory of Groups*, Oxford Univ. Press (1968).

[7] H. S. M. COXETER and W. O. J. MOSER, *Generators and Relations for Discrete Groups*, Fourth ed., Springer (1980).

[8] J. E. HUMPHREYS, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press (1990).

[9] V. G. KAC, *Infinite Dimensional Lie Algebras*, Cambridge Univ. Press (1985).

[10] V. G. KAC and A. K. RAINA, *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras*, Adv. Ser. in Math. Phys. **2** (1987).

[11] J. N. BERNSTEIN and O. V. SCHWARZMAN, Chevalley's theorem for complex crystallographic Coxeter groups, Funct. Anal. Appl. **12** (1978), 79–80.

[12] E. LOOIJENGA, Invariant theory for generalized root systems, Invent. Math. **61** (1980), 1–32.

[13] I. BARS, Vertex operators in mathematics and physics, MSRI Publ. **3** (1983) (Lepowsky et al., eds.), p. 373.

[14] M. WAKIMOTO, Extended affine Lie algebras and a certain series of Hermitian representations, preprint (1985).

[15] L. FRAPPAT, E. RAGOUCY, P. SORBA and F. THULLIER, Generalized Kac-Moody algebras and the differential group of a closed surface, Nuclear Phys. **B 334** (1990), 250–264.

[16] R. H. KROHN and B. TORRESANI, Classification and construction of quasisimple Lie algebras, J. Funct. Anal. **89** (1990), 106–136.

[17] R. V. MOODY, S. E. RAO and T. YOKONUMA, Toroidal Lie algebras and vertex representations, Geom. Dedicata **35** (1990), 283–307.

[18] J. MORITA and Y. YOSHII, Universal central extensions of Chevalley algebras over Laurent polynomial rings and GIM Lie algebras, Proc. Japan Acad. Ser. A Math. Sci. **61** (1985), 179–181.

[19] H. YAMADA, Extended affine Lie algebras and their vertex representations, Publ. Res. Inst. Math. Sci. **25** (1989), 587–603.

[20] A. PRESSLEY and G. SEGAL, *Loop Groups*, Oxford Univ. Press (1986).

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