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A Generalization of the Cauchy-Hua Integral Formula on the Lie Ball

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Introduction.

First we fix notations and review known results following [3].

Put $\mathbf{E}=\mathbf{R}^{n+1}$ and $\tilde{\mathbf{E}}=\mathbf{C}^{n+1}$ $(n\geq 2)$. We put $z\cdot w=z_1w_1+z_2w_2+\cdots+z_{n+1}w_{n+1}$ for $z\in\tilde{\mathbf{E}}$ and $w\in\tilde{\mathbf{E}}$, $\|z\|^2=z\cdot\bar{z}$, and $z^2=z\cdot z$. We denote by

$$L(z)=\sqrt{\|z\|^2+\sqrt{\|z\|^4-|z^2|^2}}$$

the Lie norm of z. We have $|z^2|\leq\|z\|^2\leq L(z)^2$ for $z\in\tilde{\mathbf{E}}$. Let $r>0$. We denote by

$$\tilde{B}(r)=\{z\in\tilde{\mathbf{E}};\ L(z)<r\}$$

the open Lie ball of radius r, or the classical domain of type 4 (see [1]). $\tilde{B}[r]=\{z\in\tilde{\mathbf{E}};\ L(z)\leq r\}$ is called the closed Lie ball of radius r.

Let $\lambda\in\mathbf{C}$. The complex variety

$$\tilde{\mathbf{S}}_\lambda=\{z\in\tilde{\mathbf{E}};\ z^2=\lambda^2\}$$

is called the complex sphere of radius λ. We also call it the complex light cone if $\lambda=0$. Suppose $0<|\lambda|<r$ and put

$$\tilde{\mathbf{S}}_{\lambda,r}=\partial(\tilde{\mathbf{S}}_\lambda\cap\tilde{B}[r])=\{z\in\tilde{\mathbf{S}}_\lambda;\ L(z)=r\}\ , \tag{1}$$

$$\Sigma_{\lambda,r}=\{e^{i\theta}z;\ \theta\in\mathbf{R},\ z\in\tilde{\mathbf{S}}_{\lambda,r}\}\ . \tag{2}$$

$\tilde{\mathbf{S}}_{\lambda,r}$ and $\Sigma_{\lambda,r}$ are real analytic manifolds, $\dim_{\mathbf{R}}\tilde{\mathbf{S}}_{\lambda,r}=2n-1$, and $\dim_{\mathbf{R}}\Sigma_{\lambda,r}=2n$.

If $|\lambda|$ tends to r, then $\tilde{\mathbf{S}}_{\lambda,r}$ reduces to $\mathbf{S}_\lambda=\{\lambda x;\ x\in\mathbf{E},\ x^2=1\}$ the n-dimensional sphere of radius λ and $\Sigma_{\lambda,r}$ reduces to $\Sigma_r=\{e^{i\theta}x;\ \theta\in\mathbf{R},\ x\in\mathbf{S}_r\}$ the Lie sphere of radius r (see [2]). Note that $\dim_{\mathbf{R}}\mathbf{S}_\lambda=n$ and $\dim_{\mathbf{R}}\Sigma_\lambda=n+1$.

Let $\mathcal{O}(\tilde{B}(r))$ be the space of holomorphic functions on $\tilde{B}(r)$ and $\mathcal{O}(\tilde{B}[r])$ the space of germs of holomorphic functions on $\tilde{B}[r]$. We endow $\mathcal{O}(\tilde{B}(r))$ with the topology of uniform convergence on compact sets and $\mathcal{O}(\tilde{B}[r])$ with the locally convex inductive limit topology:

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$$\mathcal{O}(\tilde{B}[r]) = \operatorname{ind\,lim}\{\mathcal{O}(\tilde{B}(r'); r' > r\} .$$

It is known that Σ_r is the Shilov boundary of $\tilde{B}[r]$ and we have the Cauchy-Hua integral formula for $G \in \mathcal{O}(\tilde{B}(r))$:

$$G(z) = \text{s.}\int_{\Sigma_r} H_r(z, w)G(w)d\dot{w} , \qquad z \in \tilde{B}(r) , \tag{3}$$

where

$$H_r(z, w) = H_1(z/r, w/r) \quad \text{and} \quad H_1(z, w) = \frac{1}{(1 - 2z \cdot \bar{w} + z^2\bar{w}^2)^{(n+1)/2}} \tag{4}$$

is the Cauchy-Hua kernel, and $\text{s.}\int_{\Sigma_r} \cdots d\dot{w}$ is the symbolic integral form (10) on Σ_r. Note that $H_r(z, w)$ is defined on the domain

$$\Omega_r = \{(z, w) \in \tilde{\mathbf{E}} \times \tilde{\mathbf{E}};\ L(z)L(w) < r^2\} , \tag{5}$$

and holomorphic in z, and satisfies $H_r(z, w) = \overline{H_r(w, z)}$.

Let $\mathcal{O}_\Delta(\tilde{B}(r)) = \{G \in \mathcal{O}(\tilde{B}(r));\ \Delta_z G(z) = 0\}$ be the space of complex harmonic functions on $\tilde{B}(r)$, where $\Delta_z = \partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 + \cdots + \partial^2/\partial z_{n+1}^2$ is the complex Laplacian. If $G \in \mathcal{O}_\Delta(\tilde{B}(r))$, then we have the (classical) Poisson integral formula

$$G(z) = \text{s.}\int_{\mathbf{S}_r} K_r(z, x)G(x)d\dot{x}, \qquad z \in \tilde{B}(r) ,$$

where

$$K_r(z, w) = K_1(z/r, w/r) \quad \text{and} \quad K_1(z, w) = \frac{1 - z^2\bar{w}^2}{(1 - 2z \cdot \bar{w} + z^2\bar{w}^2)^{(n+1)/2}} \tag{6}$$

is the (classical) Poisson kernel, and $\text{s.}\int_{\mathbf{S}_r} \cdots d\dot{x}$ is the symbolic integral form (17) on $\mathbf{S}_r$. Note that $K_r(z, w)$ is defined on Ω_r, holomorphic and complex harmonic in z, and satisfies $K_r(z, w) = \overline{K_r(w, z)}$.

We define the (generalized) Poisson kernel $K_{\lambda,r}(z, w)$ in Lemma 2. The kernel $K_{\lambda,r}(z, w)$ is also defined on Ω_r, holomorphic and complex harmonic in z, and satisfies $K_{\lambda,r}(z, w) = \overline{K_{\lambda,r}(w, z)}$. For $G \in \mathcal{O}_\Delta(\tilde{B}(r))$ we have the (generalized) Poisson integral formula

$$G(z) = \text{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} K_{\lambda,r}(z, w)G(w)d\dot{w} , \qquad z \in \tilde{B}(r) ,$$

where $\text{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} \cdots d\dot{w}$ denotes the symbolic integral form (16) on $\tilde{\mathbf{S}}_{\lambda,r}$ (see Theorem 3 for the precise statement).

Because the degenerated case $\lambda = 0$ was treated in [4], we suppose, in this paper, $0 < |\lambda| < r$ and consider the function

$$H_{\lambda,r}(z, w) = \frac{K_{\lambda,r}(z, w)}{1 - (z/\lambda)^2(\bar{w}/\lambda)^2} . \tag{7}$$

Then the function $H_{\lambda,r}(z, w)$ is defined on the domain

$$\Omega_{\lambda,r}=\{(z, w)\in \tilde{\mathbf{E}}\times\tilde{\mathbf{E}};\ |z^2||w^2|<|\lambda|^4,\ L(z)L(w)<r^2\}$$

and holomorphic in z, and satisfies $H_{\lambda,r}(z, w)=\overline{H_{\lambda,r}(w, z)}$. Consider the open subset

$$\tilde{B}(\lambda, r)=\{z\in\tilde{B}(r);\ |z^2|<|\lambda|^2\}$$

of $\tilde{B}(r)$. Note that $\tilde{B}(\lambda, r)$ tends to $\tilde{B}(r)$ as $|\lambda|$ tends to r. We denote by $\mathcal{O}(\tilde{B}(\lambda, r))$ the space of holomorphic functions on $\tilde{B}(\lambda, r)$ endowed with the topology of uniform convergence on compact sets. For $G\in\mathcal{O}(\tilde{B}(\lambda, r))$ we shall prove the (generalized) Cauchy-Hua integral formula

$$G(z)=\mathrm{s.}\int_{\Sigma_{\lambda,r}} H_{\lambda,r}(z, w)G(w)d\dot{w}\,,\qquad z\in\tilde{B}(\lambda, r)\,, \tag{8}$$

where $\mathrm{s.}\int_{\Sigma_{\lambda,r}}\cdots d\dot{w}$ is the symbolic integral form (9) on $\Sigma_{\lambda,r}$ (see Theorem 6).

In the last §5, we shall calculate the Cauchy-Hua kernel $H_{\lambda,r}(z, w)$ and the Poisson kernel $K_{\lambda,r}(z, w)$ in one-dimensional case.

1. Symbolic integral form on $\Sigma_{\lambda,r}$.

Let $0<|\lambda|<r$. We put

$$\tilde{B}[\lambda, r]=\{z\in\tilde{\mathbf{E}};\ |z^2|\le|\lambda|^2,\ L(z)\le r\}$$

and denote by $\mathcal{O}(\tilde{B}[\lambda, r])$ the space of germs of holomorphic functions on $\tilde{B}[\lambda, r]$ equipped with the locally convex inductive limit topology. The set $\tilde{B}(\lambda, r)$ is balanced; that is, if $z\in\tilde{B}(\lambda, r)$ and $t\in\mathbf{C}$, $|t|\le 1$, then we have $tz\in\tilde{B}(\lambda, r)$.

Let $F\in\mathcal{O}(\tilde{B}[\lambda, r])$ and $G(z)\in\mathcal{O}(\tilde{B}(\lambda, r))$. Because $\tilde{B}(\lambda, r)$ is balanced, we can expand F and G in homogeneous polynomials:

$$F(z)=\sum_{j=0}^{\infty}F_j(z)\,,\qquad G(z)=\sum_{j=0}^{\infty}G_j(z)\,,$$

where the convergence is in respective topology. Take $t>1$ sufficiently close to 1 such that $F(tz)$ and $G(t^{-1}z)$ are defined on $\Sigma_{\lambda,r}$. Then by the orthogonality of exponential functions on $[0, 2\pi]$, we have

$$\begin{aligned}\int_{\Sigma_{\lambda,r}}\overline{F(tz)}G(t^{-1}z)d\dot{z}&=\int_{\Sigma_{\lambda,r}}\sum_{j=0}^{\infty}\overline{F_j(tz)}\sum_{k=0}^{\infty}G_k(t^{-1}z)d\dot{z}\\&=\frac{1}{2\pi}\int_0^{2\pi}d\theta\int_{\tilde{\mathbf{S}}_{\lambda,r}}\sum_{j=0}^{\infty}\overline{F_j(te^{i\theta}z')}\sum_{k=0}^{\infty}G_k(t^{-1}e^{i\theta}z')d\dot{z}'\\&=\sum_{j,k}\frac{1}{2\pi}\int_0^{2\pi}e^{i(k-j)\theta}d\theta\int_{\tilde{\mathbf{S}}_{\lambda,r}}\overline{F_j(tz')}G_k(t^{-1}z')d\dot{z}'\end{aligned}$$

$$=\sum_{j=0}^{\infty}\int_{\tilde{\mathbf{S}}_{\lambda,r}}\overline{F_j(z')}G_j(z')\dot{d}z',$$

where $\dot{d}z$ and $\dot{d}z'$ denote the normalized invariant measure on $\Sigma_{\lambda,r}$ and that on $\tilde{\mathbf{S}}_{\lambda,r}$.

Therefore, for $F\in\mathcal{O}(\tilde{B}[\lambda,r])$ and $G\in\mathcal{O}(\tilde{B}(\lambda,r))$ the integral

$$\int_{\Sigma_{\lambda,r}}\overline{F(tz)}G(t^{-1}z)\dot{d}z$$

is defined for $t>1$ sufficiently close to 1 and independent of t. Define the symbolic integral form on $\Sigma_{\lambda,r}$ by

$$\text{s.}\int_{\Sigma_{\lambda,r}}\overline{F(z)}G(z)\dot{d}z=\int_{\Sigma_{\lambda,r}}\overline{F(tz)}G(t^{-1}z)\dot{d}z, \tag{9}$$

where $t>1$ is sufficiently close to 1. Note that

$$\text{s.}\int_{\Sigma_{\lambda,r}}\overline{F(z)}G(z)\dot{d}z=\int_{\Sigma_{\lambda,r}}\overline{F(z)}G(z)\dot{d}z$$

if both F and G belong to $\mathcal{O}(\tilde{B}[\lambda,r])$.

The symbolic integral form $\text{s.}\int_{\Sigma_{\lambda,r}}\overline{F(z)}G(z)\dot{d}z$ is a separately continuous sesquilinear form on $\mathcal{O}(\tilde{B}[\lambda,r])\times\mathcal{O}(\tilde{B}(\lambda,r))$.

Similarly, when $\lambda=r$, we can define the symbolic integral formula

$$\text{s.}\int_{\Sigma_r}\overline{F(z)}G(z)\dot{d}z \tag{10}$$

for $F\in\mathcal{O}(\tilde{B}[r])$ and $G\in\mathcal{O}(\tilde{B}(r))$.

REMARK. If we consider

$$\tilde{B}(0,r)=\{z\in\tilde{\mathbf{S}}_0;\ L(z)<r\}=\tilde{\mathbf{S}}_0(r),$$

$$\Sigma_{0,r}=\{z\in\tilde{\mathbf{S}}_0;\ L(z)=r\}=\tilde{\mathbf{S}}_{0,r},$$

then the results of this section still hold (see [4]).

2. Symbolic integral form on $\tilde{\mathbf{S}}_{\lambda,r}$.

We denote by $\mathscr{P}^k(\tilde{\mathbf{E}})$ the space of k-homogeneous polynomials on $\tilde{\mathbf{E}}$ and by $\mathscr{P}^k_\Delta(\tilde{\mathbf{E}})$ the space of k-homogeneous harmonic polynomials on $\tilde{\mathbf{E}}$:

$$\mathscr{P}^k_\Delta(\tilde{\mathbf{E}})=\{F\in\mathscr{P}^k(\tilde{\mathbf{E}});\ \Delta_z F=0\}.$$

Put $N(k,n)=\dim_{\mathbf{C}}\mathscr{P}^k_\Delta(\tilde{\mathbf{E}})$. It is known that

$$N(k,n)=\frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}=O(k^{n-1}). \tag{11}$$

Let $P_{k,n}(t)$ be the Legendre polynomial of degree k and of dimension $n+1$, and $\tilde{P}_{k,n}(z, w)$ the extended Legendre polynomial defined by

$$\tilde{P}_{k,n}(z, w)=(\sqrt{z^2})^k(\sqrt{\bar{w}^2})^k P_{k,n}\left(\frac{z}{\sqrt{z^2}}\cdot\frac{\bar{w}}{\sqrt{\bar{w}^2}}\right), \qquad z, w\in\tilde{\mathbf{E}} .$$

It is known (see [5] or [3]) that

$$|\tilde{P}_{k,n}(z, w)|\leq L(z)^k L(w)^k , \qquad z, w\in\tilde{\mathbf{E}} . \tag{12}$$

We have the following reproducing formula for $F_k\in\mathscr{P}^k_\Delta(\tilde{\mathbf{E}})$:

$$F_k(z)=\frac{N(k, n)}{r^{2k}}\int_{\mathbf{S}_r}\tilde{P}_{k,n}(z, x)F_k(x)d\dot{x} , \qquad z\in\tilde{\mathbf{E}} . \tag{13}$$

For $|\lambda|<r$ we put

$$L_{k,\lambda,r}=|\lambda|^{2k}P_{k,n}\left(\frac{1}{2}\left(\frac{|\lambda|^2}{r^2}+\frac{r^2}{|\lambda|^2}\right)\right), \qquad \lambda\neq 0 ,$$

$$L_{k,0,r}=\frac{r^{2k}}{2^k}\gamma_{k,n} ,$$

where $\gamma_{k,n}$ is the principal coefficient of the Legendre polynomial $P_{k,n}(t)$:

$$\gamma_{k,n}=\frac{1}{N(k, n)}\frac{\Gamma(k+(n+1)/2)}{\Gamma((n+1)/2)}\frac{2^k}{k!} . \tag{14}$$

Note that $L_{k,\lambda,r}$ is continuous at $\lambda=0$ and that

$$\lim_{|\lambda|\to r} L_{k,\lambda,r}=r^{2k} .$$

It is known (see [5] or [3]) that

$$2^{-k}r^{2k}\gamma_{k,n}\leq L_{k,\lambda,r}\leq r^{2k} . \tag{15}$$

R. Wada [6] generalized (13) as follows:

THEOREM 1. (i) *If* $k\neq j$, *then* $F_k\in\mathscr{P}^k_\Delta(\tilde{\mathbf{E}})$ *and* $F_j\in\mathscr{P}^j_\Delta(\tilde{\mathbf{E}})$ *are orthogonal*:

$$\int_{\tilde{\mathbf{S}}_{\lambda,r}}\overline{F_k(w)}F_j(w)d\dot{w}=0 .$$

(ii) *Let* $F_k\in\mathscr{P}^k_\Delta(\tilde{\mathbf{E}})$. *Then we have the reproducing formula*:

$$F_k(z)=\frac{N(k, n)}{L_{k,\lambda,r}}\int_{\tilde{\mathbf{S}}_{\lambda,r}}\tilde{P}_{k,n}(z, w)F_k(w)d\dot{w} , \qquad z\in\tilde{\mathbf{E}} .$$

For $F\in\mathscr{O}_\Delta(\tilde{B}[r])$ and $G\in\mathscr{O}_\Delta(\tilde{B}(r))$ the integral

$$\int_{\tilde{\mathbf{S}}_{\lambda,r}} \overline{F(tz)} G(t^{-1}z) d\dot{z}$$

is defined for $t > 1$ sufficiently close to 1 and independent of t. In fact, we have

$$\begin{aligned}\int_{\tilde{\mathbf{S}}_{\lambda,r}} \overline{F(tz)} G(t^{-1}z) d\dot{z} &= \int_{\tilde{\mathbf{S}}_{\lambda,r}} \sum_{j=0}^{\infty} \overline{F_j(tz)} \sum_{k=0}^{\infty} G_k(t^{-1}z) d\dot{z} \\ &= \sum_{j=0}^{\infty} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \overline{F_j(z)} G_j(z) d\dot{z}\end{aligned}$$

because of Theorem 1 (i).

Therefore, we can define the symbolic integral form on $\tilde{\mathbf{S}}_{\lambda,r}$ by

$$\text{s.} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \overline{F(z)} G(z) d\dot{z} = \int_{\tilde{\mathbf{S}}_{\lambda,r}} \overline{F(tz)} G(t^{-1}z) d\dot{z} \tag{16}$$

for $F \in \mathcal{O}_\Delta(\tilde{B}[r])$ and $G \in \mathcal{O}_\Delta(\tilde{B}(r))$, where $t > 1$ is sufficiently close to 1. Note that we have

$$\text{s.} \int_{\Sigma_{\lambda,r}} \overline{F(z)} G(z) d\dot{z} = \text{s.} \int_{\tilde{\mathbf{S}}_{\lambda,r}} \overline{F(z)} G(z) d\dot{z}$$

for $F \in \mathcal{O}_\Delta(\tilde{B}[r])$ and $G \in \mathcal{O}_\Delta(\tilde{B}(r))$.

The symbolic integral form $\text{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} \overline{F(z)} G(z) d\dot{z}$ is a separately continuous sesquilinear form on $\mathcal{O}_\Delta(\tilde{B}[r]) \times \mathcal{O}_\Delta(\tilde{B}(r))$.

Similarly, when $\lambda = r$, we can define the symbolic integral form

$$\text{s.} \int_{\mathbf{S}_r} \overline{F(x)} G(x) d\dot{x} \tag{17}$$

for $F \in \mathcal{O}_\Delta(\tilde{B}[r])$ and $G \in \mathcal{O}_\Delta(\tilde{B}(r))$.

3. Integral representation of complex harmonic functions on $\tilde{B}(r)$.

LEMMA 2. *Define the (generalized) Poisson kernel $K_{\lambda,r}(z, w)$ by*

$$K_{\lambda,r}(z, w) = \sum_{k=0}^{\infty} \frac{N(k, n)\tilde{P}_{k,n}(z, w)}{L_{k,\lambda,r}} . \tag{18}$$

Then $K_{\lambda,r}(z, w)$ is defined on Ω_r, $K_{\lambda,r}(z, w) = \overline{K_{\lambda,r}(w, z)}$, and holomorphic and complex harmonic in z:

$$\Delta_z K_{\lambda,r}(z, w) = 0 .$$

PROOF. Because of (11), (12), (14) and (15) we have

$$\left| \frac{N(k, n)\tilde{P}_{k,n}(z, w)}{L_{k,\lambda,r}} \right| \leq C \frac{k^{n-1} L(z)^k L(w)^k}{r^{2k}} .$$

Therefore, the convergence being uniform on compact sets in Ω_r, $K_{\lambda,r}(z, w)$ is holomorphic in z there. The summands are Hermitian symmetric and complex harmonic in z, so is $K_{\lambda,r}(z, w)$. (q.e.d.)

REMARK. If $\lambda=0$, then we have

$$K_{0,1}(z, w)=\sum_{k=0}^{\infty} N(k, n)2^k(z \cdot \bar{w})^k=\frac{1+2(z \cdot \bar{w})}{(1-2(z \cdot \bar{w}))^n}$$

if $L(z)L(w)<1$ and $z^2w^2=0$ (see [7] and [4]). Letting $\lambda \to r$ we have

$$K_{\lambda,r}(z, w) \to K_r(z, w)\,,$$

where $K_r(z, w)$ is the classical Poisson kernel defined by (6).

THEOREM 3. *Let $0<|\lambda|<r$. For $G \in \mathcal{O}_\Delta(\tilde{B}(r))$ we have the (generalized) Poisson integral formula:*

$$G(z)=\mathrm{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} K_{\lambda,r}(z, w)G(w)d\dot{w}\,, \qquad z \in \tilde{B}(r)\,. \tag{19}$$

PROOF. Let $G \in \mathcal{O}_\Delta(\tilde{B}(r))$ and $G(z)=\sum_{j=0}^{\infty} G_j(z)$ be the expansion in homogeneous polynomials. Then $G_j \in \mathcal{P}_\Delta^j(\tilde{\mathbf{E}})$ and we have, for $t>1$,

$$\int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{P}_{k,n}(z, tw)G(t^{-1}w)d\dot{w}=\int_{\tilde{\mathbf{S}}_{\lambda,r}} \sum_{j=0}^{\infty} \tilde{P}_{k,n}(z, tw)G_j(t^{-1}w)d\dot{w}$$

$$=\int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{P}_{k,n}(z, w)G_k(w)d\dot{w}=\frac{L_{k,\lambda,r}}{N(k, n)}\,G_k(z)\,.$$

Let $z \in \tilde{B}(r)$. Then the funcion $w \mapsto \overline{K_{\lambda,r}(z, w)}$ is complex harmonic in a neighborhood of $\tilde{B}[r]$. So the symbolic integral form in (19) is well-defined. We have

$$G(z)=\sum_{k=0}^{\infty} G_k(z)=\sum_{k=0}^{\infty} \frac{N(k, n)}{L_{k,\lambda,r}}\mathrm{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} \tilde{P}_{k,n}(z, w)G(w)d\dot{w}$$

$$=\mathrm{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} \sum_{k=0}^{\infty} \frac{N(k, n)}{L_{k,\lambda,r}}\tilde{P}_{k,n}(z, w)G(w)d\dot{w}=\mathrm{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} K_{\lambda,r}(z, w)G(w)d\dot{w} \qquad \text{(q.e.d.)}$$

COROLLARY 4. *Let $F \in \mathcal{O}_\Delta(\tilde{B}[r])$ be given. Then*

$$\Lambda_F\colon G \mapsto \mathrm{s.}\int_{\tilde{\mathbf{S}}_{\lambda,r}} \overline{F(w)}G(w)d\dot{w}$$

is a continuous linear functional on $\mathcal{O}_\Delta(\tilde{B}(r))$.

Conversely, let $\Lambda \in \mathcal{O}'_\Delta(\tilde{B}(r))$ be given. We define the Poisson transform $\mathcal{P}_{\lambda,r}\Lambda$ of Λ by

$$(\mathcal{P}_{\lambda,r}\Lambda)(w)=\overline{\langle \Lambda_z, K_{\lambda,r}(z, w)\rangle}\,.$$

Then $\mathscr{P}_{\lambda,r}\Lambda \in \mathcal{O}_\Delta(\tilde{B}[r])$ *and we have*

$$\langle \Lambda, G\rangle = \mathrm{s}.\int_{\tilde{\mathbf{S}}_{\lambda,r}} \overline{(\mathscr{P}_{\lambda,r}\Lambda)(w)}G(w)d\dot{w}\,, \qquad G\in\mathcal{O}_\Delta(\tilde{B}(r))\,.$$

Then Poisson transformation $\mathscr{P}_{\lambda,r}: \mathcal{O}'_\Delta(\tilde{B}(r)) \to \mathcal{O}_\Delta(\tilde{B}[r])$ *is a topological antilinear isomorphism.*

REMARK. The corresponding result with $\lambda=0$ can be found in [4].

4. Integral representation of holomorphic functions on $\tilde{B}(\lambda, r)$.

Let $0<|\lambda|<r$ and $G\in\mathcal{O}(\tilde{B}(\lambda, r))$. Expand G in homogeneous polynomials:

$$G(z)=\sum_{k=0}^{\infty} G_k(z)\,, \qquad z\in\tilde{B}(\lambda, r)\,,$$

where $G_k\in\mathscr{P}^k(\tilde{\mathbf{E}})$ and the convergence is uniform on compact sets in $\tilde{B}(\lambda, r)$. It is well-known that a k-homogeneous polynomial G_k can be expanded as follows:

$$G_k(z)=\sum_{l=0}^{[k/2]}(z^2)^l G_{k,k-2l}(z)\,, \qquad z\in\tilde{\mathbf{E}}\,,$$

where $G_{k,k-2l}\in\mathscr{P}^{k-2l}_\Delta(\tilde{\mathbf{E}})$.

Theorem 1 implies

$$\begin{aligned}\int_{\tilde{\mathbf{S}}_{\lambda,r}}(\bar{w}^2)^l\tilde{P}_{k-2l,n}(z, w)G_k(w)d\dot{w} &= \int_{\tilde{\mathbf{S}}_{\lambda,r}}(\bar{w}^2)^l\tilde{P}_{k-2l,n}(z, w)\sum_{l'=0}^{[k/2]}(w^2)^{l'}G_{k,k-2l'}(w)d\dot{w}\\ &=|\lambda|^{4l}\int_{\tilde{\mathbf{S}}_{\lambda,r}}\tilde{P}_{k-2l,n}(z, w)G_{k,k-2l}(w)d\dot{w}\\ &=|\lambda|^{4l}\frac{L_{k-2l,\lambda,r}}{N(k-2l, n)}G_{k,k-2l}(z)\,.\end{aligned}$$

Therefore, we have

$$\begin{aligned}G_k(z) &= \sum_{l=0}^{[k/2]}(z^2)^l G_{k,k-2l}(z)\\ &=\sum_{l=0}^{[k/2]}\frac{(z^2)^l N(k-2l,n)}{|\lambda|^{4l}L_{k-2l,\lambda,r}}\int_{\tilde{\mathbf{S}}_{\lambda,r}}(\bar{w}^2)^l\tilde{P}_{k-2l,n}(z, w)G_k(w)d\dot{w}\\ &=\int_{\tilde{\mathbf{S}}_{\lambda,r}}H_{\lambda,r,k}(z, w)G_k(w)d\dot{w}\,,\end{aligned}$$

where we put

$$H_{\lambda,r,k}(z,w)=\sum_{l=0}^{[k/2]}\frac{1}{|\lambda|^{4l}}\frac{N(k-2l,n)}{L_{k-2l,\lambda,r}}(z^2)^l(\bar{w}^2)^l\tilde{P}_{k-2l,n}(z,w)\,. \tag{20}$$

The polynomial $H_{\lambda,r,k}(z,w)$ is the reproducing kernel for $\mathscr{P}^k(\tilde{\mathbf{E}})$ with respect to the bilinear form $\int_{\tilde{\mathbf{S}}_{\lambda,r}}\cdots d\dot{w}$. $H_{\lambda,r,k}(z,w)$ is a homogeneous polynomial of degree k with respect to z and with respect to $\bar{w}$. We have

$$H_{\lambda,r,k}(z,w)=\overline{H_{\lambda,r,k}(w,z)}\,.$$

REMARK (see [1]). Because $\lim_{\lambda\to r}L_{k-2l,\lambda,r}=r^{2k-4l}$, we have

$$\begin{aligned}\lim_{\lambda\to r}H_{\lambda,r,k}(z,w)&=\frac{1}{r^{2k}}\sum_{l=0}^{[k/2]}N(k-2l,n)(z^2)^l(\bar{w}^2)^l\tilde{P}_{k-2l,n}(z,w)\\&=\frac{1}{r^{2k}}\frac{n+1}{2k+n+1}N(k,n+2)\tilde{P}_{k,n+2}(z,w)\\&=\frac{n+1}{2k+n+1}N(k,n+2)\tilde{P}_{k,n+2}(z/r,w/r)\,.\end{aligned}$$

By the orthogonality of exponential functions on $[0,2\pi]$, we have, for $G_j\in\mathscr{P}^j(\tilde{\mathbf{E}})$,

$$\frac{1}{2\pi}\int_0^{2\pi}H_{\lambda,r,k}(z,e^{i\theta}w)G_j(e^{i\theta}w)d\theta=\begin{cases}0\,, & j\neq k\,,\\ H_{\lambda,r,k}(z,w)G_k(w)\,, & j=k\,.\end{cases}$$

Therefore, for $G\in\mathcal{O}(\tilde{B}(\lambda,r))$ we have

$$\text{s.}\int_{\Sigma_{\lambda,r}}H_{\lambda,r,k}(z,w)G(w)d\dot{w}=\int_{\tilde{\mathbf{S}}_{\lambda,r}}H_{\lambda,r,k}(z,w)G_k(w)d\dot{w}=G_k(z)\,.$$

This means that the kernel $H_{\lambda,r,k}(z,w)$ gives the projection of $\mathcal{O}(\tilde{B}(\lambda,r))$ onto $\mathscr{P}^k(\tilde{\mathbf{E}})$.

LEMMA 5. *Define the (generalized) Cauchy-Hua kernel $H_{\lambda,r}(z,w)$ by*

$$H_{\lambda,r}(z,w)=\sum_{k=0}^{\infty}H_{\lambda,r,k}(z,w)\,.$$

Then we have

$$H_{\lambda,r}(z,w)=\frac{K_{\lambda,r}(z,w)}{1-(z/|\lambda|)^2(\bar{w}/|\lambda|)^2}\,.$$

$H_{\lambda,r}(z,w)$ is defined on the domain

$$\Omega_{\lambda,r}=\{(z,w)\in\tilde{\mathbf{E}}\times\tilde{\mathbf{E}};\,|z^2||w^2|<|\lambda|^4,\,L(z)L(w)<r^2\}$$

and holomorphic in z, and satisfies $H_{\lambda,r}(z,w)=\overline{H_{\lambda,r}(w,z)}$.

PROOF. We have

$$H_{\lambda,r}(z,w)=\sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}\frac{1}{|\lambda|^{4l}}\frac{N(k-2l,n)}{L_{k-2l,\lambda,r}}(z^2)^l(\bar{w}^2)^l\tilde{P}_{k-2l,n}(z,w)$$

$$=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{|\lambda|^{4l}}\frac{N(m,n)}{L_{m,\lambda,r}}(z^2)^l(\bar{w}^2)^l\tilde{P}_{m,n}(z,w)$$

$$=\sum_{l=0}^{\infty}\frac{(z^2)^l(\bar{w}^2)^l}{|\lambda|^{4l}}\sum_{m=0}^{\infty}\frac{N(m,n)}{L_{m,\lambda,r}}\tilde{P}_{m,n}(z,w)$$

$$=\frac{K_{\lambda,r}(z,w)}{1-(z/|\lambda|)^2(\bar{w}/|\lambda|)^2}.$$

The second statement is clear by Lemma 2. (q.e.d.)

THEOREM 6. *Let* $0<|\lambda|<r$. *If* $G\in\mathcal{O}(\tilde{B}(\lambda,r))$, *then we have the following (generalized) Cauchy-Hua integral formula*:

$$G(z)=\mathrm{s.}\int_{\Sigma_{\lambda,r}}H_{\lambda,r}(z,w)G(w)d\dot{w}\,,\qquad z\in\tilde{B}(\lambda,r)\,. \tag{21}$$

PROOF. Let $z\in\tilde{B}(\lambda,r)$. By Lemma 5, the function $w\mapsto H_{\lambda,r}(z,w)$ is a holomorphic function in a neighborhood of $\tilde{B}[\lambda,r]$. Therefore, the symbolic integral form in (21) is well-defined. We have

$$G(z)=\sum_{k=0}^{\infty}G_k(z)$$

$$=\sum_{k=0}^{\infty}\mathrm{s.}\int_{\Sigma_{\lambda,r}}H_{\lambda,r,k}(z,w)G(w)d\dot{w}$$

$$=\mathrm{s.}\int_{\Sigma_{\lambda,r}}\left(\sum_{k=0}^{\infty}H_{\lambda,r,k}(z,w)\right)G(w)d\dot{w}$$

$$=\mathrm{s.}\int_{\Sigma_{\lambda,r}}H_{\lambda,r}(z,w)G(w)d\dot{w}\,.$$ (q.e.d.)

REMARK. Tending $\lambda\to r$, we have

$$H_{\lambda,r}(z,w)\to H_r(z,w)\,,$$

where $H_r(z,w)$ is the Canchy-Hua kernel given by (4). In fact, we have

$$H_r(z,w)=\sum_{k=0}^{\infty}\frac{1}{r^{2k}}\frac{n+1}{2k+n+1}N(k,n+2)\tilde{P}_{k,n+2}(z,w)$$

by the Gegenbauer generating formula (see [3]).

COROLLARY 7. *Let $F \in \mathcal{O}(\tilde{B}[\lambda, r])$ be given. Then*

$$T_F : G \mapsto \mathrm{s.}\int_{\Sigma_{\lambda,r}} \overline{F(w)} G(w) d\dot{w}$$

is a continuous linear functional on $\mathcal{O}(\tilde{B}(\lambda, r))$.

Conversely, let $T \in \mathcal{O}'(\tilde{B}(\lambda, r))$ be given. If we define the Cauchy-Hua transform $\mathscr{C}_{\lambda,r}T$ of T by

$$(\mathscr{C}_{\lambda,r}T)(w) = \overline{\langle T_z, H_{\lambda,r}(z, w)\rangle} .$$

Then $\mathscr{C}_{\lambda,r}T \in \mathcal{O}(\tilde{B}[\lambda, r])$ and we have

$$\langle T, G\rangle = \mathrm{s.}\int_{\Sigma_{\lambda,r}} \overline{(\mathscr{C}_{\lambda,r}T)(w)} G(w) d\dot{w} , \qquad G \in \mathcal{O}(\tilde{B}(\lambda, r)) .$$

The Cauchy-Hua transformation $\mathscr{C}_{\lambda,r} : \mathcal{O}'(\tilde{B}(\lambda, r)) \to \mathcal{O}(\tilde{B}[\lambda, r])$ is a topological antilinear isomorphism.

REMARK. The following restriction mapping is a linear topological isomorphism.

$$\mathcal{O}_\Delta(\tilde{B}(r)) \xrightarrow{\sim} \mathcal{O}_\Delta(\tilde{B}(\lambda, r)) ,$$

where

$$\mathcal{O}_\Delta(\tilde{B}(\lambda, r)) = \{F \in \mathcal{O}(\tilde{B}(\lambda, r)); \Delta_z F(z) = 0\} .$$

5. One dimensional case.

We consider the case of $n = 1$. We put

$$\zeta_1 = z_1 + iz_2 , \qquad \zeta_2 = z_1 - iz_2 .$$

It is convenient to use the coordinate system $\zeta = (\zeta_1, \zeta_2) \in \mathbf{C}^2$ instead of the coordinate system $z = (z_1, z_2) \in \mathbf{C}^2$, for $z_1^2 + z_2^2 = \zeta_1\zeta_2$ and $L(z) = \max(|\zeta_1|, |\zeta_2|)$.

Let $0 < |\lambda| < r$. The (complex 2-dimensional) Lie ball is given by

$$\tilde{B}(r) = \{(\zeta_1, \zeta_2) \in \mathbf{C}^2; |\zeta_1| < r, |\zeta_2| < r\} .$$

Then the (complex 1-dimensional) complex sphere is given by

$$\tilde{\mathbf{S}}_\lambda = \{(\zeta_1, \zeta_2) \in \mathbf{C}^2; \zeta_1\zeta_2 = \lambda^2\} .$$

The boundary of the set

$$\tilde{\mathbf{S}}_\lambda \cap \tilde{B}(r) = \{(\zeta_1, \zeta_2) \in \mathbf{C}^2; \zeta_1\zeta_2 = \lambda^2, |\zeta_1| < r, |\zeta_2| < r\}$$

is given by

$$\tilde{\mathbf{S}}_{\lambda,r} = \{(\zeta_1, \zeta_2); \zeta_1\zeta_2 = \lambda^2, \max(|\zeta_1|, |\zeta_2|) = r\} .$$

Consider

$$\Sigma_{\lambda,r}=\{(\zeta_1,\zeta_2);\ |\zeta_1\zeta_2|=|\lambda|^2,\ \max(|\zeta_1|,|\zeta_2|)=r\}\,.$$

$\Sigma_{\lambda,r}$ is the Shilov boundary of the domain

$$\tilde{B}(\lambda,r)=\{(\zeta_1,\zeta_2);\ |\zeta_1\zeta_2|<|\lambda|^2,\ |\zeta_1|<r,\ |\zeta_2|<r\}\,.$$

REMARK. The real sphere (real dim = 1) is the circle given by

$$\lim_{\lambda\to r}\tilde{\mathbf{S}}_{\lambda,r}=\mathbf{S}_r=\{(\zeta_1,\zeta_2);\ \zeta_1\zeta_2=r^2,\ \max(|\zeta_1|,|\zeta_2|)=r\}$$
$$=\{(re^{i\theta},re^{-i\theta});\ \theta\in\mathbf{R}\}\,.$$

The Lie sphere (real dim = 2)

$$\lim_{\lambda\to r}\Sigma_{\lambda,r}=\Sigma_r=\{(\zeta_1,\zeta_2);\ |\zeta_1|=r,\ |\zeta_2|=r\}=\{(re^{i\theta},re^{i\phi});\ \theta,\phi\in\mathbf{R}\}$$

is the Shilov boundary of the Lie ball $\tilde{B}(r)$.

If F and G belong to $\mathcal{O}(\tilde{B}[\lambda,r])$, the integral on $\Sigma_{\lambda,r}$ has the following form: (we put $\dot{d}\zeta_1=d\zeta_1/(2\pi i\zeta_1)$, $\dot{d}\zeta_2=d\zeta_2/(2\pi i\zeta_2)$.)

$$\int_{\Sigma_{\lambda,r}}\overline{F(\zeta_1,\zeta_2)}G(\zeta_1,\zeta_2)\dot{d}(\zeta)=\frac{1}{2}\left(\iint_{|\zeta_1|=r,|\zeta_2|=|\lambda|^2/r}\overline{F(\zeta_1,\zeta_2)}G(\zeta_1,\zeta_2)\dot{d}\zeta_1\dot{d}\zeta_2\right.$$
$$\left.+\iint_{|\zeta_1|=|\lambda|^2/r,|\zeta_2|=r}\overline{F(\zeta_1,\zeta_2)}G(\zeta_1,\zeta_2)\dot{d}\zeta_1\dot{d}\zeta_2\right).$$

If F_k and G_k are k-homogeneous polynomials

$$F_k(\zeta_1,\zeta_2)=\sum_{l=0}^{k}a_{l,k-l}\zeta_1^l\zeta_2^{k-l}\,,\qquad G_k(\zeta_1,\zeta_2)=\sum_{l=0}^{k}b_{l,k-l}\zeta_1^l\zeta_2^{k-l}\,,$$

then we have

$$\int_{\Sigma_{\lambda,r}}\overline{F_k(\zeta_1,\zeta_2)}G_k(\zeta_1,\zeta_2)\dot{d}(\zeta)=\sum_{l,l'}\overline{a_{l,k-l}}b_{l',k-l'}\int_{\Sigma_{\lambda,r}}\overline{\zeta_1}^{\,l}\overline{\zeta_2}^{\,k-l}\zeta_1^{l'}\zeta_2^{k-l'}\dot{d}(\zeta)\,.$$

By the orthogonality of exponential functions, terms vanish unless $l=l'$. Therefore, we have

$$\int_{\Sigma_{\lambda,r}}\overline{F_k(\zeta_1,\zeta_2)}G_k(\zeta_1,\zeta_2)\dot{d}(\zeta)=\sum_{l=0}^{k}\overline{a_{l,k-l}}b_{l,k-l}\frac{|\lambda|^{2k}}{2}\left(\left(\frac{|\lambda|}{r}\right)^{2k-4l}+\left(\frac{r}{|\lambda|}\right)^{2k-4l}\right).$$

Put

$$H_{\lambda,r,k}(\eta_1,\eta_2,\zeta_1,\zeta_2)=\sum_{l=0}^{k}\frac{(\eta_1\overline{\zeta_1})^l(\eta_2\overline{\zeta_2})^{k-l}}{(|\lambda|^{2k}/2)((|\lambda|/r)^{2k-4l}+(r/|\lambda|)^{2k-4l})}\,.$$

Then we have the reproducing formula:

$$G_k(\eta_1, \eta_2)=\int_{\Sigma_{\lambda,r}} H_{\lambda,r,k}(\eta_1, \eta_2, \zeta_1, \zeta_2)G_k(\zeta_1, \zeta_2)\dot{d}(\zeta)$$

for a k-homogeneous polynomial G_k.

Let $F \in \mathcal{O}(\tilde{B}[\lambda, r])$ and $G \in \mathcal{O}(\tilde{B}(\lambda, r))$. We expand them into the Taylor series at (0, 0):

$$F(\zeta_1, \zeta_2)=\sum_{p,q=0}^{\infty} a_{p,q}\zeta_1^p\zeta_2^q , \qquad G(\zeta_1, \zeta_2)=\sum_{p,q=0}^{\infty} b_{p,q}\zeta_1^p\zeta_2^q .$$

Then we can find $t>1$ sufficiently close to 1 such that the series

$$F(t\zeta_1, t\zeta_2)=\sum_{p,q=0}^{\infty} a_{p,q}(t\zeta_1)^p(t\zeta_2)^q ,$$

$$G(t^{-1}\zeta_1, t^{-1}\zeta_2)=\sum_{p,q=0}^{\infty} b_{p,q}(t^{-1}\zeta_1)^p(t^{-1}\zeta_2)^q$$

converge uniformly on $\Sigma_{\lambda,r}$. With this t we have

$$\begin{aligned}
&\int_{\Sigma_{\lambda,r}} \overline{F(t\zeta_1, t\zeta_2)}G(t^{-1}\zeta_1, t^{-1}\zeta_2)\dot{d}(\zeta) \\
&=\int_{\Sigma_{\lambda,r}} \sum_{p,q} \overline{a_{p,q}}(t\overline{\zeta_1})^p(t\overline{\zeta_2})^q \sum_{p',q'} b_{p',q'}(t^{-1}\zeta_1)^{p'}(t^{-1}\zeta_2)^{q'}\dot{d}(\zeta) \\
&=\sum_{p,q,p',q'} \int_{\Sigma_{\lambda,r}} \overline{a_{p,q}}b_{p',q'}(t\overline{\zeta_1})^p(t\overline{\zeta_2})^q(t^{-1}\zeta_1)^{p'}(t^{-1}\zeta_2)^{q'}\dot{d}(\zeta) \\
&=\sum_{p,q}\int_{\Sigma_{\lambda,r}} \overline{a_{p,q}}b_{p,q}|\zeta_1|^{2p}|\zeta_2|^{2q}\dot{d}(\zeta) \\
&=\sum_{p,q} \frac{\overline{a_{p,q}}b_{p,q}}{2}\left(\left(\frac{|\lambda|^2}{r}\right)^{2p} r^{2q}+r^{2p}\left(\frac{|\lambda|^2}{r}\right)^{2p}\right).
\end{aligned}$$

For $F \in \mathcal{O}(\tilde{B}[\lambda, r])$ and $G \in \mathcal{O}(\tilde{B}(\lambda, r))$, the symbolic integral form on $\Sigma_{\lambda,r}$ is given by

$$\text{s.}\int_{\Sigma_{\lambda,r}} \overline{F(\zeta_1, \zeta_2)}G(\zeta_1, \zeta_2)\dot{d}(\zeta)=\int_{\Sigma_{\lambda,r}} \overline{F(t\zeta_1, t\zeta_2)}G(t^{-1}\zeta_1, t^{-1}\zeta_2)\dot{d}(\zeta)$$

where $t>1$ is sufficiently close to 1.

Let $F_k(\zeta_1, \zeta_2)=a_k\zeta_1^k+b_k\zeta_2^k$ and $G_j(\zeta_1, \zeta_2)=a_j'\zeta_1^j+b_j'\zeta_2^j$ be homogeneous harmonic polynomials. We have the orthogonality relation:

$$\int_{\Sigma_{\lambda,r}} \overline{F_k(\zeta_1, \zeta_2)}G_j(\zeta_1, \zeta_2)\dot{d}(\zeta)=0 , \qquad j\neq k .$$

If $j=k\geq 1$, then we have

$$\int_{\Sigma_{\lambda,r}} \overline{F_k(\zeta_1, \zeta_2)} G_k(\zeta_1, \zeta_2) \dot{d}(\zeta) = \int_{\Sigma_{\lambda,r}} (\overline{a_k}\overline{\zeta_1}^k + \overline{b_k}\overline{\zeta_2}^k)(a'_k\zeta_1^k + b'_k\zeta_2^k) \dot{d}(\zeta)$$

$$= \int_{\Sigma_{\lambda,r}} (\overline{a_k}a'_k |\zeta_1|^{2k} + \overline{a_k}b'_k\overline{\zeta_1}^k\zeta_2^k + \overline{b_k}a'_k\overline{\zeta_1}^k\zeta_2^k + \overline{b_k}b'_k|\zeta_2|^{2k}) \dot{d}(\zeta)$$

$$= (\overline{a_k}a'_k + \overline{b_k}b'_k) \frac{|\lambda|^{2k}}{2} \left(\left(\frac{r}{|\lambda|}\right)^{2k} + \left(\frac{|\lambda|}{r}\right)^{2k} \right).$$

Put

$$K_{\lambda,r,k}(\eta_1, \eta_2, \zeta_1, \zeta_2) = \frac{(\eta_1\overline{\zeta_1})^k + (\eta_2\overline{\zeta_2})^k}{(|\lambda|^{2k}/2)((r/|\lambda|)^{2k} + (|\lambda|/r)^{2k})}, \qquad k \geq 1$$

and $K_{\lambda,r,0}(\eta_1, \eta_2, \zeta_1, \zeta_2) = 1$. Then we have a reproducing formula

$$G_k(\eta_1, \eta_2) = \int_{\Sigma_{\lambda,r}} K_{\lambda,r,k}(\eta_1, \eta_2, \zeta_1, \zeta_2) G_k(\zeta_1, \zeta_2) \dot{d}(\zeta)$$

for a k-homogeneous harmonic polynomial G_k.

If G is a complex harmonic function on $\tilde{B}(r)$, then we can expand it as follows:

$$G(\zeta_1, \zeta_2) = \sum_{k=0}^{\infty} G_k(\zeta_1, \zeta_2) = G(0, 0) + \sum_{k=1}^{\infty} (a_k\zeta_1^k + b_k\zeta_2^k).$$

Therefore, we have

$$G_k(\eta_1, \eta_2) = \mathrm{s.} \int_{\Sigma_{\lambda,r}} K_{\lambda,r,k}(\eta_1, \eta_2, \zeta_1, \zeta_1) G(\zeta_1, \zeta_2) \dot{d}(\zeta).$$

Define the (generalized) Poisson kernel by

$$K_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2) = \sum_{k=0}^{\infty} K_{\lambda,r,k}(\eta_1, \eta_2, \zeta_1, \zeta_2). \qquad (22)$$

Because

$$\frac{|\lambda|^{2k}}{2} \left(\left(\frac{r}{|\lambda|}\right)^{2k} + \left(\frac{|\lambda|}{r}\right)^{2k} \right) \geq \frac{r^{2k}}{2},$$

(22) converges uniformly for $|\eta_1\zeta_1| < r^2$ and $|\eta_2\zeta_2| < r^2$. Therefore, the function $K_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2)$ is defined on the domain

$$\Omega'_r = \{(\eta_1, \eta_2, \zeta_1, \zeta_2); |\eta_1\zeta_1| < r^2, |\eta_2\zeta_2| < r^2\}$$

and is holomorphic and complex harmonic in (η) there.

We have the reproducing formula:

$$G(\eta_1, \eta_2) = \mathrm{s.}\int_{\Sigma_{\lambda,r}} K_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2) G(\zeta_1, \zeta_2) d\dot{}(\zeta), \qquad (\eta_1, \eta_2) \in \tilde{B}(r)$$

for $G \in \mathcal{O}_\Delta(\tilde{B}(r))$.

REMARK. We have

$$\lim_{\lambda \to r} K_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2) = K_r(\eta_1, \eta_2, \zeta_1, \zeta_2)$$

$$= \frac{1}{1 - \eta_1\bar{\zeta}_1/r^2} + \frac{1}{1 - \eta_2\bar{\zeta}_2/r^2} - 1 = \frac{1 - \eta_1\eta_2\overline{\zeta_1\zeta_2}/r^4}{(1 - \eta_1\bar{\zeta}_1/r^2)(1 - \eta_2\bar{\zeta}_2/r^2)}.$$

Put

$$H_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2) = \sum_{p,q} \frac{(\eta_1\bar{\zeta}_1)^q(\eta_2\bar{\zeta}_2)^p}{\frac{1}{2}((|\lambda|^2/r)^{2p} r^{2q} + r^{2p}(|\lambda|^2/r)^{2q})}. \tag{23}$$

Because

$$\frac{1}{2}\left(\left(\frac{|\lambda|^2}{r}\right)^{2p} r^{2q} + r^{2p}\left(\frac{|\lambda|^2}{r}\right)^{2q}\right) \geq |\lambda|^{2(p+q)},$$

we have

$$|H_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2)| \leq \sum_{p,q} \frac{|\eta_1\zeta_1|^q |\eta_2\zeta_2|^p}{|\lambda|^{2p+2q}}.$$

Therefore, the right-hand side of (23) converges uniformly on compact sets of the domain

$$\Omega'_\lambda = \{(\eta_1, \eta_2, \zeta_1, \zeta_2) \in \mathbf{C}^4;\ |\eta_1\zeta_1| < |\lambda|^2, |\eta_1\zeta_2| < |\lambda|^2\}$$

and hence, holomorphic in (η_1, η_2) there. We shall see that the function is holomorphic in (η_1, η_2) in a larger domain.

REMARK. We have

$$\lim_{\lambda \to r} H_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2) = H_r(\eta_1, \eta_2, \zeta_1, \zeta_2) = \sum_{p,q} \frac{(\eta_1\bar{\zeta}_1)^q(\eta_2\bar{\zeta}_2)^p}{r^{2p+2q}}$$

$$= \frac{1}{(1 - \eta_1\bar{\zeta}_1/r^2)(1 - \eta_2\bar{\zeta}_2/r^2)}.$$

This is the (1-dimensional) Cauchy-Hua kernel of $\mathcal{O}(\tilde{B}(r))$ with respect to the Lie sphere Σ_r.

Recall

$$H_{\lambda,r,k}(\eta_1, \eta_2, \zeta_1, \zeta_2) = \sum_{l=0}^{k} \frac{(\eta_1\bar{\zeta}_1)^l(\eta_2\bar{\zeta}_2)^{k-l}}{(|\lambda|^{2k}/2)((|\lambda|/r)^{2k-4l} + (r/|\lambda|)^{2k-4l})}$$

$$= \sum_{l=0}^{[k/2]} \frac{(\eta_1\eta_2\overline{\zeta_1\zeta_2})^l}{|\lambda|^{4l}} K_{\lambda,r,k-2l}(\eta_1, \eta_2, \zeta_1, \zeta_2).$$

Changing the order of summation we get

$$
\begin{aligned}
H_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2) &= \sum_{k=0}^{\infty} H_{\lambda,r,k}(\eta_1, \eta_2, \zeta_1, \zeta_2) \\
&= \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{(\eta_1\eta_2\overline{\zeta_1\zeta_2})^l}{|\lambda|^{4l}} K_{\lambda,r,k-2l}(\eta_1, \eta_2, \zeta_1, \zeta_2) \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\eta_1\eta_2\overline{\zeta_1\zeta_2})^l}{|\lambda|^{4l}} K_{\lambda,r,m}(\eta_1, \eta_2, \zeta_1, \zeta_2) \\
&= \sum_{l=0}^{\infty} \frac{(\eta_1\eta_2\overline{\zeta_1\zeta_2})^l}{|\lambda|^{4l}} K_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2) \\
&= \frac{K_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2)}{1-\eta_1\eta_2\overline{\zeta_1\zeta_2}/|\lambda|^4} .
\end{aligned}
$$

The (generalized) Cauchy-Hua kernel $H_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2)$ is holomorphic in (η_1, η_2) in the domain

$$
\Omega'_{\lambda,r} = \{(\eta_1, \eta_2, \zeta_1, \zeta_2);\ |\eta_1\eta_2\zeta_1\zeta_2| < |\lambda|^4,\ |\eta_1\zeta_1| < r^2,\ |\eta_2\zeta_2| < r^2\} .
$$

We have the reproducing formula

$$
G(\eta_1, \eta_2) = \mathrm{s.}\int_{\Sigma_{\lambda,r}} H_{\lambda,r}(\eta_1, \eta_2, \zeta_1, \zeta_2) G(\zeta_1, \zeta_2) d(\dot{\zeta}) , \qquad (\eta_1, \eta_2) \in \tilde{B}(\lambda, r)
$$

for $G \in \mathcal{O}(\tilde{B}(\lambda, r))$.

References

[1] L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in Classical Domains*, Moscow (1959, in Russian); Transl. Math. Monographs **6** (1963), Amer. Math. Soc.

[2] M. Morimoto, Analytic functionals on the Lie sphere, Tokyo J. Math. **3** (1980), 1–35.

[3] M. Morimoto, *Analytic Functionals on the Sphere*, Transl. Math. Monographs 178 (1998), Amer. Math. Soc.

[4] M. Morimoto and K. Fujita, Analytic functionals and entire functionals on the complex light cone, Hiroshima Math. J. **25** (1995), 493–512.

[5] C. Müller, *Spherical Harmonics*, Lecture Notes in Math. **17** (1966), Springer.

[6] R. Wada, Holomorphic functions on the complex sphere, Tokyo J. Math. **11** (1988), 205–218.

[7] R. Wada and M. Morimoto, A uniqueness set for the differential operator $\Delta_z + \lambda^2$, Tokyo J. Math. **10** (1987), 93–105.

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