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Shadowing Property of Non-Invertible Maps with Hyperbolic Measures

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Abstract. We show that if a differentiable map of a smooth manifold has a non-atomic ergodic hyperbolic measure then the topological entropy is positive and the space contains a hyperbolic horseshoe. Moreover we give some relations between hyperbolic measures and periodic points for differentiable maps. These are generalized contents of the results obtained by Katok for diffeomorphisms.

We know that for diffeomorphisms transversal homoclinic points imply the existence of (non-trivial) hyperbolic horseshoes [Sm]. Using the Pesin theory ([Pe1], [Pe2]), Katok [K] has proved that if a diffeomorphism has a non-atomic ergodic hyperbolic measure, then there exists a hyperbolic periodic point having a transversal homoclinic point, and then by the homoclinic point theorem there is a hyperbolic horseshoe and infinitely many hyperbolic periodic points of saddle type.

We consider a differentiable map $f: M \to M$ of a smooth closed manifold M. In case f is non-invertible, the manifold M may have a point x such that the derivative at x, $D_x f$, is not injective. Such a point x is called a singular point of f. If f has a singular point, then the behavior of f is quite different from that of diffeomorphisms. In this case it is not true that the forward and backward images of a submanifold I of M by iterations of f are manifolds with the same dimension as I. However, by taking advantage of results obtained by Katok [K] for diffeomorphisms, we have the following:

THEOREM A. *Let $f: M \to M$ be a $C^{1+\alpha}$ map ($\alpha > 0$). Suppose that f has a non-atomic ergodic hyperbolic measure. Then there exists a hyperbolic horseshoe of f, and the topological entropy of f, $h(f)$, is positive.*

From Theorem A we can take infinitely many hyperbolic periodic points under the assumption that f has a non-atomic ergodic hyperbolic measure. Then we must notice that these periodic points may not be of saddle type if f is non-invertible. This

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phenomenon does not occur for any diffeomorphism.

For the proof of Theorem A we need a result called the shadowing lemma for non-invertible maps (Key Lemma 2). Thus the shadowing lemma is an important ingredient of this paper.

Using the shadowing lemma we shall also give here some relations between hyperbolic measures and periodic points for differentiable maps, which are extended contents of results in [K], as follows:

THEOREM B. *Let μ be a hyperbolic measure of a $C^{1+\alpha}$ map $f: M \to M$. Then the support of μ, $supp(\mu)$, is a subset of the closure of the set $Per(f)$ of periodic points, i.e.*

$$supp(\mu) \subset \overline{Per(f)} .$$

THEOREM C. *Let μ be as in Theorem* B. *If the metric entropy of μ, $h_\mu(f)$, is positive, then for any $\varepsilon > 0$ there exists a hyperbolic horseshoe Γ_ε of f such that*

$$h(f|_{\Gamma_\varepsilon}) \geq h_\mu(f) - \varepsilon .$$

THEOREM D. *Let μ be as in Theorem* B. *Then we have*

$$h_\mu(f) \leq \max\left\{0, \limsup_{n\to\infty} \frac{1}{n} \log \sharp Fix(f^n)\right\}$$

where $Fix(g)$ denotes the set of all fixed points of a map g, and $\sharp A$ the cardinality of a set A.

1. Lyapunov chart and shadowing property.

Let $f: M \to M$ be a $C^{1+\alpha}$ map ($\alpha > 0$) of a finite dimensional smooth closed manifold M.

To definite the inverse limit of $f: M \to M$ we consider a compact metric space

$$M_f = \left\{\tilde{x} = (x_n) \in \prod_{-\infty}^{\infty} M \ : \ f(x_n) = x_{n+1} \text{ for all } n \in \mathbf{Z}\right\},$$

equipped with the distance $\tilde{d}$ defined by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \sum_{n=-\infty}^{\infty} 2^{-|n|} d(x_n, y_n)$$

for $\tilde{x} = (x_n)$, $\tilde{y} = (y_n) \in M_f$, where d is the distance on M induced by the Riemannian metric. Let $\tilde{f}: M_f \to M_f$ be defined by $\tilde{f}((x_n)) = (x_{n+1})$. Then $\tilde{f}$ is a homeomorphism and $\pi \circ \tilde{f} = f \circ \pi$ holds, where $\pi: M_f \to M$ is the projection defined by $\pi((x_n)) = x_0$. The map $\tilde{f}: M_f \to M_f$ is called the *inverse limit* of $f: M \to M$. For a periodic point $\tilde{p}$ of $\tilde{f}$ it is obvious that $\pi(\tilde{p})$ is also a periodic point of f with the same period as $\tilde{p}$. For $\Gamma \subset M$ we set

$$\Gamma_f = \{\tilde{x} = (x_n) \in M_f \ : \ x_n \in \Gamma \text{ for all } n \in \mathbf{Z}\} .$$

A compact f-invariant set Γ is called *hyperbolic* for f if there are constants $C \geq 1$ and $0 < \lambda < 1$ such that for all $\tilde{x} = (x_n) \in \Gamma_f$, the tangent space $T_{x_0}M$ at the 0-coordinate x_0 splits into a direct sum $T_{x_0}M = E^s(\tilde{x}) \oplus E^u(\tilde{x})$ satisfying:

$$D_{x_0}f(E^s(\tilde{x})) \subset E^s(\tilde{f}(\tilde{x})), \qquad D_{x_0}f(E^u(\tilde{x})) = E^u(\tilde{f}(\tilde{x})),$$

$$\|D_{x_0}f^n(v)\| \leq C\lambda^n \|v\| \qquad \text{for } v \in E^s(\tilde{x}),$$

$$\|(D_{x_{-n}}f^n|_{E^u(\tilde{f}^{-n}(\tilde{x}))})^{-1}(w)\| \leq C\lambda^n \|w\| \qquad \text{for } w \in E^u(\tilde{x})$$

where $\|\cdot\|$ denotes the Riemannian metric.

We say that a compact f-invariant set Γ is a *horseshoe* of f if there exist positive integers l, m and subsets $\Gamma_0, \cdots, \Gamma_{m-1}$ of M such that

$$\Gamma = \Gamma_0 \cup \cdots \cup \Gamma_{m-1}, \qquad f(\Gamma_i) = \Gamma_{i+1} \pmod{m},$$

and the inverse limit $\tilde{f}^m|_{(\Gamma_0)_{f^m}} : (\Gamma_0)_{f^m} \to (\Gamma_0)_{f^m}$ of $f^m|_{\Gamma_0} : \Gamma_0 \to \Gamma_0$ is topologically conjugate to a full-shift in l-symbols. For such a set Γ and integers l, m we have

$$h(f|_\Gamma) = \frac{1}{m} h(\tilde{f}^m|_{(\Gamma_0)_{f^m}}) = \frac{1}{m} \log l,$$

$$l^j \leq \# Fix(f^{mj}|_\Gamma) \leq ml^j$$

for $j \geq 1$. If f has a hyperbolic horseshoe Γ, then for a small C^1 neighborhood $\mathscr{U}$ of f and $g \in \mathscr{U}$ there exists a hyperbolic horseshoe $\Gamma(g)$ of g such that $\tilde{g}|_{\Gamma(g)_g} : \Gamma(g)_g \to \Gamma(g)_g$ is topologically conjugate to $\tilde{f}|_{\Gamma_f} : \Gamma_f \to \Gamma_f$ ([AM], [Mo]).

For $x \in M$ and $v \in T_xM$ we put

$$\chi^+(v, f) = \limsup_{n \to \infty} \frac{1}{n} \log \|D_x f^n(v)\|.$$

The number $\chi^+(v, f)$ is called the *Lyapunov exponent* for v. Then, for $x \in M$ there are at most $\dim M$ numbers $\chi_1(x), \cdots, \chi_{r(x)}(x)$ with $-\infty \leq \chi_1(x) < \cdots < \chi_{r(x)} < \infty$, and a filtration of subspaces

$$\{0\} = L_0(x) \subsetneq L_1(x) \subsetneq \cdots \subsetneq L_{r(x)}(x) = T_xM$$

such that $\chi^+(v, f) = \chi_i(x)$ holds for $v \in L_i(x) \backslash L_{i-1}(x)$, $1 \leq i \leq r(x)$. The numbers $\chi_1(x), \cdots, \chi_{r(x)}(x)$ are called the *Lyapunov exponents at x*. We set

$$k_i(x) = \dim L_i(x) - \dim L_{i-1}(x)$$

for $1 \leq i \leq r(x)$. Then $r(x)$, $\chi_i(x)$, $k_i(x)$ are measurable and f-invariant functions with respect to any f-invariant Borel probability measure μ. If μ is ergodic, then these functions are constant almost everywhere. In this case we denote these values by r^μ, χ_i^μ, k_i^μ, respectively. An f-invariant Borel probability measure μ is called *hyperbolic* if all the Lyapunov exponents are different from zero for μ-almost everywhere. If μ is ergodic and all the

Lyapunov exponents are negative almost everywhere, i.e. $\chi^{\mu}_{r^{\mu}}<0$, then μ is concentrated on the orbit of a periodic sink, and the metric entropy, $h_{\mu}(f)$, is zero.

In order to state the shadowing lemma we need some preparations.

For $0\le k\le \dim M$, $\chi>0$ and $l\ge 1$, we define a subset $\tilde{\Lambda}^k_{\chi,l}$ of M_f consisting of $\tilde{x}=(x_n)\in M_f$ for which there exists a sequence of splittings $T_{x_n}M=E^s(\tilde{x},n)\oplus E^u(\tilde{x},n)$, $n\in\mathbf{Z}$, satisfying:

(a) $\dim E^s(\tilde{x},n)=k$;

(b) $D_{x_n}f(E^s(\tilde{x},n))\subset E^s(\tilde{x},n+1)$, $D_{x_n}f(E^u(\tilde{x},n))=E^u(\tilde{x},n+1)$;

(c) for $m\ge 0$

$$\|D_{x_n}f^m(v)\|\le \exp\{-\chi m\}\cdot\exp\{(\chi/100)(l+|n|)\}\|v\| \quad \text{for } v\in E^s(\tilde{x},n)\,,$$

$$\|(D_{x_{n-m}}f^m|_{E^u(\tilde{x}_{n-m})})^{-1}(w)\|\le \exp\{-\chi m\}\cdot\exp\{(\chi/100)(l+|n|)\}\|w\| \quad \text{for } w\in E^u(\tilde{x},n)\,;$$

and if $1\le k\le \dim M-1$ then

(d) $\sin\angle(E^s(\tilde{x},n),E^u(\tilde{x},n))\ge \exp\{-(\chi/100)(l+|n|)\}$.

It is easy to see that for $0\le k\le\dim M$, $\chi>0$ and $l\ge 1$, $\tilde{\Lambda}^k_{\chi,l}$ is compact, and that $\tilde{\Lambda}^k_{\chi,l}\subset\tilde{\Lambda}^k_{\chi,l+1}$ and $\tilde{f}^{\pm 1}(\tilde{\Lambda}^k_{\chi,l})\subset\tilde{\Lambda}^k_{\chi,l+1}$ hold. Therefore $\tilde{\Lambda}^k_{\chi}=\bigcup_{l=1}^{\infty}\tilde{\Lambda}^k_{\chi,l}$ is an $\tilde{f}$-invariant Borel set of M_f, and

$$E^s(\tilde{f}(\tilde{x}),n)\subset E^s(\tilde{x},n+1)\,,\qquad E^u(\tilde{f}(\tilde{x}),n)=E^u(\tilde{x},n+1)$$

hold for $\tilde{x}=(x_n)\in\tilde{\Lambda}^k_{\chi}$ and $n\in\mathbf{Z}$. The subspaces $E^s(\tilde{x},n)$, $E^u(\tilde{x},n)$ of $T_{x_n}M$ depend on $\tilde{x}$ continuously on the set $\tilde{\Lambda}^k_{\chi,l}$.

For a differentiable map f we know the following:

MULTIPLICATIVE ERGODIC THEOREM ([PS], Theorem 5.2). *Let μ be an f-invariant Borel probability measure on M. We denote by $\tilde{\mu}$ the $\tilde{f}$-invariant Borel probability measure on M_f such that $\pi_*\tilde{\mu}=\mu$, i.e. $\tilde{\mu}(\pi^{-1}E)=\mu(E)$ holds for each Borel set $E\subset M$. Then for $\tilde{\mu}$-almost all $\tilde{x}=(x_n)\in M_f$ and $n\in\mathbf{Z}$ the tangent space $T_{x_n}M$ splits into a direct sum*

$$T_{x_n}M=E_1(\tilde{x},n)\oplus\cdots\oplus E_{r(x_0)}(\tilde{x},n)$$

satisfying:

(1) $\dim E_i(\tilde{x},n)=k_i(x_0)$;

(2) $D_{x_n}f(E_i(\tilde{x},n))\subset E_i(\tilde{x},n+1)$, *and for* $v\in E_i(\tilde{x},n)\setminus\{0\}$

$$\lim_{m\to\infty}\frac{1}{m}\log\|D_{x_n}f^m(v)\|=\chi_i(x_0)\,;$$

(3) *if* $\chi_i(x_0)\ne-\infty$, *then* $D_{x_n}f|_{E_i(\tilde{x},n)}:E_i(\tilde{x},n)\to E_i(\tilde{x},n+1)$ *is an isomorphism, and*

for $v \in E_i(\tilde{x}, n)\setminus\{0\}$

$$\lim_{m\to\infty} -\frac{1}{m}\log\|(D_{x_{n-m}}f^m|_{E_i(\tilde{x},n-m)})^{-1}(v)\| = \chi_i(x_0)\ ;$$

(4) *if* $i \neq j$ *then*

$$\lim_{n\to\pm\infty} \frac{1}{n}\log\sin\angle(E_i(\tilde{x}, n), E_j(\tilde{x}, n)) = 0\ ,$$

where $\angle(V, W)$ *denotes the angle between subspaces* V *and* W.

For a hyperbolic measure μ by the multiplicative ergodic theorem we have

$$\tilde{\mu}\left(\bigcup_{k=0}^{\dim M}\bigcup_{\chi>0}\tilde{\Lambda}_\chi^k\right) = 1\ . \tag{1.1}$$

In particular, if μ is ergodic then we have

$$\tilde{\mu}(\tilde{\Lambda}_\chi^k) = 1 \tag{1.2}$$

for $k = \sum_{\chi_i^\mu < 0} k_i^\mu$ and $0 < \chi < \min_{1\le i\le r^\mu}|\chi_i^\mu|$.

For $1 \le k \le \dim M$ and $\eta > 0$ we put

$$\mathbf{D}^k(\eta) = \{z \in \mathbf{R}^k : \ |z| \le \eta\}$$

where $|\cdot|$ denotes the usual norm on $\mathbf{R}^k$.

We take a finite open cover $\mathscr{U} = \{U_1, \cdots, U_d\}$ of M such that each U_i is C^∞ diffeomorphic to an open set of $\mathbf{R}^{\dim M}$. Then we have the following:

LEMMA 1 (Existence of Lyapunov charts). *For* $0 \le k \le \dim M$ *and* $\chi > 0$, *put*

$$\lambda = \lambda(\chi) = \exp\{-(9\chi)/10\} \in (0, 1)\ ,$$

and fix a small number $\beta = \beta(\chi) > 0$ *such that*

$$(1+\beta)(\lambda + 2\beta)\ , \quad (1-\beta)^{-1}(\lambda^{-1} - 2\beta)^{-1} < 1 - \beta\ .$$

Then there exists a family of C^∞ *diffeomorphisms*

$$\{\Phi_{\tilde{x},i} : \mathbf{D}^{\dim M}(\alpha_0) \to M\ :\ 1 \le i \le d,\ \tilde{x} \in \tilde{\Lambda}_\chi^k \cap \pi^{-1}U_i\}\ ,$$

where $\alpha_0 > 1$ *depends only on* f, *and a family of* C^1 *maps*

$$\{F_{\tilde{x},i,j} : \mathbf{R}^{\dim M} \to \mathbf{R}^{\dim M}\ :\ 1 \le i, j \le d,\ \tilde{x} \in \tilde{\Lambda}_\chi^k \cap \pi^{-1}(U_i \cap f^{-1}U_j)\}$$

satisfying the following properties:

(1) $\Phi_{\tilde{x},i}(0) = \pi(\tilde{x})$;

(2) $F_{\tilde{x},i,j}(\mathbf{z}) = \Phi_{\tilde{f}(\tilde{x}),j}^{-1} \circ f \circ \Phi_{\tilde{x},i}(\mathbf{z})$ *if* $\mathbf{z} \in \mathbf{D}^{\dim M}(1)$;

(3) $F_{\tilde{x},i,j} : \mathbf{R}^{\dim M} \to \mathbf{R}^{\dim M}$ *has the form*

$$F_{\tilde{x},i,j}(\mathbf{z}^s, \mathbf{z}^u) = (A_{\tilde{x},i,j}^s \mathbf{z}^s + h_{\tilde{x},i,j}^s(\mathbf{z}^s, \mathbf{z}^u)\ ,\quad A_{\tilde{x},i,j}^u \mathbf{z}^u + h_{\tilde{x},i,j}^u(\mathbf{z}^s, \mathbf{z}^u))\ ,$$

$$|A^s_{\tilde{x},i,j}\mathbf{Z}^s| \le \lambda|\mathbf{z}^s|, \quad |A^u_{\tilde{x},i,j}\mathbf{z}^u| \ge \lambda^{-1}|\mathbf{z}^u|, \quad |D_\mathbf{z}h^s_{\tilde{x},i,j}| \le \beta, \quad |D_\mathbf{z}h^u_{\tilde{x},i,j}| \le \beta$$

for $\mathbf{z}=(\mathbf{z}^s, \mathbf{z}^u) \in \mathbf{R}^k \times \mathbf{R}^{\dim M - k} = \mathbf{R}^{\dim M}$, *and*

$$h^s_{\tilde{x},i,j}(0,0)=0, \quad h^u_{\tilde{x},i,j}(0,0)=0, \quad D_{(0,0)}h^s_{\tilde{x},i,j}=0, \quad D_{(0,0)}h^u_{\tilde{x},i,j}=0;$$

(4) *if* $\tilde{x} \in \tilde{\Lambda}^k_{\chi,l} \cap \pi^{-1}U_i$ *then*

$$c_0^{-1}\|D_\mathbf{z}\Phi_{\tilde{x},i}(\mathbf{v})\| \le |\mathbf{v}| \le c_l\|D_\mathbf{z}\Phi_{\tilde{x},i}(\mathbf{v})\|$$

for $\mathbf{z} \in \mathbf{D}^{\dim M}(\alpha_0)$, $\mathbf{v} \in \mathbf{R}^{\dim M}$, *where* c_0, $c_l = c_l(k, \chi) \ge 1$ *are independent of* $\tilde{x}$;

(5) *for* $l \ge 1$, $\Phi_{\tilde{x},i}$ *depends on* $\tilde{x}$ *continuously on the set* $\tilde{\Lambda}^k_{\chi,l} \cap \pi^{-1}U_i$.

PROOF. For $\tilde{x}=(x_n) \in \tilde{\Lambda}^k_\chi$ we define a norm $\|\cdot\|'_{\tilde{x}}$ on $T_{x_0}M$ as follows: for $v = v_s + v_u \in T_{x_0}M$ with $v_s \in E^s(\tilde{x}, 0)$, $v_u \in E^u(\tilde{x}, 0)$

$$\|v_s\|'_{\tilde{x}} = \sum_{n=0}^{\infty} \exp\left(\frac{99}{100}\chi n\right)\|D_{x_0}f^n(v_s)\|,$$

$$\|v_u\|'_{\tilde{x}} = \sum_{n=0}^{\infty} \exp\left(\frac{99}{100}\chi n\right)\|(D_{x_{-n}}f^n|_{E^u(\tilde{x},-n)})^{-1}(v_u)\|,$$

$$\|v\|'_{\tilde{x}} = \max\{\|v_s\|'_{\tilde{x}}, \|v_u\|'_{\tilde{x}}\}.$$

Then we have

$$\|D_{x_0}f(v_s)\|'_{\tilde{f}(\tilde{x})} \le \exp\left(-\frac{99}{100}\chi\right)\|v_s\|'_{\tilde{x}},$$

$$\|D_{x_0}f(v_u)\|'_{\tilde{f}(\tilde{x})} \ge \exp\left(\frac{99}{100}\chi\right)\|v_u\|'_{\tilde{x}}, \tag{1.3}$$

$$\tfrac{1}{2}\|v\| \le \|v\|'_{\tilde{x}} \le a_l\|v\| \quad \text{if} \quad \tilde{x}=(x_n) \in \tilde{\Lambda}^k_{\chi,l} \tag{1.4}$$

where

$$a_l = a_l(\chi) = \left\{\sum_{n=0}^{\infty} \exp\left(-\frac{1}{100}\chi n\right)\right\} \cdot \exp\left(\frac{1}{50}\chi l\right).$$

We denote by $\exp_x$ the exponential map associated to the Riemannian metric $\|\cdot\|$, and defined on a neighborhood of the origin in the tangent space T_xM at $x \in M$. Since M is compact, there exists $\alpha_1 > 0$ such that for every $x \in M$, $\exp_x$ is a C^∞ diffeomorphism from $T_xM(\alpha_1)$ (the closed α_1-ball around the origin in T_xM), onto $B(x, \alpha_1)$ (the closed α_1-ball around x in M). Then

$$d(x, \exp_x(v)) = \|v\|$$

holds for $v \in T_xM(\alpha_1)$. By retaking α_1 small enough if necessary, we may assume that $0 < \alpha_1 < 1$ and

$$\|D_v \exp_x\| \le 2 \qquad \text{for } v \in T_xM(\alpha_1)\,,$$
$$\|D_y \exp_x^{-1}\| \le 2 \qquad \text{for } y \in B(x, \alpha_1)\,. \tag{1.5}$$

By the uniform continuity of f there exists a constant α_2 with $0<\alpha_2<\alpha_1$ such that $d(f(x), f(y)) \le \alpha_1$ holds whenever $d(x, y) \le \alpha_2$. Since $f: M \to M$ is of class $C^{1+\alpha}$, by applying the same method to the proof of Proposition 7 obtained in [FHY] we can take a family of C^1 maps $\{f_{\tilde{x}}: T_{x_0}M \to T_{x_1}M : \tilde{x}=(x_n) \in \tilde{\Lambda}_\chi^k\}$, and a measurable function $\xi: \tilde{\Lambda}_\chi^k \to (0, \alpha_2/2)$ such that, for $\tilde{x}=(x_n) \in \tilde{\Lambda}_{\chi,l}^k$:

(α) $\|D_w f_{\tilde{x}}(v) - D_{x_0} f(v)\|'_{\tilde{f}(\tilde{x})} \le 2^{-1} \exp(-\chi/100)\beta \|v\|'_{\tilde{x}}$ for $v, w \in T_{x_0}M$;

(β) $f_{\tilde{x}}(v) = (\exp_{x_1}^{-1} \circ f \circ \exp_{x_0})(v)$ if $\|v\|'_{\tilde{x}} \le \xi(\tilde{x})$;

(γ) $\xi(\tilde{f}^n(\tilde{x})) \le \exp(\frac{1}{100}\chi |n|)\xi(\tilde{x})$ for all $n \in \mathbf{Z}$;

(δ) for $l \ge 1$, $f_{\tilde{x}}$ and $\xi(\tilde{x})$ depend on $\tilde{x}$ continuously on the set $\tilde{\Lambda}_{\chi,l}^k$.

For $1 \le i \le d$ and $\tilde{x}=(x_n) \in \tilde{\Lambda}_\chi^k \cap \pi^{-1} U_i$ we take an isomorphism $L_{\tilde{x},i}: \mathbf{R}^{\dim M} \to T_{x_0}M$ such that:

$$L_{\tilde{x},i}(\mathbf{R}^k \times \{0\}) = E^s(\tilde{x}, 0)\,, \quad L_{\tilde{x},i}(\{0\} \times \mathbf{R}^{\dim M - k}) = E^u(\tilde{x}, 0)\,;$$
$$\|L_{\tilde{x},i}(\mathbf{y})\|'_{\tilde{x}} = \xi(\tilde{x})|\mathbf{y}| \qquad \text{for } \mathbf{y} \in \mathbf{R}^k \times \{0\}\,;$$
$$\|L_{\tilde{x},i}(\mathbf{z})\|'_{\tilde{x}} = \xi(\tilde{x})|\mathbf{z}| \qquad \text{for } \mathbf{z} \in \{0\} \times \mathbf{R}^{\dim M - k}\,;$$

and that depends on $\tilde{x}$ continuously on the set $\tilde{\Lambda}_{\chi,l}^k \cap \pi^{-1} U_i$ for $l \ge 1$. Then by the definition of $L_{\tilde{x},i}$,

$$\tfrac{1}{2}\xi(\tilde{x})|\mathbf{z}| \le \|L_{\tilde{x},i}(\mathbf{z})\|'_{\tilde{x}} \le \xi(\tilde{x})|\mathbf{z}| \le |\mathbf{z}| \tag{1.6}$$

holds for $\mathbf{z} \in \mathbf{R}^{\dim M}$. We put $\alpha_0 = \alpha_1/\alpha_2 > 1$. Then by (1.4) and (1.6) for $1 \le i \le d$ and $\tilde{x} \in \tilde{\Lambda}_\chi^k \cap \pi^{-1} U_i$ we have

$$\|L_{\tilde{x},i}(\mathbf{z})\| \le 2\|L_{\tilde{x},i}(\mathbf{z})\|'_{\tilde{x}} \le 2\xi(\tilde{x})|\mathbf{z}| \le 2 \cdot \alpha_2/2 \cdot \alpha_0 = \alpha_1$$

whenever $\mathbf{z} \in \mathbf{D}^{\dim M}(\alpha_0)$, and so

$$L_{\tilde{x},i}(\mathbf{D}^{\dim M}(\alpha_0)) \subset T_{x_0}M(\alpha_1)\,.$$

Thus a C^∞ diffeomorphism

$$\Phi_{\tilde{x},i} = \exp_{x_0} \circ (L_{\tilde{x},i}|_{\mathbf{D}^{\dim M}(\alpha_0)}): \ \mathbf{D}^{\dim M}(\alpha_0) \to M$$

is well-defined. We define a C^1 map $F_{\tilde{x},i,j}: \mathbf{R}^{\dim M} \to \mathbf{R}^{\dim M}$ by

$$F_{\tilde{x},i,j} = L_{\tilde{f}(\tilde{x}),j}^{-1} \circ f_{\tilde{x}} \circ L_{\tilde{x},i}$$

for $1 \le i, j \le d$ and $\tilde{x} \in \tilde{\Lambda}_\chi^k \cap \pi^{-1}(U_i \cap f^{-1} U_j)$.

We want to show that families $\{\Phi_{\tilde{x},i}\}$ and $\{F_{\tilde{x},i,j}\}$ satisfy the properties (1)–(5) stated in Lemma 1. The properties (1) and (2) are obvious from the definitions. The property (3) is checked as follows. By the definition of $F_{\tilde{x},i,j}$ we have

$$\begin{aligned}
D_0F_{\tilde{x},i,j}(\mathbf{R}^k\times\{0\}) &= (L^{-1}_{\tilde{f}(\tilde{x}),j}\circ D_{x_0}f\circ L_{\tilde{x},i})(\mathbf{R}^k\times\{0\}) \\
&= (L^{-1}_{\tilde{f}(\tilde{x}),j}\circ D_{x_0}f)(E^s(\tilde{x},0)) \\
&\subset L^{-1}_{\tilde{f}(\tilde{x})j}(E^s(\tilde{x},1)) \\
&= L^{-1}_{\tilde{f}(\tilde{x}),j}(E^s(\tilde{f}(\tilde{x}),0)) = \mathbf{R}^k\times\{0\}\,, \qquad (1.7)\\
D_0F_{\tilde{x},i,j}(\{0\}\times\mathbf{R}^{\dim M-k}) &= (L^{-1}_{\tilde{f}(\tilde{x}),j}\circ D_{x_0}f\circ L_{\tilde{x},i})(\{0\}\times\mathbf{R}^{\dim M-k}) \\
&= (L^{-1}_{\tilde{f}(\tilde{x}),j}\circ D_{x_0}f)(E^u(\tilde{x},0)) \\
&= L^{-1}_{\tilde{f}(\tilde{x}),j}(E^u(\tilde{x},1)) \\
&= L^{-1}_{\tilde{f}(\tilde{x}),j}(E^u(\tilde{f}(\tilde{x}),0)) = \{0\}\times\mathbf{R}^{\dim M-k}\,. \qquad (1.8)
\end{aligned}$$

From (1.3) and (1.7) it follows that for $\mathbf{v}\in\mathbf{R}^k\times\{0\}$

$$\begin{aligned}
|D_0F_{\tilde{x},i,j}(\mathbf{v})| &= |L^{-1}_{\tilde{f}(\tilde{x}),j}\circ D_{x_0}f\circ L_{\tilde{x},i}(\mathbf{v})| \\
&= \frac{1}{\xi(\tilde{f}(\tilde{x}))}\,\|D_{x_0}f(L_{\tilde{x},i}(\mathbf{v}))\|'_{\tilde{f}(\tilde{x})} \\
&\le \frac{1}{\xi(\tilde{f}(\tilde{x}))}\exp\left(-\frac{99}{100}\chi\right)\|L_{\tilde{x},i}(\mathbf{v})\|'_{\tilde{x}} \\
&\le \frac{1}{\xi(\tilde{f}(\tilde{x}))}\exp\left(-\frac{99}{100}\chi\right)\xi(\tilde{x})|\mathbf{v}| \\
&\le \exp\left(-\frac{98}{100}\chi\right)|\mathbf{v}| \le \lambda|\mathbf{v}|\,, \qquad (1.9)
\end{aligned}$$

and from (1.3) and (1.8) for $\mathbf{w}\in\{0\}\times\mathbf{R}^{\dim M-k}$

$$\begin{aligned}
|D_0F_{\tilde{x},i,j}(\mathbf{w})| &= |L^{-1}_{\tilde{f}(\tilde{x}),j}\circ D_{x_0}f\circ L_{\tilde{x},i}(\mathbf{w})| \\
&= \frac{1}{\xi(\tilde{f}(\tilde{x}))}\,\|D_{x_0}f(L_{\tilde{x},i}(\mathbf{w}))\|'_{\tilde{f}(\tilde{x})} \\
&\ge \frac{1}{\xi(\tilde{f}(\tilde{x}))}\exp\left(\frac{99}{100}\chi\right)\|L_{\tilde{x},i}(\mathbf{w})\|'_{\tilde{x}} \\
&\ge \frac{1}{\xi(\tilde{f}(\tilde{x}))}\exp\left(\frac{99}{100}\chi\right)\xi(\tilde{x})|\mathbf{w}| \\
&\ge \exp\left(\frac{98}{100}\chi\right)|\mathbf{w}| \ge \lambda^{-1}|\mathbf{w}|\,. \qquad (1.10)
\end{aligned}$$

Combining (1.7), (1.8), (1.9) and (1.10) we can express $D_0F_{\tilde{x},i,j}$ as follows:

$$D_0F_{\tilde{x},i,j}(\mathbf{z}^s,\mathbf{z}^u) = (A^s_{\tilde{x},i,j}\mathbf{z}^s, A^u_{\tilde{x},i,j}\mathbf{z}^u)\,;$$

$$|A^s_{\tilde{x},i,j}\mathbf{z}^s| \le \lambda|\mathbf{z}^s|\,,\qquad |A^u_{\tilde{x},i,j}\mathbf{z}^u| \ge \lambda^{-1}|\mathbf{z}^u| \qquad (1.11)$$

for $\mathbf{z}=(\mathbf{z}^s,\mathbf{z}^u)\in\mathbf{R}^k\times\mathbf{R}^{\dim M-k}=\mathbf{R}^{\dim M}$. On the other hand, by the definition of $F_{\tilde{x},i,j}$ we have

$$F_{\tilde{x},i,j}(0)=(L^{-1}_{\tilde{f}(\tilde{x}),j}\circ f_{\tilde{x}}\circ L_{\tilde{x},i})(0)$$
$$=(L^{-1}_{\tilde{f}(\tilde{x}),j}\circ \exp^{-1}_{x_1}\circ f\circ \exp_{x_0}\circ L_{\tilde{x},i})(0)=0\,, \tag{1.12}$$

and by (1.6) for $\mathbf{z}$, $\mathbf{v}\in\mathbf{R}^{\dim M}$

$$\begin{aligned}
&|D_{\mathbf{z}}F_{\tilde{x},i,j}(\mathbf{v})-D_0F_{\tilde{x},i,j}(\mathbf{v})|\\
&=|(L^{-1}_{\tilde{f}(\tilde{x}),j}\circ D_{L_{\tilde{x},i}(\mathbf{z})}f_{\tilde{x}}\circ L_{\tilde{x},i})(\mathbf{v})-(L^{-1}_{\tilde{f}(\tilde{x}),j}\circ D_{x_0}f\circ L_{\tilde{x},i})(\mathbf{v})|\\
&\le\frac{2}{\xi(\tilde{f}(\tilde{x}))}\|(D_{L_{\tilde{x},i}(\mathbf{z})}f_{\tilde{x}}\circ L_{\tilde{x},i})(\mathbf{v})-(D_{x_0}f\circ L_{\tilde{x},i})(\mathbf{v})\|'_{\tilde{f}(\tilde{x})}\\
&\le\frac{2}{\xi(\tilde{f}(\tilde{x}))}2^{-1}\exp(-\chi/100)\beta\|L_{\tilde{x},i}(\mathbf{v})\|'_{\tilde{x}}\\
&\le\frac{\xi(\tilde{x})}{\xi(\tilde{f}(\tilde{x}))}\exp(-\chi/100)\beta|\mathbf{v}|\le\beta|\mathbf{v}|\,.
\end{aligned}\tag{1.13}$$

By (1.11), (1.12) and (1.13), $F_{\tilde{x},i,j}$ has the form as in the property (3). If $1\le i\le d$ and $\tilde{x}\in\tilde{\Lambda}^k_{\chi,l}\cap\pi^{-1}U_i$ for some $l\ge1$, then by (1.4), (1.5) and (1.6) we have

$$\begin{aligned}
\tfrac14\|D_{\mathbf{z}}\Phi_{\tilde{x},i}(\mathbf{v})\|&=\tfrac14\|D_{\mathbf{z}}(\exp_{x_0}\circ L_{\tilde{x},i})(\mathbf{v})\|\\
&=\tfrac14\|(D_{L_{\tilde{x},i}(\mathbf{z})}\exp_{x_0})\circ L_{\tilde{x},i}(\mathbf{v})\|\\
&\le\tfrac12\|L_{\tilde{x},i}(\mathbf{v})\|\le\|L_{\tilde{x},i}(\mathbf{v})\|'_{\tilde{x}}\le|\mathbf{v}|\\
&\le2\xi(\tilde{x})^{-1}\|L_{\tilde{x},i}(\mathbf{v})\|'_{\tilde{x}}\le2\xi(\tilde{x})^{-1}a_l\|L_{\tilde{x},i}(\mathbf{v})\|\\
&\le2\xi_la_l\|L_{\tilde{x},i}(\mathbf{v})\|\le4\xi_la_l\|(D_{L_{\tilde{x},i}(\mathbf{z})}\exp_{x_0})\circ L_{\tilde{x},i}(\mathbf{v})\|\\
&=4\xi_la_l\|D_{\mathbf{z}}(\exp_{x_0}\circ L_{\tilde{x},i})(\mathbf{v})\|=4\xi_la_l\|D_{\mathbf{z}}\Phi_{\tilde{x},i}(\mathbf{v})\|
\end{aligned}$$

for $\mathbf{z}\in\mathbf{D}^{\dim M}(\alpha_0)$ and $\mathbf{v}\in\mathbf{R}^{\dim M}$, where

$$\xi_l=\xi_l(k,\chi)=\max\{\xi(\tilde{x})^{-1}:\tilde{x}\in\tilde{\Lambda}^k_{\chi,l}\}\,.$$

Letting $c_0=4$ and $c_l=c_l(k,\chi)=4\xi_la_l$, we have the property (4). From the continuity of $L_{\tilde{x},i}$ on $\tilde{\Lambda}^k_{\chi,l}\cap\pi^{-1}U_i$ for $l\ge1$, the property (5) follows. Lemma 1 was proved. □

We denote by $r_0>0$ the Lebesgue number of the finite open cover $\mathscr{U}=\{U_1,\cdots U_d\}$ of M. For each $x\in M$ take $i(x)$ with $1\le i(x)\le d$ such that $B(x,r_0/2)\subset U_{i(x)}$.

For a sequence δ_l ($l\ge1$) of positive real numbers we say that

$$\mathscr{A}=\{\tilde{x}^m=(x^m_n)\in\tilde{\Lambda}^k_\chi\,:\,m\in\mathbf{Z}\}$$

is a $(\delta_l)^\infty_{l=1}$*-pseudo orbit* of $\tilde{\Lambda}^k_\chi$ if there is a sequence $l_m\ge1$ ($m\in\mathbf{Z}$) of positive integers such that for $m\in\mathbf{Z}$:

$$\tilde{x}^m\in\tilde{\Lambda}^k_{\chi,l_m}\,,\quad \tilde{d}(\tilde{f}(\tilde{x}^{m-1}),\tilde{x}^m)\le\delta_{l_m}\quad\text{and}\quad l_{m\pm1}\le l_m+1\,.$$

For $0<\eta\le1/4$ we say that a point $\tilde{y}=(y_n)\in M_f$ is an *η-shadowing point* of a

$(\delta_l)_{l=1}^{\infty}$-pseudo orbit $\mathscr{A}$ if

$$\tilde{y} \in \bigcap_{n=-\infty}^{\infty} \tilde{f}^{-n}((\pi^{-1} \circ \Phi_{\tilde{x}^n, i(x_0^n)})(\mathbf{B}^k(\eta))),$$

where

$$\mathbf{B}^k(\eta) = \begin{cases} \mathbf{D}^k(\eta) \times \mathbf{D}^{\dim M - k}(\eta) & \text{if } 1 \leq k \leq \dim M - 1, \\ \mathbf{D}^{\dim M}(\eta) & \text{if } k = 0 \text{ or } \dim M. \end{cases}$$

Now we give the shadowing lemma as follows:

KEY LEMMA 2 (Shadowing lemma). *For* $0 \leq k \leq \dim M$, $\chi > 0$ *and* $0 < \eta \leq 1/4$ *there exists a sequence* $\delta_l(\eta) = \delta_l(k, \chi, \eta)$ $(l \geq 1)$ *of positive numbers such that every* $(\delta_l(\eta))_{l=1}^{\infty}$*-pseudo orbit* $\mathscr{A}$ *of* $\tilde{\Lambda}_{\chi}^k$ *has a unique* η*-shadowing point* $\tilde{y} = (y_n) \in M_f$. *For* $n \in \mathbf{Z}$ *the tangent space* $T_{y_n}M$ *splits into a direct sum* $T_{y_n}M = E^s(\tilde{y}, n) \oplus E^u(\tilde{y}, n)$ *such that*:

(1) $\dim E^s(\tilde{y}, n) = k$;

(2) $D_{y_n}f(E^s(\tilde{y}, n)) \subset E^s(\tilde{y}, n+1)$, $D_{y_n}f(E^u(\tilde{y}, n)) = E^u(\tilde{y}, n+1)$;

(3) *for* $m \geq 0$

$$\|D_{y_n}f^m(v)\| \leq c'_{l_n}\lambda_0^m \|v\| \qquad \text{for } v \in E^s(\tilde{y}, n),$$

$$\|(D_{y_{n-m}}f^m|_{E^u(\tilde{y}, n-m)})^{-1}(w)\| \leq c'_{l_n}\lambda_0^m \|w\| \qquad \text{for } w \in E^u(\tilde{y}, n)$$

where l_m $(m \in \mathbf{Z})$ *is a sequence as in the definition of pseudo orbit, and* $c'_l = c'_l(k, \chi) \geq 1$ $(l \geq 1)$, $\lambda_0 = \lambda_0(\chi)$ *with* $0 < \lambda_0 < 1$ *are numbers independent of* η, $\mathscr{A}$.

If we establish Key Lemma 2, then we have the following:

LEMMA 3 (Closing lemma). *For* $0 \leq k \leq \dim M$, $\chi > 0$, $l \geq 1$ *and* $\rho > 0$ *there exists a number* $\gamma_l(\rho) = \gamma_l(k, \chi, \rho) > 0$ *such that, if* $\tilde{x} = (x_n) \in \tilde{\Lambda}_{\chi,l}^k$ *satisfies*

$$\tilde{f}^m(\tilde{x}) \in \tilde{\Lambda}_{\chi,l}^k, \qquad \tilde{d}(\tilde{f}^m(\tilde{x}), \tilde{x}) \leq \gamma_l(\rho)$$

for some $m \geq 1$, *then there is a hyperbolic periodic point* $p = p(\tilde{x}) \in M$ *of* f *with* $f^m(p) = p$ *such that*

$$d(f^j(p), x_j) \leq \rho$$

for all $0 \leq j \leq m-1$.

PROOF. For $l \geq 1$ and $\rho > 0$ we put

$$\eta = \min\{\rho/(2c_0), 1/4\} > 0 \quad \text{and} \quad \gamma_l(\rho) = \delta_l(\eta),$$

where $c_0 \geq 1$ is as in Lemma 1, and $\delta_l(\eta) = \delta_l(k, \chi, \eta) > 0$ the number found from Key Lemma 2. We show that $\gamma_l(\rho)$ satisfies the assertion of Lemma 3. To do that, we assume that $\tilde{x} = (x_n) \in \tilde{\Lambda}_{\chi,l}^k$ satisfies

$$\tilde{f}^m(\tilde{x}) \in \tilde{\Lambda}^k_{\chi,l} \quad \text{and} \quad \tilde{d}(\tilde{f}^m(\tilde{x}), \tilde{x}) \le \gamma_l(\rho)$$

for some $m \ge 1$. Then the sequence

$$\mathscr{A} = \{\tilde{x}^{nm+j} = \tilde{f}^j(\tilde{x}) \ : \ n \in \mathbf{Z}, 0 \le j \le m-1\}$$

is a $(\delta_l(\eta))_{l=1}^{\infty}$-pseudo orbit of $\tilde{\Lambda}^k_\chi$. Therefore there is an η-shadowing point $\tilde{y} = (y_n) \in M_f$ of $\mathscr{A}$. It is easy to see that $\tilde{f}^m(\tilde{y})$ is also an η-shadowing point of $\mathscr{A}$. By the uniqueness of shadowing point we have $\tilde{f}^m(\tilde{y}) = \tilde{y}$, and thus $p = p(\tilde{x}) = y_0$ is a hyperbolic periodic point of f with $f^m(p) = p$. For $0 \le j \le m-1$ since

$$x_j, y_j \in \Phi_{\tilde{f}^j(\tilde{x}), i(x_j)}(\mathbf{B}^k(\eta)),$$

by Lemma 1 (4) we have

$$\begin{aligned} d(f^j(p), x_j) = d(y_j, x_j) &\le diam\, \Phi_{\tilde{f}^j(\tilde{x}), i(x_j)}(\mathbf{B}^k(\eta)) \\ &\le c_0 \cdot diam(\mathbf{B}^k(\eta)) \le 2c_0\eta \le \rho \end{aligned}$$

where $diam A$ denotes the diameter of a set A. Lemma 3 was proved. □

2. Proof of Key Lemma 2.

Let $0 \le k \le \dim M$, $\chi > 0$ and $0 < \eta \le 1/4$. By Lemma 1 (5), for $l \ge 1$ we can choose a number $\delta_l(\eta) = \delta_l(k, \chi, \eta)$ with $0 < \delta_l(\eta) < r_0/2$ so small that if $\tilde{x} = (x_n)$, $\tilde{y} = (y_n) \in \tilde{\Lambda}^k_{\chi,l}$ and $\tilde{d}(\tilde{x}, \tilde{y}) \le \delta_l(\eta)$ then

$$(\Phi^{-1}_{\tilde{y}, i(y_0)} \circ \Phi_{\tilde{x}, i(y_0)})(\mathbf{B}^k(\eta)) \subset \mathbf{D}^d(1),$$

$$|(\Phi^{-1}_{\tilde{y}, i(y_0)} \circ \Phi_{\tilde{x}, i(y_0)})(0)| \le \beta\eta, \tag{2.1}$$

$$|D_{\mathbf{z}}(\Phi^{-1}_{\tilde{y}, i(y_0)} \circ \Phi_{\tilde{x}, i(y_0)}) - id| \le \beta \qquad \text{for } \mathbf{z} \in \mathbf{B}^k(\eta) \tag{2.2}$$

where $\beta = \beta(\chi) > 0$ is the number as in Lemma 1 and id the identity map of $\mathbf{R}^{\dim M}$.

From now on, we show that the sequence $\delta_l(\eta)$ $(l \ge 1)$ satisfies the assertion of Key Lemma 2. We put

$$c'_l = c'_l(k, \chi) = 2c_0c_l \ge 1 \qquad (l \ge 1),$$

$$\lambda_0 = \lambda_0(\chi) = \max\{(1+\beta)(\lambda+2\beta), (1-\beta)^{-1}(\lambda^{-1}-2\beta)^{-1}\} \in (0, 1-\beta),$$

where c_0, $c_l \ge 1$ $(l \ge 1)$ and $0 < \lambda < 1$ are numbers as in Lemma 1.

Let $\mathscr{A} = \{\tilde{x}^m = (x^m_n) : m \in \mathbf{Z}\}$ be a $(\delta_l(\eta))_{l=1}^{\infty}$-pseudo orbit of $\tilde{\Lambda}^k_\chi$, and l_m $(m \in \mathbf{Z})$ the sequence as in the definition of pseudo orbit. We check that there exists a unique η-shadowing point of $\mathscr{A}$, and which has the property stated in Key Lemma 2. To simplify the notations we write

$$\Phi_n = \Phi_{\tilde{x}^n, i(x^n_0)},$$

$$G_n = (\Phi^{-1}_{\tilde{x}^{n+1}, i(x^{n+1}_0)} \circ \Phi_{\tilde{f}(\tilde{x}^n), i(x^{n+1}_0)}) \circ F_{\tilde{x}^n, i(x^n_0), i(x^{n+1}_0)}$$

for $n \in \mathbf{Z}$.

The proof is done by dividing into three cases:

(I) $1 \leq k \leq \dim M - 1$;

(II) $k = 0$;

(III) $k = \dim M$.

Case (I). We define the stable and unstable cones C^s, $C^u \subset \mathbf{R}^{\dim M}$ by

$$C^s = \{\mathbf{z} = (\mathbf{z}^s, \mathbf{z}^u) \in \mathbf{R}^k \times \mathbf{R}^{\dim M - k} \ : \ |\mathbf{z}^s| \geq |\mathbf{z}^u|\},$$
$$C^u = \{\mathbf{z} = (\mathbf{z}^s, \mathbf{z}^u) \in \mathbf{R}^k \times \mathbf{R}^{\dim M - k} \ : \ |\mathbf{z}^s| \leq |\mathbf{z}^u|\}.$$

Then from Lemma 1 (3) and (2.2) it follows that for $\mathbf{z} \in \mathbf{B}^k(\eta)$, $\mathbf{v} = (\mathbf{v}^s, \mathbf{v}^u) \in \mathbf{R}^k \times \mathbf{R}^{\dim M - k}$ and $n \in \mathbf{Z}$:

(1) if $\mathbf{v} \in C^u$, then

$$D_{\mathbf{z}} G_n(\mathbf{v}) \in C^u \quad \text{and} \quad |\mathbf{v}^u| \leq \lambda_0 |\pi^u D_{\mathbf{z}} G_n(\mathbf{v})|; \tag{2.3}$$

(2) if $D_{\mathbf{z}} G_n(\mathbf{v}) \in C^s$, then

$$\mathbf{v} \in C^s \quad \text{and} \quad |\pi^s D_{\mathbf{z}} G_n(\mathbf{v})| \leq \lambda_0 |\mathbf{v}^s|, \tag{2.4}$$

where $\pi^s : \mathbf{R}^{\dim M} \to \mathbf{R}^k$, $\pi^u : \mathbf{R}^{\dim M} \to \mathbf{R}^{\dim M - k}$ are natural projections defined by

$$\pi^\sigma(\mathbf{v}^s, \mathbf{v}^u) = \mathbf{v}^\sigma \qquad (\sigma = s, u).$$

We say that a $(\dim M - k)$ dimensional submanifold $I \subset \mathbf{B}^k(\eta)$ is an admissible (u, η)-manifold if I has the form

$$I = graph(\psi) = \{(\psi(\mathbf{v}), \mathbf{v}) \in \mathbf{D}^k(\eta) \times \mathbf{D}^{\dim M - k}(\eta) \ : \ \mathbf{v} \in \mathbf{D}^{\dim M - K}(\eta)\}$$

where $\psi : \mathbf{D}^{\dim M - k}(\eta) \to \mathbf{D}^k(\eta)$ is a C^1 map such that $|D_{\mathbf{v}} \psi| \leq 1$ for $\mathbf{v} \in \mathbf{D}^{\dim M - k}(\eta)$. Combining Lemma 1 (3) with (2.1), (2.3) and (2.4), it can be checked that if I is an admissible (u, η)-manifold then $G_n(I) \cap \mathbf{B}^k(\eta)$ is also an admissible (u, η)-manifold for $n \in \mathbf{Z}$, even if f is non-invertible.

For $m \geq 1$ subsets

$$I^m_{-m} = \{0\} \times \mathbf{D}^{\dim M - k}(\eta) \subset \mathbf{B}^k(\eta),$$
$$I^m_{n+1} = G_n(I^m_n) \cap \mathbf{B}^k(\eta) \qquad (-m \leq n \leq m-1)$$

are admissible (u, η)-manifolds which satisfy

$$\begin{aligned} \Phi_{n+1}(I^m_{n+1}) &= \Phi_{n+1}(G_n(I^m_n) \cap \mathbf{B}^k(\eta)) \\ &= (\Phi_{\tilde{f}(\tilde{x}^n), i(x_0^{n+1})} \circ F_{\tilde{x}^n, i(x_0^n), i(x_0^{n+1})})(I^m_n) \cap \Phi_{n+1}(\mathbf{B}^k(\eta)) \\ &= f(\Phi_n(I^m_n)) \cap \Phi_{n+1}(\mathbf{B}^k(\eta)) \end{aligned}$$

for $-m \leq n \leq m-1$. If $y^m_m \in \Phi_m(I^m_m)$, then by induction on $1 \leq j \leq 2m$ there is $y^m_{m-j} \in \Phi_{m-j}(I^m_{m-j})$ such that $f(y^m_{m-j}) = y^m_{m-j+1}$. Then we have

$$y_n = \lim_{m\to\infty} y_n^m \in \Phi_n(I_n^m) \subset \Phi_n(\mathbf{B}^k(\eta))$$

(take a subsequence if necessary) for $n \in \mathbf{Z}$. Thus $\tilde{y} = (y_n) \in M_f$ is an η-shadowing point of $\mathscr{A}$.

The uniqueness of η-shadowing point follows immediately from:

CLAIM (*).

$$diam(\bigcap_{n=-m}^{m} \tilde{f}^{-n}((\pi^{-1} \circ \Phi_n)(\mathbf{B}^k(\eta)))) \le 12c_0\eta\lambda_0^{m/2} + 4 \cdot diam M \cdot 2^{-m/2}.$$

In order to prove Claim (*), we take

$$\tilde{y} = (y_n), \qquad \tilde{z} = (z_n) \in \bigcap_{n=-m}^{m} \tilde{f}^{-n}((\pi^{-1} \circ \Phi_n)(\mathbf{B}^k(\eta))),$$

and show that

$$\tilde{d}(\tilde{y}, \tilde{z}) \le 12c_0\eta\lambda_0^{m/2} + 4 \cdot diam M \cdot 2^{-m/2}.$$

For $-m \le n \le m$ put

$$\mathbf{y}_n = (\mathbf{y}_n^s, \mathbf{y}_n^u) = \Phi_n^{-1}(y_n), \qquad \mathbf{z}_n = (\mathbf{z}_n^s, \mathbf{z}_n^u) = \Phi_n^{-1}(z_n) \in \mathbf{B}^k(\eta).$$

Then

$$|\mathbf{y}_n - \mathbf{z}_n| \le 4\eta \max\{\lambda_0^{m+n}, \lambda_0^{m-n}\}.$$

This follows from the fact that: if $|\mathbf{y}_n^s - \mathbf{z}_n^s| \le |\mathbf{y}_n^u - \mathbf{z}_n^u|$, then since $\mathbf{y}_m^u, \mathbf{z}_m^u \in \mathbf{D}^{\dim M - k}(\eta)$ by (2.3) we have

$$|\mathbf{y}_n^u - \mathbf{z}_n^u| \le \lambda_0^{m-n} |\mathbf{y}_m^u - \mathbf{z}_m^u| \le 2\eta\lambda_0^{m-n},$$

$$|\mathbf{y}_n - \mathbf{z}_n| \le |\mathbf{y}_n^s - \mathbf{z}_n^s| + |\mathbf{y}_n^u - \mathbf{z}_n^u| \le 4\eta\lambda_0^{m-n};$$

if $|\mathbf{y}_n^s - \mathbf{z}_n^s| \ge |\mathbf{y}_n^u - \mathbf{z}_n^u|$, then by (2.4)

$$|\mathbf{y}_n^s - \mathbf{z}_n^s| \le \lambda_0^{m+n} |\mathbf{y}_{-m}^s - \mathbf{z}_{-m}^s| \le 2\eta\lambda_0^{m+n},$$

$$|\mathbf{y}_n - \mathbf{z}_n| \le 4\eta\lambda_0^{m+n}.$$

Thus we have

$$|\mathbf{y}_n - \mathbf{z}_n| \le 4\eta \max\{\lambda_0^{m+n}, \lambda_0^{m-n}\}.$$

Combining this and Lemma 1 (4), we have

$$d(y_n, z_n) = d(\Phi_n(\mathbf{y}_n), \Phi_n(\mathbf{z}_n)) \le c_0 \cdot |\mathbf{y}_n - \mathbf{z}_n| \le 4c_0\eta \max\{\lambda_0^{m-n}, \lambda_0^{m+n}\}$$

for $-m \le n \le m$. Then

$$\tilde{d}(\tilde{y}, \tilde{z}) = \sum_{n=-\infty}^{\infty} 2^{-|n|} d(y_n, z_n)$$

$$
\begin{aligned}
&= \sum_{n=-[m/2]}^{[m/2]} 2^{-|n|} d(y_n, z_n) + \left(\sum_{n<-[m/2]} + \sum_{n>[m/2]} \right) 2^{-|n|} d(y_n, z_n) \\
&\le \sum_{n=-[m/2]}^{[m/2]} 2^{-|n|} 4c_0\eta \max\{\lambda_0^{m-n}, \lambda_0^{m+n}\} + 2 \cdot \sum_{n=[m/2]+1}^{\infty} 2^{-n} \cdot diam M \\
&\le \sum_{n=-[m/2]}^{[m/2]} 2^{-|n|} 4c_0\eta \lambda_0^{m-[m/2]} + 2^{-[m/2]+1} \cdot diam M \\
&\le 12 c_0 \eta \lambda_0^{m/2} + 4 \cdot diam M \cdot 2^{-m/2} .
\end{aligned}
$$

where [·] denotes the Gauss' symbols. Claim (*) was proved.

For the η-shadowing point $\tilde{y} = (y_n)$ of $\mathscr{A}$ and $n \in \mathbf{Z}$ we denote

$$E^s(\tilde{y}, n) = \bigcap_{m=0}^{\infty} D_{\mathbf{y}_n} \Phi_n((D_{\mathbf{y}_n}(G_{n+m-1} \circ \cdots \circ G_n))^{-1}(C^s)) ,$$

$$E^u(\tilde{y}, n) = \bigcap_{m=0}^{\infty} D_{\mathbf{y}_n} \Phi_n(D_{\mathbf{y}_{n-m}}(G_{n-1} \circ \cdots \circ G_{n-m})(C^u)) ,$$

where $\mathbf{y}_n = \Phi_n^{-1}(y_n) \in \mathbf{B}^k(\eta)$. Then these are subspaces of $T_{y_n}M$ such that

$$T_{y_n}M = E^s(\tilde{y}, n) \oplus E^u(\tilde{y}, n) , \quad \dim E^s(\tilde{y}, n) = k ,$$

$$D_{y_n} f(E^s(\tilde{y}, n)) \subset E^s(\tilde{y}, n+1) , \quad D_{y_n} f(E^u(\tilde{y}, n)) = E^u(\tilde{y}, n+1) .$$

By Lemma 1 (4) and (2.4) we have that for $v \in E^s(\tilde{y}, n)$

$$
\begin{aligned}
\|D_{y_n} f^m(v)\| &= \|D_{y_n}(\Phi_{n+m} \circ (G_{n+m-1} \circ \cdots \circ G_n) \circ \Phi_n^{-1})(v)\| \\
&\le c_0 | D_{\mathbf{y}_n}((G_{n+m-1} \circ \cdots \circ G_n) \circ \Phi_n^{-1})(v) | \\
&\le 2c_0 | \pi^s \circ D_{\mathbf{y}_n}((G_{n+m-1} \circ \cdots \circ G_n) \circ \Phi_n^{-1})(v) | \\
&\le 2c_0 \lambda_0^m | \pi^s (D_{y_n} \Phi_n^{-1}(v)) | \le 2c_0 \lambda_0^m | D_{y_n} \Phi_n^{-1}(v) | \\
&\le 2c_0 \lambda_0^m c_{l_n} \|v\| = c'_{l_n} \lambda_0^m \|v\| ,
\end{aligned}
$$

and by (2.3) for $w \in E^u(\tilde{y}, n)$

$$
\begin{aligned}
\|w\| &= \|(D_{y_{n-m}} f^m) \circ (D_{y_{n-m}} f^m |_{E^u(\tilde{y}, n-m)})^{-1}(w)\| \\
&= \|(D_{y_{n-m}}(\Phi_n \circ (G_{n-1} \circ \cdots \circ G_{n-m}) \circ \Phi_{n-m}^{-1})) \circ (D_{y_{n-m}} f^m |_{E^u(\tilde{y}, n-m)})^{-1}(w)\| \\
&\ge c_{l_n}^{-1} | (D_{y_{n-m}}((G_{n-1} \circ \cdots \circ G_{n-m}) \circ \Phi_{n-m}^{-1})) \circ (D_{y_{n-m}} f^m |_{E^u(\tilde{y}, n-m)})^{-1}(w) | \\
&\ge c_{l_n}^{-1} | \pi^u \circ (D_{y_{n-m}}((G_{n-1} \circ \cdots \circ G_{n-m}) \circ \Phi_{n-m}^{-1})) \circ (D_{y_{n-m}} f^m |_{E^u(\tilde{y}, n-m)})^{-1}(w) | \\
&\ge c_{l_n}^{-1} \lambda_0^{-m} | \pi^u \circ (D_{y_{n-m}} \Phi_{n-m}^{-1}) \circ (D_{y_{n-m}} f^m |_{E^u(\tilde{y}, n-m)})^{-1}(w) | \\
&\ge \tfrac{1}{2} c_{l_n}^{-1} \lambda_0^{-m} | (D_{y_{n-m}} \Phi_{n-m}^{-1}) \circ (D_{y_{n-m}} f^m |_{E^u(\tilde{y}, n-m)})^{-1}(w) | \\
&\ge \tfrac{1}{2} c_{l_n}^{-1} \lambda_0^{-m} c_0^{-1} \|(D_{y_{n-m}} f^m |_{E^u(\tilde{y}, n-m)})^{-1}(w)\| \\
&= c'^{-1}_{l_n} \lambda_0^{-m} \|(D_{y_{n-m}} f^m |_{E^u(\tilde{y}, n-m)})^{-1}(w)\|
\end{aligned}
$$

for all $m \ge 1$. Key Lemma 2 was concluded in the case of (I).

CASE (II). By Lemma 1 (3) and (2.2) we have

$$|D_{\mathbf{z}}G_n(\mathbf{v})|\geq\lambda_0^{-1}|\mathbf{v}|$$

for $\mathbf{z}\in\mathbf{D}^{\dim M}(\eta)$, $\mathbf{v}\in\mathbf{R}^{\dim M}$ and $n\in\mathbf{Z}$. Combining this and (2.1), for $n\in\mathbf{Z}$ we have

$$\mathbf{D}^{\dim M}(\eta)\subset\mathbf{D}^{\dim M}((\lambda_0^{-1}-\beta)\eta)\subset G_n(\mathbf{D}^{\dim M}(\eta))\,,$$

and thus

$$\begin{aligned}\Phi_{n+1}(\mathbf{D}^{\dim M}(\eta))&\subset(\Phi_{n+1}\circ G_n)(\mathbf{D}^{\dim M}((\lambda_0^{-1}-\beta)\eta))\\&=f(\Phi_n(\mathbf{D}^{\dim M}(\eta)))\,.\end{aligned}\tag{2.5}$$

For $m\geq 1$ take $y_m^m\in\Phi_m(\mathbf{D}^{\dim M}(\eta))$. By (2.5) and induction on $j\geq 1$ we can choose $y_{m-j}^m\in\Phi_{m-j}(\mathbf{D}^{\dim M}(\eta))$ such that $f(y_{m-j}^m)=y_{m-j+1}^m$. For $n\in\mathbf{Z}$ we have

$$y_n=\lim_{m\to\infty}y_n^m\in\Phi_n(\mathbf{D}^{\dim M}(\eta))$$

(take a subsequence if necessary). Then $\tilde{y}=(y_n)\in M_f$ is an η-shadowing point of $\mathscr{A}$. By similar calculation as in the case of (I), it can be checked that the assertion of Claim(*) holds even if $k=0$. Thus we have the uniqueness of shadowing point. By Lemma 1 (4) and (2.2) the η-shadowing point $\tilde{y}=(y_n)\in M_f$ ensures that for $n\in\mathbf{Z}$

$$\|D_{y_n}f(v)\|=\|D_{y_n}(\Phi_{n+1}\circ G_n\circ\Phi_n^{-1})(v)\|\geq c_{l_{n+1}}^{-1}\lambda_0^{-1}c_0^{-1}\|v\|$$

for $v\in T_{y_n}M$. Thus, $D_{y_n}f:T_{y_n}M\to T_{y_{n+1}}M$ is an isomorphism, and for $v\in T_{y_n}M$ and $m\geq 1$

$$\begin{aligned}\|v\|&=\|(D_{y_{n-m}}f^m)\circ(D_{y_{n-m}}f^m)^{-1}(v)\|\\&=\|\{D_{y_{n-m}}(\Phi_n\circ(G_{n-1}\circ\cdots\circ G_{n-m})\circ\Phi_{n-m}^{-1})\}\circ(D_{y_{n-m}}f^m)^{-1}(v)\|\\&\geq c_{l_n}^{-1}\lambda_0^{-m}c_0^{-1}\|(D_{y_{n-m}}f^m)^{-1}(v)\|\geq c_{l_n}'^{-1}\lambda_0^{-m}\|(D_{y_{n-m}}f^m)^{-1}(v)\|\,.\end{aligned}$$

Since $k=0$, we have the conclusion of Key Lemma 2 if $E^s(\tilde{y},n)=\{0\}$ and $E^u(\tilde{y},n)=T_{y_n}M$.

CASE (III). The proof is similar to that of (II), and so we omit it. Key Lemma 2 was proved. □

3. Proofs of theorems.

Using Key Lemma 2 and Lemma 3 we prove our theorems.

PROOF OF THEOREM A. We conclude the proof using a technique as follows:

CLAIM. *Under the assumption of Theorem A, there are* $N\geq 1$ *and* $\tilde{f}^{2N}$*-invariant set* $\tilde{\Gamma}_N$ *such that*

(1) $\pi\tilde{\Gamma}_N$ *is a hyperbolic set of* f^{2N};

(2) $\tilde{f}^{2N}|_{\tilde{\Gamma}_N}:\tilde{\Gamma}_N\to\tilde{\Gamma}_N$ *is topologically conjugate to a full shift in* 2*-symbols,*

$\sigma_2 : \Sigma_2 \to \Sigma_2$.

In fact, if it is true, then $\Gamma = \pi\tilde{\Gamma}_N \cup \cdots \cup f^{2N-1}(\pi\tilde{\Gamma}_N)$ is a hyperbolic horseshoe of f, and

$$h(f) = h(\tilde{f}) = \frac{1}{2N} h(\tilde{f}^{2N}) \geq \frac{1}{2N} h(\tilde{f}^{2N}|_{\tilde{\Gamma}_N}) = \frac{1}{2N} \log 2 > 0 .$$

Theorem A holds.

Thus it is enough to see the claim. Let μ be a non-atomic ergodic hyperbolic measure of f and $\tilde{\mu}$ the $\tilde{f}$-invariant Borel probability measure on M_f such that $\mu = \pi_* \tilde{\mu}$. Then by (1.2) there are $0 \leq k \leq \dim M$, $\chi > 0$ and $l \geq 1$ such that $\tilde{\mu}(\tilde{\Lambda}^k_{\chi,l}) > 0$. To simplify the notation we write $\Phi_{\tilde{y}}$ instead of $\Phi_{\tilde{y},i(y_0)}$ for $\tilde{y} = (y_n) \in \tilde{\Lambda}^k_\chi$. Take $\tilde{K} \subset \tilde{\Lambda}^k_{\chi,l}$ satisfying

$$\tilde{\mu}(\tilde{K}) > 0 \quad \text{and} \quad diam(\tilde{K}) \leq \delta_l(1/4) \tag{3.1}$$

where $\delta_l(1/4) = \delta_l(k, \chi, 1/4) > 0$ is a number found from Key Lemma 2. Since $\tilde{\mu}$ is non-atomic, by Poincaré's recurrence theorem, there are $\tilde{x}^1 = (x^1_n)$, $\tilde{x}^2 = (x^2_n) \in \tilde{K}$ with $\tilde{x}^1 \neq \tilde{x}^2$ such that for $i = 1, 2$

$$\tilde{f}^{-m^i_j}(\tilde{x}^i), \quad \tilde{f}^{n^i_j}(\tilde{x}^i) \in \tilde{K} \qquad (j \geq 1) \tag{3.2}$$

where $1 \leq m^i_1 < m^i_2 < \cdots$, $1 \leq n^i_1 < n^i_2 < \cdots$. Then,

$$\bigcap_{j=-\infty}^{\infty} \tilde{f}^{-j}((\pi^{-1} \circ \Phi_{\tilde{f}^j(\tilde{x}^1)})(\mathbf{B}^k(1/4)) \cap (\pi^{-1} \circ \Phi_{\tilde{f}^j(\tilde{x}^2)})(\mathbf{B}^k(1/4))) = \emptyset . \tag{3.3}$$

Indeed, if (3.3) is not true, then there exists a point

$$\tilde{y} \in \bigcap_{j=-\infty}^{\infty} \tilde{f}^{-j}((\pi^{-1} \circ \Phi_{\tilde{f}^j(\tilde{x}^1)})(\mathbf{B}^k(1/4)) \cap (\pi^{-1} \circ \Phi_{\tilde{f}^j(\tilde{x}^2)})(\mathbf{B}^k(1/4))) .$$

On the other hand, for $i = 1, 2$ the orbit

$$\mathcal{O}_{\tilde{f}}(\tilde{x}^i) = \{\cdots, \tilde{f}^{-1}(\tilde{x}^i), \tilde{x}^i, \tilde{f}(\tilde{x}^i), \cdots\}$$

of $\tilde{x}^i$ by $\tilde{f}$ is a $(\delta_l(1/4))_{l=1}^{\infty}$-pseudo orbit of $\tilde{\Lambda}^k_\chi$. Since the $(1/4)$-shadowing point of $\mathcal{O}_{\tilde{f}}(\tilde{x}^i)$ is unique, we have $\tilde{x}^i = \tilde{y}$. This contradicts $\tilde{x}^1 \neq \tilde{x}^2$. Thus (3.3) holds. By (3.3) we can choose $N_1 \geq 1$ such that

$$\bigcap_{j=-N_1}^{N_1-1} \tilde{f}^{-j}((\pi^{-1} \circ \Phi_{\tilde{f}^j(\tilde{x}^1)})(\mathbf{B}^k(1/4)) \cap (\pi^{-1} \circ \Phi_{\tilde{f}^j(\tilde{x}^2)})(\mathbf{B}^k(1/4))) = \emptyset . \tag{3.4}$$

Take $j_0 \geq 1$ so large that $m_i = m^i_{j_0}$, $n_i = n^i_{j_0} \geq N_1$ hold for $i = 1, 2$, and put

$$N = (m_1 + n_1)(m_2 + n_2) \geq N_1 .$$

For $i = 1, 2$ define a $(\delta_l(1/4))_{l=1}^{\infty}$-pseudo orbit $\{\tilde{y}^{i,n} : n \in \mathbf{Z}\}$ of $\tilde{\Lambda}^k_\chi$ by

$$\tilde{y}^{i,m(n_i+m_i)+j} = \tilde{f}^j(\tilde{x}^i) \qquad (m \in \mathbf{Z},\ -m_i \leq j \leq n_1 - 1) \tag{3.5}$$

and put

$$\tilde{\Gamma}_N^i = \bigcap_{j=-N}^{N-1} \tilde{f}^{-j}((\pi^{-1} \circ \Phi_{\tilde{y}^{i,j}})(\mathbf{B}^k(1/4))) .$$

By (3.4) we have

$$\begin{aligned}\tilde{\Gamma}_N^1 \cap \tilde{\Gamma}_N^2 &= \bigcap_{j=-N}^{N-1} \tilde{f}^{-j}((\pi^{-1} \circ \Phi_{\tilde{y}^{1,j}})(\mathbf{B}^k(1/4)) \cap (\pi^{-1} \circ \Phi_{\tilde{y}^{2,j}})(\mathbf{B}^k(1/4))) \\ &\subset \bigcap_{j=-N_1}^{N_1-1} \tilde{f}^{-j}((\pi^{-1} \circ \Phi_{\tilde{f}^j(\tilde{x}^1)})(\mathbf{B}^k(1/4)) \cap (\pi^{-1} \circ \Phi_{\tilde{f}^j(\tilde{x}^2)})(\mathbf{B}^k(1/4))) \\ &= \varnothing . \end{aligned} \tag{3.6}$$

We define a compact $\tilde{f}^{2N}$-invariant set $\tilde{\Gamma}_N$ of M_f by

$$\tilde{\Gamma}_N = \bigcap_{m=-\infty}^{\infty} \tilde{f}^{-2mN}(\tilde{\Gamma}_N^1 \cup \tilde{\Gamma}_N^2) ,$$

and show that $\tilde{\Gamma}_N$ satisfies the properties (1) and (2) of the claim. To do that we define a map $\varphi : \Sigma_2 \to \tilde{\Gamma}_N$ as follows. For $\bar{q} = (q_n) \in \Sigma_2$ we put

$$\tilde{z}^{2mN+j}(\bar{q}) = \tilde{y}^{q_m,j} \qquad (m \in \mathbf{Z},\ -N \le j \le N-1) .$$

Then by (3.1), (3.2) and (3.5)

$$\mathscr{A}(\bar{q}) = \{\tilde{z}^n(\bar{q}) : n \in \mathbf{Z}\}$$

is a $(\delta_l(1/4))_{l=1}^{\infty}$-pseudo orbit of $\tilde{\Lambda}_\chi^k$. Thus there is an unique (1/4)-shadowing point $\varphi(\bar{q}) \in M_f$ of $\mathscr{A}(\bar{q})$. Then,

$$\begin{aligned}\varphi(\bar{q}) &\in \bigcap_{n=-\infty}^{\infty} \tilde{f}^{-n}((\pi^{-1} \circ \Phi_{\tilde{z}^n(\bar{q})})(\mathbf{B}^k(1/4))) \\ &= \bigcap_{m=-\infty}^{\infty} \bigcap_{j=-N}^{N-1} \tilde{f}^{-2mN-j}((\pi^{-1} \circ \Phi_{\tilde{y}^{q_m,j}})(\mathbf{B}^k(1/4))) \\ &= \bigcap_{m=-\infty}^{\infty} \tilde{f}^{-2mN} \tilde{\Gamma}_N^{q_m} \subset \tilde{\Gamma}_N .\end{aligned}$$

On the other hand, for $\tilde{x} \in \tilde{\Gamma}_N$ take $\bar{q} = (q_n) \in \Sigma_2$ satisfying $\tilde{x} \in \bigcap_{m=-\infty}^{\infty} \tilde{f}^{-2mN} \tilde{\Gamma}_N^{q_m}$, then we have $\varphi(\bar{q}) = \tilde{x}$. Thus $\varphi(\Sigma_2) = \tilde{\Gamma}_N$. The continuity of φ follows from Claim (*) in the proof of Key Lemma 2. To see the injectivity of φ, we assume that $\bar{q} = (q_m)$, $\bar{q}' = (\bar{q}_m') \in \Sigma_2$ satisfy $q_m \neq q_m'$ for some $m \in \mathbf{Z}$. Since

$$\varphi(\bar{q}) \in \tilde{f}^{-2mN} \tilde{\Gamma}_N^{q_m} , \qquad \phi(\bar{q}') \in \tilde{f}^{-2mN} \tilde{\Gamma}_N^{q_m'} ,$$

by (3.6) we have $\varphi(\bar{q}) \neq \varphi(\bar{q}')$. Thus $\varphi : \Sigma_2 \to \tilde{\Gamma}_N$ is a homeomorphism. Obviously, $(\tilde{f}^{2N}|_{\tilde{\Gamma}_N}) \circ \varphi = \varphi \circ \sigma_2$ holds, and thus $\tilde{f}^{2N}|_{\tilde{\Gamma}_N} : \tilde{\Gamma}_N \to \tilde{\Gamma}_N$ is topologically conjugate to $\sigma_2 : \Sigma_2 \to \Sigma_2$. We obtained the property (2) of the claim. To get (1), we fix $\tilde{y} = (y_n) \in \tilde{\Gamma}_N$.

Since $\varphi:\Sigma_2\to\tilde{\Gamma}_N$ is surjective, there is $\bar{q}=(q_m)\in\Sigma_2$ such that $\varphi(\bar{q})=\tilde{y}$. By the definition of φ, $\tilde{y}$ is a $(1/4)$-shadowing point of a $(\delta_l(1/4))_{l=1}^{\infty}$-pseudo orbit of $\mathscr{A}(\bar{q})=\{\tilde{z}^n(\bar{q}):n\in\mathbf{Z}\}$ of $\tilde{\Lambda}_\chi^k$. For $n\in\mathbf{Z}$ let $T_{y_n}M=E^s(\tilde{y},n)\oplus E^u(\tilde{y},n)$ be the splitting as in Key Lemma 2. Then we have

$$D_{y_0}f^{2N}(E^s(\tilde{y},0))\subset E^s(\tilde{y},2N)=E^s(\tilde{f}^{2N}(\tilde{y}),0)\,,$$
$$D_{y_0}f^{2N}(E^u(\tilde{y},0))=E^u(\tilde{y},2N)=E^u(\tilde{f}^{2N}(\tilde{y}),0)\,.$$

For $m\in\mathbf{Z}$ since

$$\tilde{z}^{2mN}(\bar{q})=\tilde{y}^{q_m,0}=\tilde{x}^{q_m}\in\tilde{K}\subset\tilde{\Lambda}_{\chi,l}^k\,,$$

we have

$$\|D_{y_0}f^{2mN}(v)\|\le c_l'\lambda_0^{2mN}\|v\|\qquad\text{for }v\in E^s(\tilde{y},0)\,,$$
$$\|(D_{y_{-2mN}}f^{2mN}|_{E^u(\tilde{f}^{-2mN}(\tilde{y}),0)})^{-1}(w)\|\le c_l'\lambda_0^{2mN}\|w\|\qquad\text{for }w\in E^u(\tilde{y},0)\,.$$

Thus $\pi\tilde{\Gamma}_N$ is a hyperbolic set of f^{2N}. We proved the property (1), and thus the claim. This completes the proof of Theorem A. □

REMARK. In case f is a diffeomorphism, every non-atomic ergodic hyperbolic measure μ is of saddle type $(\chi_1^\mu<0<\chi_{r_\mu}^\mu)$. Thus we have a hyperbolic horseshoe of saddle type. However, if f is non-invertible, then it may have a non-atomic ergodic hyperbolic measure μ such that all the Lyapunov exponents are positive almost everywhere. Then the hyperbolic horseshoe obtained in Theorem A is a repellor, and it has infinitely many source periodic points.

REMARK. The homoclinic point theorem for differentiable maps can be found in [SW]. However, it was not used in our proof.

PROOF OF THEOREM B. Let μ be a hyperbolic measure of f. We denote $\tilde{\mu}$ as the $\tilde{f}$-invariant Borel probability measure on M_f with $\pi_*\tilde{\mu}=\mu$. For $x\in\operatorname{supp}(\mu)$ and $\rho>0$, by (1.1) we have

$$\tilde{\mu}(\pi^{-1}B(x,\rho/2)\cap(\textstyle\bigcup_{k=0}^{\dim M}\bigcup_{\chi>0}\tilde{\Lambda}_\chi^k))=\tilde{\mu}(\pi^{-1}B(x,\rho/2))$$
$$=\mu(B(x,\rho/2))>0\,.$$

Thus there are $0\le k\le\dim M$, $\chi>0$ and $l\ge1$ such that

$$\tilde{\mu}(\pi^{-1}B(x,\rho/2)\cap\tilde{\Lambda}_{\chi,l}^k)>0\,.$$

Let $\gamma_l(\rho/2)=\gamma_l(k,\chi,\rho/2)>0$ be as in Lemma 3, and take $\tilde{K}\subset\pi^{-1}B(x,\rho/2)\cap\tilde{\Lambda}_{\chi,l}^k$ satisfying

$$diam\,\tilde{K}\le\gamma_l(\rho/2)\,,\qquad\tilde{\mu}(\tilde{K})>0\,.$$

By Poincaré's recurrence theorem, there exist $\tilde{y}=(y_n)\in\tilde{K}$ and $m\ge1$ such that $\tilde{f}^m(\tilde{y})\in\tilde{K}$. Since $\tilde{y}$, $\tilde{f}^m(\tilde{y})\in\tilde{\Lambda}_{\chi,l}^k$ and

$$\tilde{d}(\tilde{f}^m(\tilde{y}), \tilde{y}) \leq diam\,\tilde{K} \leq \gamma_l(\rho/2),$$

by Lemma 3 there exists a hyperbolic periodic point $p = p(\tilde{y})$ of f with $f^m(p) = p$ such that $d(y_0, p) \leq \rho/2$. Since

$$\tilde{y} = (y_n) \in \tilde{K} \subset \pi^{-1}B(x, \rho/2),$$

we have $d(x, y_0) \leq \rho/2$ and so

$$d(x, p) \leq d(x, y_0) + d(y_0, p) \leq \rho .$$

Therefore, $\mathrm{supp}(\mu) \subset \overline{\mathrm{Per}(f)}$. □

PROOF OF THEOREM C. We assume without loss of generality that μ is ergodic. By (1.2), $\tilde{\mu}(\tilde{\Lambda}_\chi^k) = 1$ holds for some $0 \leq k \leq \dim M$ and $\chi > 0$. We choose $l \geq 1$ so large that $\tilde{\mu}(\tilde{\Lambda}_{\chi,l}^k) \geq 3/4$, and write $\tilde{\Lambda} = \tilde{\Lambda}_{\chi,l}^k$ to simplify the notation. For $n \geq 1$ we define the distance d_n^f on M by

$$d_n^f(x, y) = \max\{d(f^i(x), f^i(y)) : 0 \leq i \leq n-1\}$$

for $x, y \in M$. Denote by $N_\mu(n, \rho)$ the minimal number of ρ-balls with respect to the d_n^f-distance which cover the set with the value of μ-measure more than or equal to 1/2. Then it holds that

$$h_\mu(f) = \lim_{\rho \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N_\mu(n, \rho) .$$

(Katok [K] has proved this fact for homeomorphisms. We know that it is valid for all continuous maps.)

Thus, for $\gamma > 0$ we can choose $\rho > 0$ such that

$$\liminf_{n \to \infty} \frac{1}{n} \log N_\mu(n, \rho) \geq h_\mu(f) - \gamma . \tag{3.7}$$

Put $\eta = \min\{\rho/(8c_0), 1/4\} > 0$, where $c_0 = c_0 \geq 1$ is a number as in Lemma 1, and take a finite partition $\xi = \{\tilde{A}_1, \cdots, \tilde{A}_t\}$ of M_f such that

$$diam\,\xi \leq \delta_l(\eta), \qquad \xi \leq \{\tilde{\Lambda}, M_f \backslash \tilde{\Lambda}\}$$

where $\delta_l(\eta) = \delta_l(k, \chi, \eta) > 0$ is a number found from Key Lemma 2. For $n \geq 1$ we write

$$\tilde{\Lambda}_n = \{\tilde{x} \in \tilde{\Lambda} \ : \ \tilde{f}^m(\tilde{x}) \in \xi(\tilde{x}) \text{ for some } n \leq m \leq (1+\gamma)n\},$$

where $\xi(\tilde{x})$ denotes the element of ξ containing $\tilde{x}$. By Birkhöff's ergodic theorem we have

$$\tilde{\mu}(\tilde{\Lambda}_n) \geq \tilde{\mu}(\tilde{\Lambda}) - 1/4 \geq 1/2$$

for $n \geq 1$ large enough. Let $E_n = \{y^1, \cdots, y^{p_n}\}$ be an (n, ρ)-separated set of $\pi\tilde{\Lambda}_n$ for f with the maximal cardinality, and choose $\tilde{y}^q \in \tilde{\Lambda}_n$ such that $\pi(\tilde{y}^q) = y^q$ for $1 \leq q \leq p_n$. Then we have

$$\pi\tilde{\Lambda}_n \subset \bigcup_{q=1}^{p_n} B_n^f(y^q, \rho)\,,$$

$$\mu(\textstyle\bigcup_{q=1}^{p_n} B_n^f(y^q, \rho)) = \tilde{\mu}(\pi^{-1}(\bigcup_{q=1}^{p_n} B_n^f(y^q, \rho))) \ge \tilde{\mu}(\tilde{\Lambda}_n) \ge 1/2\,,$$

where $B_n^f(y, \rho) = \{z \in M : d_n^f(y, z) \le \rho\}$. Thus,

$$\sharp E_n = p_n \ge N_\mu(n, \rho)\,.$$

By (3.7) there exists $n_0 \ge 1$ such that

$$\sharp E_n \ge N_\mu(n, \rho) \ge \exp\{n(h_\mu(f) - 2\gamma)\} \tag{3.8}$$

for $n \ge n_0$. We fix $n \ge n_0$, and for m with $n \le m \le [(1+\gamma)n]$ put

$$X_m = \{1 \le q \le p_n : \tilde{f}^m(\tilde{y}^q) \in \xi(\tilde{y}^q)\}\,.$$

Since $\sharp(\bigcup_{m=n}^{[(1+\gamma)n]} X_m) = p_n = \sharp E_n$, by (3.8) we can choose $n \le m_0 \le [(1+\gamma)n]$ such that

$$\sharp X_{m_0} \ge \frac{1}{n\gamma + 1} \sharp E_n \ge \exp\{n(h_\mu(f) - 3\gamma)\}\,. \tag{3.9}$$

For $1 \le j \le t$ we put

$$Y_j = \{q \in X_{m_0} : \xi(\tilde{y}^q) = \tilde{A}_j\}\,.$$

Since $\bigcup_{j=1}^t Y_j = X_{m_0}$, by (3.9) there exists $1 \le j_0 \le t$ such that

$$\sharp Y_{j_0} \ge \frac{1}{t} \sharp X_{m_0} \ge \frac{1}{t} \exp\{n(h_\mu(f) - 3\gamma)\}\,. \tag{3.10}$$

For $q \in Y_{j_0}$ we put

$$\tilde{\Gamma}_0^q = \bigcap_{j=0}^{m_0-1} \tilde{f}^{-j}(\pi^{-1} \circ \Phi_{\tilde{f}^j(\tilde{y}^q)}(\mathbf{B}^k(\eta)))\,,$$

where $\Phi_{\tilde{y}} = \Phi_{\tilde{y}, i(y_0)}$ for $\tilde{y} = (y_n) \in \tilde{\Lambda}_\chi^k$. We claim that for a pair $q, q' \in Y_{j_0}$, $q \ne q'$ implies $\tilde{\Gamma}_0^q \cap \tilde{\Gamma}_0^{q'} = \varnothing$. Indeed, if $\tilde{z} = (z_n) \in \tilde{\Gamma}_0^q \cap \tilde{\Gamma}_0^{q'}$, then for $0 \le j \le m_0 - 1$ we have

$$z_j \in \Phi_{\tilde{f}^j(\tilde{y}^q)}(\mathbf{B}^k(\eta)) \cap \Phi_{\tilde{f}^j(\tilde{y}^{q'})}(\mathbf{B}^k(\eta))\,,$$

and hence, by Lemma 1 (4)

$$\begin{aligned} d(f^j(y^q), f^j(y^{q'})) &\le d(f^j(y^q), z_j) + d(f^j(y^{q'}), z_j) \\ &\le 2c_0\eta + 2c_0\eta = 4c_0\eta \le \rho/2\,. \end{aligned}$$

On the other hand, since $\{y^q : q \in Y_{j_0}\} \subset E_n$ is an (n, ρ)-separated set of f, we have

$$d(f^j(y^q), f^j(y^{q'})) > \rho$$

for some $0 \le j \le n-1 \le m_0 - 1$. It is a contradiction, and so $\tilde{\Gamma}_0^q \cap \tilde{\Gamma}_0^{q'} = \varnothing$.

Let

$$\tilde{\Gamma}_0 = \bigcap_{n=-\infty}^{\infty} \tilde{f}^{-nm_0}\left(\bigcup_{q \in Y_{j_0}} \tilde{\Gamma}_0^q\right).$$

Then it is a compact $\tilde{f}^{m_0}$-invariant set of M_f. By the same fashion as the proof of Theorem A, it is checked that $\tilde{f}^{m_0}|_{\tilde{\Gamma}_0} : \tilde{\Gamma}_0 \to \tilde{\Gamma}_0$ is topologically conjugate to a full shift in $\# Y_{j_0}$-symbols, and that

$$\Gamma = \pi\tilde{\Gamma}_0 \cup \cdots \cup f^{m_0-1}(\pi\tilde{\Gamma}_0) \subset M$$

is a hyperbolic horseshoe of f. Then by (3.10) we have

$$\begin{aligned} h(f|_\Gamma) &= \frac{1}{m_0} h(\tilde{f}^{m_0}|_{\tilde{\Gamma}_0}) = \frac{1}{m_0} \log \# Y_{j_0} \\ &\geq \frac{n}{m_0}(h_\mu(f) - 3\gamma) - \frac{1}{m_0} \log t \\ &\geq \frac{1}{1+\gamma}(h_\mu(f) - 3\gamma) - \frac{1}{n} \log t . \end{aligned}$$

By choosing $n \geq 1$ large and $\gamma > 0$ small enough, we have $h(f|_\Gamma) \geq h_\mu(f) - \varepsilon$. This completes the proof of Theorem C. □

PROOF OF THEOREM D. It is enough to see the case when $h_\mu(f) > 0$. For $\varepsilon > 0$ take a hyperbolic horseshoe Γ_ε of f as in Theorem C, and integers l, m as in the definition of horseshoe. Then,

$$h(f|_{\Gamma_\varepsilon}) = \frac{1}{m} \log l$$

and for $j \geq 1$

$$l^j \leq \# Fix(f^{mj}|_{\Gamma_\varepsilon}) \leq ml^j .$$

Then,

$$\begin{aligned} h_\mu(f) &\leq h(f|_{\Gamma_\varepsilon}) + \varepsilon = \frac{1}{m} \log l + \varepsilon \\ &= \lim_{j\to\infty} \frac{1}{mj} \log \# Fix(f^{mj}|_{\Gamma_\varepsilon}) + \varepsilon \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \# Fix(f^n) + \varepsilon . \end{aligned}$$

By letting $\varepsilon \to 0$, we have

$$h_\mu(f) \leq \limsup_{n\to\infty} \frac{1}{n} \log \# Fix(f^n) .$$

Theorem D was proved. □

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