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On p and q-Additive Functions

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1. Introduction.

Let q be an integer greater than 1. Let $a(n)$ be a complex-valued arithmetical function. $a(n)$ is said to be *q-additive* if

$$a(n)=\sum_{i\geq 0} a(b_i q^i)$$

for any positive integer $n=\sum_{i\geq 0} b_i q^i$ with $b_i \in \{0, 1, \cdots, q-1\}$, and $a(0)=0$. It follows from the definition that $a(n)$ is q-additive if and only if

$$a(nq^k+r)=a(nq^k)+a(r)$$

for any integer $n\geq 0$ and $k\geq 0$ with $0\leq r<q^k$. $a(n)$ is said to be *q-multiplicative* if

$$a(n)=\prod_{i\geq 0} a(b_i q^i)$$

for any positive integer n as above, and $a(0)=1$. $a(n)$ is a q-multiplicative function if and only if

$$a(nq^k+r)=a(nq^k)a(r)$$

for any $n\geq 0$ and $k\geq 0$ with $0\leq r<q^k$. If q-additive or q-multiplicative function $a(n)$ satisfies

$$a(bq^i)=a(b) \qquad (b\in\{0, 1, \cdots, q-1\}, i\geq 0)\,, \tag{1}$$

then $a(n)$ is said to be *strongly q-additive* or *strongly q-multiplicative*, respectively. We say $a(n)$ is p and q-additive if it is p-additive and also q-additive. Similarly, a p and q-multiplicative function is defined. The notion of q-additive functions and q-multiplicative functions were introduced by Gel'fond [2] and Delange [1] respectively and has been investigated by many authors (eg. [3], [4], [5]).

If $a(n)$ is a q-additive or q-multiplicative function, $a(n)$ is q^l-additive or q^l-

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multiplicative for any positive integer l.

Recently, Toshimitsu [8] proved that any strongly p and q-additive functions is identically zero, if $\log p/\log q$ is irrational. He also obtained a similar result for strongly p and q-multiplicative functions (see Corollary 1 and 2 below). His proofs based on the deep results in the transcendence theory of Mahler functions (cf. Nishioka [6], [7]). Elementary proofs of these results are given in [9]. In this paper, we determine explicitly the form of p and q-additive or multiplicative functions without the 'strongly' condition (1).

THEOREM 1. *Let p and q be integers greater than* 1 *such that* $\log p/\log q$ *is irrational. Let $a(n)$ be a p and q-additive function. Then there exist positive integers l, m,* and $g=\text{g.c.d.}(p^l, q^m)$ *such that $a(ng)=na(g)$ for each $n\geq 1$. If g is greater than* 1, *then $a(n)$ is g-additive.*

COROLLARY 1 (Toshimitsu [8; Theorem 3], [9]). *Let p and q be as in Theorem* 1. *Let $a(n)$ be a strongly p and q-additive function. Then $a(n)=0$ $(n\geq 0)$.*

THEOREM 2. *Let p and q be integers greater than* 1 *such that* $\log p/\log q$ *is irrational. Let $a(n)$ be a p and q-multiplicative function. If p and q are relatively prime, then $a(n)=a(1)^n$ $(n\geq 1)$ or there exists a positive integer l such that $a(np^l)=0$ $(n\geq 1)$. If p and q are not relatively prime, then there exist positive integers l, m, and $g=\text{g.c.d.}(p^l, q^m)$ such that $a(ng)=a(g)^n$ for each $n\geq 1$ and $a(n)$ is g-multiplicative.*

COROLLARY 2 (Toshimitsu [8; Theorem 4], [9]). *Let p and q be as in Theorem* 2. *Let $a(n)$ be a strongly p and q-multiplicative function. Then $a(n)=0$ $(n\geq 1)$ or $a(n)=\gamma^n$ $(n\geq 1)$, where $\gamma^{p-1}=\gamma^{q-1}=1$.*

PROOF OF COROLLARY 1. Let g, l, and m be as in Theorem 1. Since $a(n)$ is strongly p-additive, we have by Theorem 1, $a(g)=a(pg)=pa(g)$. So $a(g)=0$, noting that $p\geq 2$. Hence we get by Theorem 1 and strongly p-additivity,

$$a(n)=a(np^l)=a\left(\frac{np^l}{g}g\right)=\frac{np^l}{g}a(g)=0 \qquad (n\geq 0).$$

PROOF OF COROLLARY 2. Assume that p and q are relatively prime. Since $a(n)$ is strongly p-multiplicative, we have $a(n)=a(1)^n$ $(n\geq 1)$ by Theorem 2. Let p and q are not relatively prime. Let g, l, and m be as in Theorem 2. We write $p^l=p_1g$. Since $a(n)$ is strongly p-multiplicative, we have $a(1)=a(p^l)=a(p_1g)=a(g)^{p_1}$, so that $a(n)=a(np^l)=a(g)^{np_1}=a(1)^n$ $(n\geq 1)$. In any case, we get $a(n)=a(1)^n$ $(n\geq 1)$. In particular,

$$a(1)=a(p)=a(1)^p, \quad a(1)=a(q)=a(1)^q.$$

Hence we get $a(1)^{p-1}=a(1)^{q-1}=1$ if $a(1)\neq 0$.

2. A lemma.

In this section, we shall prove the key lemma for the proof of Theorems 1 and 2. Let p and q be as in Theorem 1.

LEMMA 1. *Let L be an infinite set of positive integers and m_0 be a positive integer. Then there exist integers $l \in L$ and $m \geq m_0$ satisfying the following two conditions*;

(i) *$p^l > g$ and $q^m > g$, where $g = \text{g.c.d.}(p^l, q^m)$,*

(ii) *$bp^h \neq cq^k$ for any integers b, c, h, and k with $1 \leq b \leq p-1$, $1 \leq c \leq q-1$, $h \geq l$, and $k \geq m$.*

PROOF. *First step.* We show that there exists a sequence $\{(l_n, m_n)\}_{n \geq 0}$ with $l_n \in L$, $l_0 < l_1 < \cdots$, and $m_n \geq m_0$ such that (i) holds for any $l = l_n$ and $m = m_n$ $(n \geq 0)$.

For any $l \in L$, let $\mu(l)$ denote the smallest integer $m \geq m_0$ such that $p^l < q^m$. Let

$$p = p_1^{e_1} \cdots p_s^{e_s}, \quad q = q_1^{f_1} \cdots q_t^{f_t},$$

be the factorization of p and q into distinct primes, where $e_1, \cdots, e_s, f_1, \cdots, f_t$ are positive integers.

Case 1. Let $\{p_1, \cdots, p_s\} \neq \{q_1, \cdots, q_t\}$. If $p_i \notin \{q_1, \cdots, q_t\}$ for some i, then $p^l \nmid q^{\mu(l)}$ for any $l \in L$. So we can choose $\{l_0, l_1, \cdots\} = L$ and $m_n = \mu(l_n)$ $(n \geq 0)$. Otherwise, we have $q_j \notin \{p_1, \cdots, p_s\}$ for some j. Let $l \in L$ be an integer such that $\mu(l) > m_0$. Since $\log p / \log q$ is irrational, we get $q^{\mu(l)-1} < p^l$, so that $q^{\mu(l)-1} \nmid p^l$. Then we choose $\{l_0, l_1, \cdots\} = \{l \in L \mid \mu(l) > m_0\}$ and $m_n = \mu(l_n) - 1$ $(n \geq 0)$.

Case 2. Let $\{p_1, \cdots, p_s\} = \{q_1, \cdots, q_t\}$. We may put $q_i = p_i$ $(1 \leq i \leq s = t)$. We show that

$$p^l \nmid q^{\mu(l)} \quad \text{or} \quad q^{\mu(l)-1} \nmid p^l \qquad \text{for infinitely many } l \in L\,. \tag{2}$$

Assume to the contrary that there exists $l_0 \in L$ such that $p^l \mid q^{\mu(l)}$ and $q^{\mu(l)-1} \mid p^l$ for any $l_0 \leq l \in L$. Then we have

$$le_i \leq \mu(l) f_i\,, \quad (\mu(l) - 1) f_i \leq le_i \qquad (1 \leq i \leq s)$$

for any l with $l_0 \leq l \in L$, and so

$$\frac{\mu(l) - 1}{l} \leq \frac{e_i}{f_i} \leq \frac{\mu(l)}{l} \qquad (1 \leq i \leq s,\, l_0 \leq l \in L)\,. \tag{3}$$

Let $\gamma = \log_p q$. Since $p^l < q^{\mu(l)}$, we get $l < \mu(l) \log_p q = \mu(l)\gamma$ for any $l \in L$. We define the sequence $\{l_n\}_{n \geq 0}$ inductively as in the following. Let $n \geq 1$. Suppose that $l_0, \cdots, l_{n-1}$ are defined. Noting that $\mu(l_{n-1})\gamma / l_{n-1} > 1$. We can choose $l_{n-1} < l_n \in L$ and $m > m_0$ such that

$$l_n < m\gamma < \frac{\mu(l_{n-1})}{l_{n-1}} \gamma l_n\,.$$

Since

$$p^{l_n} < p^{m\gamma} = q^m < q^{\mu(l_{n-1})l_n/l_{n-1}},$$

we get

$$\mu(l_n) \le m < \frac{\mu(l_{n-1})}{l_{n-1}} l_n,$$

and so

$$0 < \frac{\mu(l_n)}{l_n} < \frac{\mu(l_{n-1})}{l_{n-1}} \qquad (n \ge 1).$$

Then the sequence $\{\mu(l_n)/l_n\}_{n\ge 0}$ converges to the limit $\alpha = \lim_{n\to\infty} \mu(l_n)/l_n$. It follows from (3) that $\alpha = e_i/f_i$ for any i, and so $e_i f_1 = e_1 f_i$ for any i $(1 \le i \le s)$. This contradicts the irrationality of $\log p/\log q$, and (2) is proved.

Now by (2), we can choose an infinite subset $\{l_0, l_1, \cdots\}$ of L such that

$$p^{l_n} \nmid q^{\mu(l_n)} \quad \text{or} \quad q^{\mu(l_n)-1} \nmid p^{l_n} \qquad (n \ge 0).$$

We put $m_n = \mu(l_n)$ if $p^{l_n} \nmid q^{\mu(l_n)}$ and $m_n = \mu(l_n) - 1$ if $q^{\mu(l_n)-1} \nmid p^{l_n}$. Then $l = l_n$ and $m = m_n$ satisfy the condition (i).

Second step. Let $\{(l_n, m_n)\}_{n\ge 0}$ be the sequence constructed in the first step. It remains to show that there exists an integer $n \ge 0$ such that (ii) holds for $l = l_n$ and $m = m_n$. We assume, to the contrary, that for any integer $n \ge 0$, there exist integers b_n, c_n, h_n, and k_n with $1 \le b_n \le p-1$, $1 \le c_n \le q-1$, $h_n \ge l_n$, and $k_n \ge m_n$ such that $b_n p^{h_n} = c_n q^{k_n}$. Since $\{b_n\}_{n\ge 0}$, $\{c_n\}_{n\ge 0}$ are bounded, there exist integers n_1, n_2 such that

$$b_{n_1} = b_{n_2}, \quad c_{n_1} = c_{n_2}, \quad h_{n_1} < h_{n_2}.$$

Then we have

$$p^{h_{n_2} - h_{n_1}} = \frac{b_{n_2} p^{h_{n_2}}}{b_{n_1} p^{h_{n_1}}} = \frac{c_{n_2} q^{k_{n_2}}}{c_{n_1} q^{k_{n_1}}} = q^{k_{n_2} - k_{n_1}}.$$

This contradicts the irrationality of $\log p/\log q$, and the lemma is proved.

3. Some formulas for p and q-additive functions.

Let p, q and $a(n)$ be as in Theorem 1. In this section, we may assume without loss of generality that $p < q$ and write

$$q = dp + r, \qquad r \in \{0, 1, \cdots, p-1\}. \tag{4}$$

In the following Lemmas 2-7, we shall prove some formulas for p and q-additive functions which are necessary for the proof of Theorem 1.

LEMMA 2. *We have*

$$a(q)=a(dp)+a(r)\,, \tag{5}$$

$$a((d+1)p)=a(dp)+a(p)\,. \tag{6}$$

PROOF. (5) is obvious. We prove only (6). Since $a(n)$ is p and q-additive, we have by (4)

$$a(q+p)=a((d+1)p+r)=a((d+1)p)+a(r)\,,$$

and so by (5)

$$\begin{aligned} a((d+1)p)&=a(q+p)-a(r)\\ &=a(q)+a(p)-a(r)=a(dp)+a(p)\,. \end{aligned}$$

LEMMA 3. *Let f $(\leq p-1)$, h, and k be nonnegative integers such that $0\leq f+hp-kr<p$. Then*

$$a(f+hp-kr)=a(f)+ha(p)-ka(r)\,.$$

PROOF. By induction on $h+k$. This is true if $h+k=0$. Let $h+k>0$ and suppose that $a(f+h'p-k'r)=a(f)+h'a(p)-k'a(r)$ for any nonnegative integers h', k' with $h'+k'<h+k$ and $0\leq f+h'p-k'r<p$. Since $0\leq f+hp-kr<p$, we have

$$r\leq f+hp-(k-1)r<p+r\,.$$

Case 1. Assume first that f, h, k satisfy $p\leq f+hp-(k-1)r<p+r$. Then we have $0\leq f+(h-1)p-(k-1)r<r$, and so

$$a(q+f+hp-kr)=a((d+1)p+f+(h-1)p-(k-1)r)\,,$$

using (4). Here we note that $h\geq 1$ and $k\geq 1$. So we have by p and q-additivity

$$\begin{aligned} a(q)+a(f+hp-kr)&=a((d+1)p)+a(f+(h-1)p-(k-1)r)\\ &=a(dp)+a(p)+a(f)+(h-1)a(p)-(k-1)a(r)\\ &=a(q)+a(f)+ha(p)-ka(r) \end{aligned}$$

by (5), (6), and the induction hypothesis. Therefore we obtain

$$a(f+hp-kr)=a(f)+ha(p)-ka(r)\,.$$

Case 2. Let $r\leq f+hp-(k-1)r<p$. Then we have

$$\begin{aligned} a(q+f+hp-kr)&=a(dp+f+hp-(k-1)r)\\ &=a(dp)+a(f+hp-(k-1)r)\\ &=a(q)+a(f)+ha(p)-ka(r) \end{aligned}$$

by (5), $k\geq 1$, and the induction hypothesis. Since $a(q+f+hp-kr)=a(q)+a(f+hp-kr)$, we get

$$a(f+hp-kr)=a(f)+ha(p)-ka(r)\,.$$

LEMMA 4. *If $r\neq 0$, then $a(np)=na(p)$ $(0\leq n\leq d)$.*

PROOF. This is true if $n=0$. Let $1\leq n\leq d$. We note that $q>np$, since $r\neq 0$. Then we have by p and q-additivity

$$a(q)+a(np)=a(q+np)=a((d+n)p+r)=a((d+n)p)+a(r)\,;$$

namely

$$a(np)=a((d+n)p)-a(q)+a(r)\,.$$

By (4), Lemma 3, and $r\neq 0$, we have

$$\begin{aligned}a((d+n)p)&=a(dp+r+(n-1)p+p-r)\\&=a(q)+a((n-1)p)+a(p-r)\\&=a((n-1)p)+a(q)+a(p)-a(r)\,.\end{aligned}$$

Hence we get

$$a(np)=a((n-1)p)+a(p)=\cdots=na(p)\,.$$

LEMMA 5. *Assume that $r\neq 0$. Let $f\,(\leq p-1)$, h, and k be nonnegative integers such that $0\leq f+hq-kp<p$. Then*

$$a(f+hq-kp)=a(f)+ha(q)-ka(p)\,.$$

PROOF. By induction on $h+k$. This is true if $h+k=0$. Let $h+k>0$ and suppose that $a(f+h'q-k'p)=a(f)+h'a(q)-k'a(p)$ for any nonnegative integers h', k' such that $h'+k'<h+k$ and $0\leq f+h'q-k'p<p$. We have to show that

$$a(f+hq-kp)=a(f)+ha(q)-ka(p)\,. \tag{7}$$

Case 1. Let $0\leq f+hq-kp<r$. Since $q=dp+r$, we have $h\geq 1$, $k\geq d+1$, and

$$(d+1)p\leq f+hq-(k-(d+1))p<(d+1)p+r\,,$$

and so

$$p-r\leq f+(h-1)q-(k-(d+1))p<p\,.$$

Hence we get

$$a((d+1)p+f+hq-kp)=a(q+f+(h-1)q-(k-(d+1))p)\,.$$

Since $a(n)$ is p and q-additive, we have by the induction hypothesis

$$a((d+1)p)+a(f+hq-kp)=a(q)+a(f+(h-1)q-(k-(d+1))p)$$

$$=a(q)+a(f)+(h-1)a(q)-(k-(d+1))a(p)$$
$$=a(f)+ha(q)-ka(p)+da(p)+a(p)$$

using Lemma 4 and (6). Therefore we obtain (7).

Case 2. Let $r\le f+hq-kp<p$. Since $h\ge 1$, $k\ge d$ and

$$0\le f+(h-1)q-(k-d)p<p-r\,,$$

we have

$$\begin{aligned}a(dp+f+hq-kp)&=a(q+f+(h-1)q-(k-d)p)\\&=a(q)+a(f+(h-1)q-(k-d)p)\\&=a(f)+ha(q)-ka(p)+da(p)\end{aligned}$$

by the induction hypothesis. Using Lemma 4, we obtain (7).

LEMMA 6. *Assume that $r\ne 0$. Let n be a positive integer such that $a(hp)=ha(p)$ for any h $(0\le h\le n-1)$. Let k be a nonnegative integer such that $k<np/q$. Then $a(kq)=ka(q)$.*

PROOF. Let h be a nonnegative integer such that $0\le kq-hp<p$. Since $0\le kq-hp$, we have $h\le kq/p$. Noting that $k<np/q$, we get $h<n$. Then we have by Lemma 5,

$$a(kq)=a(hp)+a(kq-hp)=a(hp)+ka(q)-ha(p)\,.$$

Hence we obtain $a(kq)=ka(q)$ since $h<n$.

LEMMA 7. *Assume that $r\ne 0$ and $bp^h\ne cq^k$ for any integers b, c, h, and k with $1\le b\le p-1$, $1\le c\le q-1$, $h\ge 1$, and $k\ge 1$. Then*

$$a(np)=na(p)\,,\quad a(nq)=na(q)\qquad (n\ge 1)\,.$$

PROOF. We show only the first formula

$$a(np)=na(p)\qquad (n\ge 1)\,, \tag{8}$$

since the second formula follows from the first and Lemma 6. The proof will be carried on by induction on n. (8) holds for any $n\le d$ by Lemma 4. Let $n\ge d+1$ and assume that

$$a(hp)=ha(p)\qquad (0\le h\le n-1)\,. \tag{9}$$

Then we have by Lemma 6

$$a(kq)=ka(q)\qquad (0\le k<np/q)\,. \tag{10}$$

We have to prove that $a(np)=na(p)$.

Case 1. Let $q\,|\,np$. We expand np to base p and q;

$$np=\sum_{i=s_p}^{t_p} b_ip^i\qquad (b_i\in\{0,1,\cdots,p-1\},\ b_{s_p}\ne 0,\ b_{t_p}\ne 0)\,,$$

$$= \sum_{i=s_q}^{t_q} c_i q^i \qquad (c_i \in \{0, 1, \cdots, q-1\}, c_{s_q} \neq 0, c_{t_q} \neq 0),$$

so that $s_p \geq 1$ and $s_q \geq 1$. By the assumption of this lemma, we have $s_p \neq t_p$ or $s_q \neq t_q$. We assume first that $s_p \neq t_p$. Noting that $b_i p^{i-1} < n$ $(s_p \leq i \leq t_p)$, we have by (9) $a(b_i p^i) = b_i p^{i-1} a(p)$. Using this we get

$$a(np) = \sum_{i=s_p}^{t_p} a(b_i p^i) = \sum_{i=s_p}^{t_p} b_i p^{i-1} a(p) = na(p).$$

Next we consider the case $s_q \neq t_q$. Since $c_i q^{i-1} < np/q$ $(s_q \leq i \leq t_q)$, we have by (10) $a(c_i q^i) = c_i q^{i-1} a(q)$. Hence we get

$$a(np) = \sum_{i=s_q}^{t_q} a(c_i q^i) = \sum_{i=s_q}^{t_q} c_i q^{i-1} a(q) = \frac{np}{q} a(q).$$

Noting that $q \mid np$, we have by Lemma 5

$$0 = a\left(\frac{np}{q} q - np\right) = \frac{np}{q} a(q) - na(p),$$

and so

$$a(np) = \frac{np}{q} a(q) = na(p).$$

Case 2. Let $q \nmid np$. Let h and k be nonnegative integers such that $0 \leq np - kq < q$ and $0 \leq np - kq - hp < p$. We note that $k \geq 1$, since $np \geq (d+1)p > q > np - kq$, and so $0 \leq h \leq n-1$, since $np - hp > np - kq - hp \geq 0$. Also $k < np/q$, since $q \nmid np$ implies $0 < np - kq$. Hence we have by p and q-additivity, Lemma 3, (9) and (10),

$$\begin{aligned} a(np) &= a(kq + (np - kq)) \\ &= a(kq) + a(hp + (np - kq - hp)) \\ &= a(kq) + a(hp) + a((n - dk - h)p - kr) \\ &= ka(q) + ha(p) + (n - dk - h)a(p) - ka(r) \\ &= na(p) + ka(q) - k(da(p) + a(r)), \end{aligned}$$

and so using (4) and Lemma 4

$$a(np) = na(p) + ka(q) - k(a(dp) + a(r)) = na(p).$$

In both cases, we obtain $a(np) = na(p)$, and so (8) is proved.

4. Proof of Theorem 1.

Proof of Theorem 1. Let $L = \{1, 2, \cdots\}$ and $m_0 = 1$. Then there exist positive integers l and m satisfying the conditions (i), (ii) in Lemma 1. We may assume that $p^l < q^m$, since, otherwise, we exchange p, l by q, m, respectively. We write

$$q^m = dp^l + r \qquad (r \in \{1, 2, \cdots, p^l - 1\}) . \tag{11}$$

Note that $r \neq 0$, because of (i). In what follows, we use Lemmas 2–7, with p^l and q^m in place of p and q, respectively.

We prove the first statement of Theorem 1; namely,

$$a(ng) = na(g) \qquad (n \geq 1, g = \text{g.c.d.}(p^l, q^m)) . \tag{12}$$

We put $p^l = p_1 g$, so that $p_1 \geq 2$ by (i) in Lemma 1. Let h, k be positive integers such that

$$kq^m - hp^l = g . \tag{13}$$

We show that

$$a(ng) = na(g) \qquad (1 \leq n \leq p_1 - 1) , \tag{14}$$

$$a(p^l) = a(p_1 g) = p_1 a(g) . \tag{15}$$

Indeed, we have for n with $1 \leq n \leq p_1 - 1$

$$a(ng) = a(knq^m - hnp^l) = n(ka(q^m) - ha(p^l))$$

by Lemma 5. In particular, $a(g) = ka(q^m) - ha(p^l)$. Combining these we get (14). Next we show (15). Since $a(p^l q^m) = p^l a(q^m)$ and $a(q^m p^l) = q^m a(p^l)$ by Lemma 7, we have by (11)

$$a(q^m) = \frac{q^m}{p^l} a(p^l) = da(p^l) + \frac{r}{p^l} a(p^l) .$$

On the other hand, we get $a(q^m) = da(p^l) + a(r)$ by (11) and Lemma 4. Comparing the right-hand side, we find

$$\frac{r}{p^l} a(p^l) = a(r) = \frac{r}{g} a(g) ,$$

noting that g divides r; which yields (15).

Now we prove (12) using (14) and (15). Let n be a positive integer. We write $n = sp_1 + t$ with $s \geq 0$ and $0 \leq t \leq p_1 - 1$. Then we have by p-additivity

$$a(ng) = a((sp_1 + t)g) = a(sp^l + tg) = a(sp^l) + a(tg) ,$$

and so

$$a(ng) = sa(p^l) + ta(g) = (sp_1 + t)a(g) = na(g)$$

using Lemma 7, (14), and (15). Therefore, (12) is proved.

It remains to show that $a(n)$ is g-additive provided $g\geq 2$. Let $n\geq 0$ be an integer. We write

$$n=sg+t \qquad (s\geq 0,\, t\in\{0, 1, \cdots, g-1\})\,,$$

$$s=s_1p_1+s_2 \qquad (s_1\geq 0,\, s_2\in\{0, 1, \cdots, p_1-1\})\,.$$

Then we have

$$a(n)=a((s_1p_1+s_2)g+t)=a(s_1p^l+s_2g+t)=a(s_1p^l)+a(s_2g+t)\,,$$

and so by (15)

$$a(n)=s_1p_1a(g)+a(s_2g+t)\,.$$

Since $0\leq s_2g+t=ks_2q^m-hs_2p^l+t<p^l$ by (13), we have by Lemma 5

$$a(s_2g+t)=a(ks_2q^m-hs_2p^l+t)=ks_2a(q^m)-hs_2a(p^l)+a(t)\,.$$

Hence we get by (12), (13)

$$a(s_2g+t)=s_2\left(\frac{kq^m}{g}-\frac{hp^l}{g}\right)a(g)+a(t)=s_2a(g)+a(t)\,,$$

and so $a(n)=(s_1p_1+s_2)a(g)+a(t)=sa(g)+a(t)$. Therefore $a(n)$ is g-additive, and the proof is completed.

5. Additional conditions to Lemma 1 in multiplicative case.

In order to apply Lemma 1 for p and q-multiplicative functions, we need additional conditions that $a(p^l)\neq 0$ and $a(q^m)\neq 0$, which is insured by Lemma 9 below. Let p, q, and $a(n)$ be as in Theorem 2.

LEMMA 8. *Let b $(1\leq b\leq p-1)$ and $l\geq 1$ be integers such that $a(bp^l)\neq 0$ and*

$$bp^l=c_tq^t+u \qquad (c_t\geq b,\, 1\leq u<q^t)\,. \tag{16}$$

Then $a(q^t)\neq 0$.

PROOF. We expand u to base q

$$u=\sum_{i=0}^{h}c_iq^i \qquad (c_i\in\{0, 1, \cdots, q-1\},\, c_h\neq 0)\,, \tag{17}$$

so that $0\leq h\leq t-1$. Since $a(bp^l)\neq 0$, $c_t\geq b$, and $u\geq 1$, we have

$$a(c_iq^i)\neq 0 \qquad (0\leq i\leq h,\, i=t)\,, \tag{18}$$

$$q^t<p^l\,. \tag{19}$$

Let f be a positive integer such that $(f-1)c_h<p\leq fc_h$.

We show first that

$$a((f-1)c_h q^h)\neq 0\,. \tag{20}$$

It is enough to show that $a(jc_h q^h)\neq 0$ for all $1\leq j\leq f-1$ by induction on j. This holds for $j=1$ by (18). Suppose that $a((j-1)c_h q^h)\neq 0$ for some $2\leq j\leq f-1$. We have by (16), (17)

$$a(bp^l+(j-1)c_h q^h)=a\left(\sum_{i=s}^{h-1} c_i q^i+jc_h q^h+c_t q^t\right),$$

and so by (19)

$$a(bp^l)a((j-1)c_h q^h)=\left(\prod_{i=0}^{h-1} a(c_i q^i)\right)a(jc_h q^h)a(c_t q^t)\,,$$

which together with (18) leads to $a(jc_h q^h)\neq 0$, and hence (20) follows.

We put

$$k_{c,j}=(f-1)c_h q^h+(q-1)q^{h+1}+\cdots+(q-1)q^{j-1}+cq^j\,,$$

where c and j are integers with $0\leq c\leq q-1$ and $h+1\leq j\leq t$. We show that if $h<t-1$,

$$a(k_{q-1,t-1})\neq 0\,. \tag{21}$$

It is enough to show that

$$a(k_{c,j})\neq 0 \qquad (0\leq c\leq q-1,\ h+1\leq j\leq t-1) \tag{22}$$

by induction on c and j. By (20), we have $a(k_{0,h+1})\neq 0$. Assume that $a(k_{c,j})\neq 0$ for some $0\leq c\leq q-2$ and $h+1\leq j\leq t-1$. Then it follows from (16), (17), and (19) that

$$a(bp^l)a(k_{c,j})=a(bp^l+k_{c,j})=\left(\prod_{i=0}^{h-1} a(c_i q^i)\right)a(nq^h)a((c+1)q^j)a(c_t q^t)\,,$$

where $n=fc_h-q$. Hence we have $a((c+1)q^j)\neq 0$, so that $a(k_{c+1,j})\neq 0$. Noting that $k_{q-1,j}=k_{0,j+1}$, we obtain (22), and so (21).

It follows from (16), (17), and (19) that

$$a(bp^l)a(k_{0,t})=a(bp^l+k_{0,t})=\left(\prod_{i=0}^{h-1} a(c_i q^i)\right)a(nq^h)a((c_t+1)q^t)\,.$$

Noting that $k_{0,t}=(f-1)c_h q^h$ if $h=t-1$, $=k_{q-1,t-1}$ if $h<t-1$ and using (20) or (21), respectively, we have

$$a((c_t+1)q^t)\neq 0\,. \tag{23}$$

It follows from (16) and (19) that

$$a(bp^l)a(q^t)=a(bp^l+q^t)=a((c_t+1)q^t)a(u)\,.$$

This together with (23) leads to $a(q^t)\neq 0$.

REMARK. Exchanging p by q, in Lemma 8, we have the following: *Let m be a positive integer such that $a(q^m)\neq 0$ and*

$$q^m=b_t p^t+v \qquad (1\leq b_t<p,\ 1\leq v<p^t)\,.$$

Then $a(p^t)\neq 0$.

LEMMA 9. *If $a(np)\neq 0$ for infinitely many $n\geq 1$, then there exist positive integer l and m satisfying* (i) *and* (ii) *in Lemma* 1 *and* (iii) $a(p^l)\neq 0$, $a(q^m)\neq 0$.

PROOF. Let $a(np)\neq 0$ for infinitely many $n\geq 1$. Then $a(nq)\neq 0$ for infinitely many $n\geq 1$. So we may assume that $q>p$, since, otherwise, we can exchange p by q.

By Lemma 1, it is enough to show that there exist an infinite set L of positive integers and a positive integer m_0 such that $a(p^l)\neq 0$ and $a(q^m)\neq 0$ for any $l\in L$ and $m\geq m_0$.

Let

$$L=\{h\geq 1\mid a(p^h)\neq 0\}\,,\quad M=\{k\geq 1\mid a(q^k)\neq 0\}\,.$$

We show that both L and M are infinite sets. First we prove that M is infinite. Let h_0 be a positive integer with $p^{h_0}\geq q$. For any b $(1\leq b\leq p-1)$ and $h\geq h_0$, we can write bp^h as in the following form:

$$bp^h=c_s q^s+u\,, \tag{24}$$

where

$$b\leq c_s=c_s(b,h)<q^2\,,\quad 0\leq u=u(b,h)<q^s\,,\quad s=s(b,h)\geq 0\,.$$

Indeed, if the first digit d_k in the q-adic expansion

$$bp^h=\sum_{i=0}^{k} d_i q^i \qquad (d_i\in\{0,1,\cdots,q-1\},\ d_k\neq 0)$$

is not less than b, we put $s=k$, $c_s=d_k$, and $u=\sum_{i=0}^{k-1} d_i q^i$; otherwise, we put $s=k-1$, $c_s=d_k q+d_{k-1}$, and $u=\sum_{i=0}^{k-2} d_i q^i$, noting that $k\geq 1$ since $p^h\geq q$.

Assume that $u(b,h)=0$ for infinitely many pairs (b,h). Then there exist integers b $(1\leq b\leq p-1)$, h_2, and h_3 $(h_2<h_3)$ such that

$$c_{s(b,h_2)}(b,h_2)=c_{s(b,h_3)}(b,h_3) \quad\text{and}\quad u(b,h_2)=u(b,h_3)=0\,,$$

since $\{c_s(b,h)\}_{1\leq b\leq p-1,\,h\geq h_0}$ is bounded; so that we have

$$p^{h_3-h_2}=\frac{bp^{h_3}}{bp^{h_2}}=\frac{c_{s(b,h_3)}(b,h_3)q^{s(b,h_3)}}{c_{s(b,h_2)}(b,h_2)q^{s(b,h_2)}}=q^{s(b,h_3)-s(b,h_2)}\,.$$

This contradicts the irrationality of $\log p/\log q$.

Hence there exists an integer $h_1\geq h_0$ such that $u(b,h)\geq 1$ for any $1\leq b\leq p-1$ and $h\geq h_1$. Also we note that $a(bp^h)\neq 0$ for infinitely many pairs (b,h), since $a(np)\neq 0$ for infinitely many $n\geq 1$. These facts with (24) and Lemma 8 imply that $a(q^s)\neq 0$ for infinitely many s; and therefore M is an infinite set.

To show that L is infinite, we write

$$q^k = b_t p^t + v \qquad (1 \le b_t = b_t(k) < p,\ 0 \le v = v(k) < p^t,\ t = t(k) \ge 0) \tag{25}$$

for any $k \ge 1$. In the similar way as above, there exists $k_0 \ge 1$ such that $v(k) \ge 1$ for any $k \ge k_0$. Since M is an infinite set, we get $a(q^k) \ne 0$ for infinitely many $k \ge k_0$. Therefore, L is also an infinite set by (25) and the remark of Lemma 8.

Next we show that $M \supset \{m_0, m_0+1, m_0+2, \cdots\}$ for some integer m_0. Let $l_0 \in L$ satisfy $l_0 \ge h_1$ and let $m_0 \in M$ satisfy $m_0 \ge k_0$ and $p^{l_0} < q^{m_0}$. We write $\{m \in M \mid m \ge m_0\} = \{m_0, m_1, m_2, \cdots\}$ $(m_0 < m_1 < m_2 < \cdots)$ and put $l_n = [m_n \gamma]$ $(n \ge 1)$, where $\gamma = \log_p q$. We note that $\gamma > 1$ since $p < q$. Let n be a positive integer. Since $m_n - m_{n-1} \ge 1$, we have

$$1 < m_n\gamma - m_{n-1}\gamma = l_n - l_{n-1} + (m_n\gamma - l_n) - (m_{n-1}\gamma - l_{n-1}),$$

and so $l_n > l_{n-1}$, noting that $0 < m_{n-1}\gamma - l_{n-1}$, $m_n\gamma - l_n < 1$. Assume that $[l_n/\gamma] < m_{n-1}$. Then we get $l_n/\gamma < m_{n-1}$, and so $l_n \le [m_{n-1}\gamma] = l_{n-1}$. It is a contradiction to $l_{n-1} < l_n$. Hence we obtain

$$m_{n-1} \le [l_n/\gamma] < m_n \qquad (n \ge 1), \tag{26}$$

noting that $l_n < m_n\gamma < l_n + 1$. Since $k_0 \le m_n \in M$ and $p^{l_n} < p^{m_n\gamma} = q^{m_n} < p^{l_n+1}$, we have $a(p^{l_n}) \ne 0$ by the remark of Lemma 8, and so $l_n \in L$. Since $h_1 \le l_n$ and $q^{[l_n/\gamma]} < q^{l_n/\gamma} = p^{l_n} < q^{[l_n/\gamma]+1}$, we get $a(q^{[l_n/\gamma]}) \ne 0$ by Lemma 8, and so $[l_n/\gamma] \in M$. Then we have $m_{n-1} = [l_n/\gamma]$ by (26). Hence we obtain

$$1 \le m_n - m_{n-1} < m_n - \left(\frac{l_n}{\gamma} - 1\right) < m_n - \left(\frac{m_n\gamma - 1}{\gamma} - 1\right) = 1 + \frac{1}{\gamma},$$

and so $m_n - m_{n-1} = 1$ since $\gamma > 1$, so that $M \supset \{m_0, m_0+1, m_0+2, \cdots\}$.

Therefore, by Lemma 1, there exist integers $l \in L$ and $m \ge m_0$ satisfying (i), (ii), and (iii), and the proof is completed.

6. Some formulas for p and q-multiplicative functions.

In this section, we assume as we may that $p < q$ and write

$$q = dp + r, \qquad r \in \{0, 1, \cdots, p-1\}.$$

The following lemmas can be proved by transforming the arguments in Section 3 into q-multiplicative case. So we omit the proofs.

LEMMA 10. *If $a(r) \ne 0$, then*

$$a(q) = a(dp)a(r), \quad a((d+1)p) = a(dp)a(p).$$

LEMMA 11. *Assume that $a(q) \ne 0$. Let $f\,(\le p-1)$, h and k be nonnegative integers* *such that* $0 \leq f + hp - kr < p$. *Then*

$$a(f+hp-kr) = \frac{a(f)a(p)^h}{a(r)^k}.$$

LEMMA 12. *If* $r \neq 0$ *and* $a(q) \neq 0$, *then* $a(np) = a(p)^n$ $(1 \leq n \leq d)$.

REMARK. *By Lemma* 10 *and* 12, *if* $r \neq 0$ *and* $a(q) \neq 0$, *then* $a(p) \neq 0$.

LEMMA 13. *Assume that* $r \neq 0$ *and* $a(q) \neq 0$. *Let* f $(\leq p-1)$, h *and* k *be nonnegative integers such that* $0 \leq f + hq - kp < p$. *Then*

$$a(f+hq-kp) = \frac{a(f)a(q)^h}{a(p)^k}.$$

LEMMA 14. *Assume that* $r \neq 0$ *and* $a(q) \neq 0$. *Let* n *be a positive integer such that* $a(hp) = a(p)^h$ *for any* h $(0 \leq h \leq n-1)$. *Let* k *be a nonnegative integer such that* $k < np/q$. *Then* $a(kq) = a(q)^k$.

LEMMA 15. *Assume that* $r \neq 0$, $a(q) \neq 0$, *and* $bp^h \neq cq^k$ *for any integers* b, c, h, *and* k *with* $1 \leq b \leq p-1$, $1 \leq c \leq q-1$, $h \geq 1$, *and* $k \geq 1$. *Then*

$$a(np) = a(p)^n, \quad a(nq) = a(q)^n \qquad (n \geq 1).$$

7. Proof of Theorem 2.

PROOF OF THEOREM 2. Case 1. Assume first that there exists a positive integer h such that $a(np^h) = 0$ for $n \geq 1$. If p and q are relatively prime, then Theorem 2 holds for $l = h$. Let p and q are not relatively prime. Since $a(np^h) = 0$ for $n \geq 1$, we have $a(nq^k) = 0$ for some $k \geq 1$ and any $n \geq 1$. Then there exists a positive integer j such that $\text{g.c.d.}(p^{jh}, q^{jk}) = \text{g.c.d.}(p^h, q^k)^j > p^h$, noting that p and q are not relatively prime. Hence we obtain $a(ng) = 0 = a(g)^n$ for $n \geq 1$, and so $a(n)$ is g-multiplicative, where $g = \text{g.c.d.}(p^{jh}, q^{jk})$. Therefore Theorem 2 holds for $l = jh$ and $m = jk$.

Case 2. Next we assume that $a(np) \neq 0$ for infinitely many $n \geq 1$. By Lemma 9, there exist positive integers l and m satisfying (i), (ii), and (iii). Hence Lemmas 10–15 hold for $p = p^l$ and $q = q^m$. We put $g = \text{g.c.d.}(p^l, q^m)$. In the same way as the proof of Theorem 1, we can prove that $a(ng) = a(g)^n$ $(n \geq 1)$ and $a(n)$ is g-multiplicative provided that $g \geq 2$, using Lemmas 10–15 in place of Lemmas 2–7 respectively. The proof is completed.

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