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Extremal 2-Connected Graphs with Given Diameter

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Dedicated to the memory of Paul Erdős

Abstract. Suppose G is a 2-connected graph of order n with diameter $d\geq 2$. We prove that

$$|E(G)|\geq\frac{dn-2d-1}{d-1}.$$

We also characterize the extremal graphs for $d\geq 5$.

1. Introduction.

In this paper, we consider finite undirected graphs without loops or multiple edges. (Terminologies not defined here can be found in [4] or [8]). The set of vertices (resp. the set of edges) of a graph G is denoted by $V(G)$ (resp. $E(G)$). The edge joining two vertices x and y is denoted by xy, and for subsets A and B of $V(G)$,

$$E(A, B):=\{xy\in E(G) \mid x\in A, y\in B\}$$

denotes the set of edges joining A and B. The set of vertices adjacent to a vertex x is called the neighbourhood of x, and is denoted by $N(x)$. The degree of a vertex x is denoted by $\deg(x)$. The minimum degree (resp. the maximum degree) of G is denoted by $\delta(G)$ (resp. $\Delta(G)$). A subset A is often identified with the induced subgraph $\langle A\rangle$, and $G-x:=\langle V(G)-\{x\}\rangle$ for $x\in V(G)$. For x and y in $V(G)$, $d(x, y)$ denotes the distance between x and y, and $\mathrm{diam}(G)$ is the diameter of G. The length of a path P is denoted by $l(P)$. A path $P=(v_0, v_1, \cdots, v_l)$ is called an *ear* if $\deg_G(v_i)=2$ for $1\leq i\leq l-1$. Two paths P and Q connecting distinct vertices u and v are called internally disjoint if $V(P)\cap V(Q)=\{u, v\}$. For a set X, $|X|$ denotes the cardinality of X. For a real number z, the greatest integer not exceeding z is denoted by $\lfloor z\rfloor$, and $\lceil z\rceil:=-\lfloor -z\rfloor$ is the least integer not less than z.

Let

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$$\mathscr{G}(n, d, d') := \left\{ G \;\middle|\; \begin{array}{l} |V(G)|=n, \ \mathrm{diam}(G) \le d, \\ \mathrm{diam}(G-x) \le d' \text{ for any } x \in V(G) \end{array} \right\},$$

$$f(n, d, d') := \min\{|E(G)| \mid G \in \mathscr{G}(n, d, d')\}.$$

Bollobás [2, 3] proved

$$\lim_{n\to\infty} \frac{f(n, d, d')}{n} = \frac{d}{d-1}$$

for $d' \ge 2d-1$. The exact value of $f(n, d, d')$ is determined if $d \le 4$ and n is not small [10, 1, 6, 5]. Applying the method introduced in [9], we prove the following theorem ([7, Conjecture 3]).

THEOREM 1. *Suppose $n > d' \ge 2d-1$. Then*

$$f(n, d, d') = \left\lceil \frac{dn-2d-1}{d-1} \right\rceil = \left\lfloor \frac{dn-d-3}{d-1} \right\rfloor$$

unless $n=4$ and $d=2$.

Define the graph $G(a, b; c, d)$ for positive integers a and b, an integer $d \ge 2$, and an integer c with $2 \le c \le d$ as follows: $G = G(a, b; c, d)$ consists of internally disjoint paths $P_1, \cdots, P_a$ connecting u and v, internally disjoint paths $Q_1, \cdots, Q_b$ connecting u and w, and an edge vw, where u, v and w are distinct vertices, $l(P_i)=d$ for $1 \le i \le a$, $l(Q_j)=d$ for $1 \le j \le b-1$, $l(Q_b)=c$, and $V(P_i) \cap V(Q_j) = \{u\}$ for $1 \le i \le a$, $1 \le j \le b$. Then

$$n := |V(G)| = (a+b-1)(d-1)+c-1+3,$$

$$|E(G)| = (a+b-1)d+c+1 = \left\lfloor \frac{dn-d-3}{d-1} \right\rfloor,$$

and $G \in \mathscr{G}(n, d, 2d-1)$. This implies the inequality $f(n, d, d') \le \lceil (dn-2d-1)/(d-1) \rceil$. Note that when d and n are given, such a, b and c exist if $n \ge d+3$. Define $G(a, b; d) := G(a, b; d, d)$. Since Theorem 1 was proved for the case $d \le 4$, $f(n, d, n-1) \le f(n, d, d')$ and

$$\mathscr{G}(n, d, n-1) = \left\{ G \;\middle|\; \begin{array}{l} |V(G)|=n, \ \mathrm{diam}(G) \le d, \\ G \text{ is 2-connected} \end{array} \right\},$$

Theorem 1 follows from the following theorem.

THEOREM 2. *Suppose G is a 2-connected graph of order n with diameter $d \ge 5$. Then*

$$|E(G)| \ge \frac{dn-2d-1}{d-1}.$$

Furthermore, equality holds if and only if G is isomorphic to some $G(a, b; d)$.

In Section 2, we prove preliminary results that estimate the number of edges. We prove Theorem 2 in Section 3.

2. Preliminaries.

In this section, we assume that $|V(G)|=n$, $\delta(G)=2$, $\Delta(G)\geq 3$, $d\geq 5$, l (≥ 2) is the length of longest ears, and a subset D_0 of $V(G)$ is given. Set

$$D_r := \{v\in V(G) \mid d(v, D_0)=r\},$$

where $d(v, D_0) := \min\{d(v, u) \mid u\in D_0\}$.

Define functions s and t on $V(G)$ as follows: Let v be a vertex in D_r. If either $r=0$, $\deg(v)\geq 3$, or $|N(v)\cap D_{r-1}|\geq 2$, define $s(v):=v$ and $t(v):=0$. If $r>0$, $\deg(v)=2$ and $N(v)\cap D_{r-1}=\{u\}$, then define $s(v):=s(u)$ and $t(v):=t(u)+1$. Note that

$$t(v)=d(v, s(v))\leq \min\{r, l-1\}.$$

If the shortest path from v to D_0 is unique, v is called of type U. If v is not of type U, v is called of type M. Note that $t(v)\leq r-1$ if v is of type M.

Define a function $w(u, v)$ for $uv\in E(D_r, D_{r+1})$, and a function $w(C)$ for a connected component C of D_r inductively as follows:

(1) For $uv\in E(D_0, D_1)$, $w(u, v):=0$.

(2) For a connected component C of $D_r (r\geq 1)$,

$$w(C) := |E(C)| + \sum_{xy\in E(D_{r-1}, C)} (1-w(x, y)).$$

(3) For $uv\in E(D_r, D_{r+1}) (r\geq 1)$, let C be the connected component of D_r that contains u. If $w(C)\geq \frac{d}{d-1}|C|$, then $w(u, v):=0$. Otherwise,

$$w(u, v) := \frac{\frac{d}{d-1}|C| - w(C)}{|E(C, D_{r+1})|}.$$

LEMMA 3. *Suppose $r\leq d-1-\lfloor l/2 \rfloor$ and $uv\in E(D_r, D_{r+1})$.*

(1) $w(u, v)\leq \dfrac{r+t(u)}{2(d-1)}$. *In particular,* $w(u, v)\leq \min\left\{\dfrac{r}{d-1}, \dfrac{r+l-1}{2(d-1)}\right\}$.

(2) *If u is of type M, then $w(u, v)\leq (r+t(u)-1)/(2(d-1))$. In particular, $w(u, v)\leq$ $\min\{(r-1)/(d-1), (r+l-2)/(2(d-1))\}$.*

(3) *If $\deg(u)\geq 4$, then $w(u, v)\leq \max\{r/(3(d-1)), (r-1)/(2(d-1))\}$.*

PROOF. We use induction on r. It is easily seen that the lemma holds for $r=0$. Suppose $r\geq 1$, and let C be the connected component of D_r that contains u, and set $\alpha:=|C|$ and $\beta:=|E(D_{r-1}, C)|-\alpha$. If $w(C)\geq \alpha d/(d-1)$, then $w(u, v)=0$. Hence we may assume $w(C)<\alpha d/(d-1)$. If C is not a tree, we have

$$w(C) \geq |E(C)| + |E(D_{r-1}, C)|\left(1 - \frac{r-1}{d-1}\right)$$
$$\geq \alpha\left(2 - \frac{r-1}{d-1}\right) > \frac{\alpha d}{d-1}$$

by induction. Hence C is a tree. Then

$$\frac{\alpha d}{d-1} > w(C) \geq \alpha - 1 + (\alpha+\beta)\left(1 - \frac{r-1}{d-1}\right) \tag{2.1}$$

$$\frac{\alpha d}{d-1} > w(C) \geq \alpha - 1 + (\alpha+\beta)\left(1 - \frac{r+l-2}{2(d-1)}\right) \tag{2.2}$$

by induction. From (2.1), we get

$$d-1-\left\lfloor \frac{l}{2} \right\rfloor \geq r \geq d - \frac{\alpha+d-2}{\alpha+\beta},$$

and from (2.2)

$$d-1-\left\lfloor \frac{l}{2} \right\rfloor \geq r \geq 2d - l - \frac{2\alpha+2d-3}{\alpha+\beta}.$$

Combining these inequalities, we get

$$\frac{\alpha+d-2}{\alpha+\beta} \geq \left\lfloor \frac{l}{2} \right\rfloor + 1 \geq \left\lceil \frac{l}{2} \right\rceil = l - \left\lfloor \frac{l}{2} \right\rfloor \geq d + 1 - \frac{2\alpha+2d-3}{\alpha+\beta}.$$

This implies $\alpha(d-2)+\beta(d+1) \leq 3d-5$. Since $\alpha \geq 1$, we conclude $\beta < 2$. More precisely, $\alpha = 1$ if $\beta = 1$, and $\alpha \leq 3 + 1/(d-2) < 4$ if $\beta = 0$.

First, suppose $\alpha = \beta = 1$. Then

$$w(u, v) \leq \frac{d}{d-1} - 2\left(1 - \min\left\{\frac{r-1}{d-1}, \frac{r+l-2}{2(d-1)}\right\}\right)$$
$$\leq \frac{d}{d-1} - \left(1 - \frac{r-1}{d-1}\right) - \left(1 - \frac{r+l-2}{2(d-1)}\right)$$
$$= \frac{3r-2d+l}{2(d-1)} \leq \frac{r+2(d-1-\lfloor l/2 \rfloor)-2d+l}{2(d-1)}$$
$$\leq \frac{r-1}{2(d-1)}.$$

This proves (1), (2) and (3), since $t(u) = 0$ in this case.

Next, suppose $\beta = 0$. First, suppose $\alpha = 3$. Then $t(u) = 0$ since $\deg(u) \geq 3$, and

$$w(u, v) \leq \frac{3d}{d-1} - \left\{2 + 2\left(1 - \frac{r-1}{d-1}\right) + \left(1 - \frac{r+l-2}{2(d-1)}\right)\right\}$$
$$= \frac{5r - 4d + l + 4}{2(d-1)} \leq \frac{r + 4(d-1-\lfloor l/2 \rfloor) - 4d + l + 4}{2(d-1)}$$
$$= \frac{r + l - 4\lfloor l/2 \rfloor}{2(d-1)} \leq \frac{r-1}{2(d-1)}.$$

Next, suppose $\alpha = 2$ and let $C = \{u, u'\}$. Then $t(u) = 0$ since $\deg(u) \geq 3$. If $|E(C, D_{r+1})| \geq 2$,

$$w(u, v) \leq \frac{1}{2}\left(\frac{2d}{d-1} - 3 + \frac{2(r-1)}{d-1}\right) \leq \frac{r-1}{2(d-1)}.$$

Hence we may assume that $|E(C, D_{r+1})| = 1$. This implies $\deg(u') = 2$, and then

$$w(u, v) \leq \frac{2d}{d-1} - \left\{1 + \left(1 - \frac{r-1}{d-1}\right) + \left(1 - \frac{r+l-3}{2(d-1)}\right)\right\}$$
$$\leq \frac{r - 2\lfloor l/2 \rfloor + l - 1}{2(d-1)} \leq \frac{r}{2(d-1)}.$$

Suppose furthermore that u is of type M. Then

$$w(u, v) \leq \frac{2d}{d-1} - \left\{1 + \left(1 - \frac{r-2}{d-1}\right) + \left(1 - \frac{r+l-3}{2(d-1)}\right)\right\}$$
$$\leq \frac{r-2}{2(d-1)}.$$

Finally, suppose $\alpha = 1$. Let $k := \deg(u)$ and $N(u) \cap D_{r-1} = \{x\}$. If $k = 2$, then $t(x) = t(u) - 1$, and

$$w(u, v) = \frac{d}{d-1} - (1 - w(x, u))$$
$$\leq \frac{1}{d-1} + \frac{r - 1 + t(u) - 1}{2(d-1)} = \frac{r + t(u)}{2(d-1)}.$$

Note that u is of type M if and only if x is of type M. Therefore, if u is of type M,

$$w(u, v) \leq \frac{1}{d-1} + \frac{r + t(x) - 2}{2(d-1)} = \frac{r + t(u) - 1}{2(d-1)}.$$

If $k \geq 3$,

$$w(u, v) \leq \frac{1}{2}\left(\frac{1}{d-1} + w(x, u)\right) \leq \frac{r}{2(d-1)}.$$

If u is of type M, then x is of type M and $t(x) \le r-2$. Hence

$$w(u, v) \le \frac{1}{2}\left(\frac{1}{d-1}+\frac{r-2}{d-1}\right)=\frac{r-1}{2(d-1)}.$$

If $k \ge 4$,

$$w(u, v) \le \frac{1}{3}\left(\frac{1}{d-1}+\frac{r-1}{d-1}\right)=\frac{r}{3(d-1)}. \qquad \square$$

A connected component C of D_r is called an *end-component* if $E(C, D_{r+1})=\varnothing$.

LEMMA 4. *Suppose C is an end-component of D_r.*

(1) *If $r \le d-1-\lfloor l/2 \rfloor$, then $w(C) \ge \frac{d}{d-1}|C|$. Furthermore, suppose $l \ge d-1$ and $w(C)=\frac{d}{d-1}|C|$. Then l is odd, $r=d-1-\lfloor l/2 \rfloor$, $|C|=2$, and $\sum_{u \in C} t(u)=l-1$.*

(2) *If $r \le d-\lfloor l/2 \rfloor$, then $w(C) \ge \dfrac{d|C|+\lfloor l/2 \rfloor-r-1}{d-1}$. Equality holds only if $r=d-\lfloor l/2 \rfloor$, $|C|=2$, and $t(u)=r$ for all $u \in C$.*

(3) *Suppose $r \le d-\lfloor l/2 \rfloor$ and all the vertices in C are of type M. Then $w(C) \ge \dfrac{d|C|+\lfloor l/2 \rfloor-r}{d-1}$. Equality holds only if $r=d-\lfloor l/2 \rfloor$, $|C|=1$, $|N(C)|=2$ and $t(x)=r-1$ for all $x \in N(C)$.*

(4) *Suppose l is even, $r=d-\lfloor l/2 \rfloor$, and all the vertices in C are of type M. Then $w(C) \ge \frac{d}{d-1}|C|$. Furthermore, suppose $l=d-1$ and $w(C)=\frac{d}{d-1}|C|$. Then $|C|=1$, $|N(C)|=2$, and $\sum_{x \in N(C)} t(x)=l-2$.*

PROOF. Let $\alpha := |C|$ and $\beta := |E(D_{r-1}, C)|-\alpha$.

(1) Suppose $w(C) \le \alpha d/(d-1)$. Then C is a tree and $\alpha(d-2)+\beta(d+1) \le 3d-3$. Hence $\beta \le 2$. More precisely, $\alpha \le 2$ if $\beta=1$, and $\alpha \le 3+3/(d-2) \le 4$ if $\beta=0$. It is easily checked that $\alpha=2$ and $\beta=1$ cannot occur. Suppose $\alpha=\beta=1$, and let $C=\{u\}$, $N(u)=\{x_1, x_2\}$. Then

$$\frac{d}{d-1} \ge w(C) \ge \sum_{i=1}^{2}\left(1-\frac{r-1+t(x_i)}{2(d-1)}\right).$$

This implies that

$$t(x_1)+t(x_2) \ge 2d-2r-2 \ge 2\lfloor l/2 \rfloor \ge l-1.$$

On the other hand, u is contained in a ear of length $t(x_1)+t(x_2)+2$, which contradicts the definition of l. Next, suppose $\beta=0$. It is easily checked that $\alpha=4$ is not possible. If $\alpha=3$, two vertices in C are of degree 2. Hence

$$w(C)\geq 2+\left(1-\frac{r-1}{d-1}\right)+2\left(1-\frac{r+l-3}{2(d-1)}\right)$$
$$\geq\frac{3d-l+2\lfloor l/2\rfloor+1}{d-1}\geq\frac{3d}{d-1}\,.$$

If $l\geq d-1$,

$$w(C)\geq 2+3\left(1-\frac{r-1}{d-1}\right)\geq\frac{2d+1+3\lfloor l/2\rfloor}{d-1}>\frac{3d}{d-1}\,.$$

Suppose $\alpha=2$ and $C=\{u_1, u_2\}$. Then

$$w(C)\geq 1+\sum_{i=1}^{2}\left(1-\frac{r+t(u_i)-2}{2(d-1)}\right).$$

This implies that

$$l-1\geq t(u_1)+t(u_2)\geq 2d-2r-2\geq 2\lfloor l/2\rfloor\,.$$

This is possible only if $w(C)=2d/(d-1)$, l is odd, $r=d-1-\lfloor l/2\rfloor$, and $t(u_1)+t(u_2)=l-1$.

(2) Suppose

$$w(C)\leq\frac{\alpha d+\lfloor l/2\rfloor-r-1}{d-1}\,. \tag{2.3}$$

Since

$$w(C)\geq\alpha-1+(\alpha+\beta)\left(1-\frac{r-1}{d-1}\right),$$

we have

$$(\alpha+\beta-2)\lfloor l/2\rfloor+(\alpha+\beta-1)(d-\lfloor l/2\rfloor-r)\leq\alpha-2\,.$$

This is possible only if equality holds in (2.3), $\alpha=2$, $\beta=0$, $r=d-\lfloor l/2\rfloor$, and $t(u)=r$ for all $u\in C$.

(3) Suppose

$$w(C)\leq\frac{\alpha d-\lfloor l/2\rfloor-r}{d-1}\,. \tag{2.4}$$

Then we have

$$(\alpha+\beta-2)\lfloor l/2\rfloor\leq\alpha-1\,.$$

This is possible only if $\beta\leq 1$. Suppose $\beta=0$. Then $|N(u)\cap D_{r-1}|=1$ for all $u\in C$, and all the vertices in $N(C)\cap D_{r-1}$ are of type M. Hence we have

$$w(C) \geq \alpha - 1 + \alpha\left(1 - \frac{r-2}{d-1}\right),$$

which implies that

$$(\alpha - 1)(d - \lfloor l/2 \rfloor - r) + (\alpha - 2)\lfloor l/2 \rfloor + 1 \leq 0,$$

a contradiction. Suppose $\beta = 1$. Then $|N(u_1) \cap D_{r-1}| = 2$ for some $u_1 \in C$ and $|N(u) \cap D_{r-1}| = 1$ for all $u \in C - \{u_1\}$. Furthermore, all the vertices in $N(C - \{u_1\}) \cap D_{r-1}$ are of type M. Hence we have

$$w(C) \geq \alpha - 1 + 2\left(1 - \frac{r-1}{d-1}\right) + (\alpha - 1)\left(1 - \frac{r-2}{d-1}\right),$$

which implies

$$\alpha(d - \lfloor l/2 \rfloor - r) + (\alpha - 1)\lfloor l/2 \rfloor \leq 0.$$

This is possible only if $r = d - \lfloor l/2 \rfloor$ and $\alpha = 1$. In this case, we have

$$\frac{d + \lfloor l/2 \rfloor - r}{d-1} \geq w(C) \geq \sum_{x \in N(C)} \left(1 - \frac{r - 1 + t(x)}{2(d-1)}\right),$$

which implies

$$\sum_{x \in N(C)} t(x) \geq 2d - 2\lfloor l/2 \rfloor - 2 \geq 2(r-1).$$

This is possible only if $t(x) = r - 1$ for all $x \in N(C)$.

(4) Suppose $w(C) \leq \frac{d}{d-1}|C|$. Then, as in the proof of (1), we get $\alpha(d-3) + \beta d \leq 3d - 3$. This implies $\beta \leq 2$. Suppose $\beta = 2$. Then $\alpha = 1$, and

$$w(C) \geq 1 - \frac{r-1}{d-1} + 2\left(1 - \frac{r+l-2}{2(d-1)}\right)$$
$$= \frac{3d - l - 2r}{d-1} = \frac{d}{d-1}.$$

Equality holds only if $t(x) = r - 1 = l - 1$ for all $x \in N(C)$, but this cannot happen when $l = d - 1$.

Suppose $\beta = 1$. Using the fact that $\alpha - 1$ vertices in $N(C) \cap D_{r-1}$ are of type M, we easily get $\alpha \leq 2$. Suppose $\alpha = 2$ and let $C = \{u_1, u_2\}$ with $N(u_1) = \{x_1, u_2\}$. Since x_1 is of type M and $t(x_1) \leq l - 1$,

$$w(C) \geq 1 + \left(1 - \frac{r-1}{d-1}\right) + \left(1 - \frac{r+l-2}{2(d-1)}\right) + \left(1 - \frac{r+l-4}{2(d-1)}\right) = \frac{2d}{d-1}.$$

Equality holds only if $t(x) = r - 1 = l - 1$ for all $x \in N(u_2) \cap D_{r-1}$. This cannot happen when $l = d - 1$. Suppose $\alpha = 1$ and let $C = \{u\}$ and $N(u) = \{x_1, x_2\}$. Since $t(x_1) + t(x_2) \leq l - 2$,

$$w(C) \geq \sum_{i=1}^{2}\left(1-\frac{r-1+t(x_i)}{2(d-1)}\right)$$

$$=\frac{4d-2r-2-t(x_1)-t(x_2)}{2(d-1)} \geq \frac{d}{d-1}.$$

Equality holds only if $t(x_1)+t(x_2)=l-1$. Finally, suppose $\beta=0$. Using the fact that all the vertices in $N(C) \cap D_{r-1}$ are of type M, and the fact that at least two vertices in C have degree 2, it is easily checked that the only possibility is $\alpha=2$. Let $C=\{u_1, u_2\}$ and $N(u_i) \cap D_{r-1}=\{x_i\}$, $i=1, 2$. Since $t(x_1)+t(x_2) \leq l-3$,

$$w(C) \geq 1+\sum_{i=1}^{2}\left(1-\frac{r-2+t(x_i)}{2(d-1)}\right)$$

$$=\frac{6d-2r-2-t(x_1)-t(x_2)}{2(d-1)}$$

$$\geq \frac{4d+1}{2(d-1)} > \frac{2d}{d-1}. \qquad \square$$

3. Proof of Theorem 2.

Let G be a 2-connected graph of order n with $\operatorname{diam}(G) \leq d$ $(d \geq 5)$. Then $\delta(G) \geq 2$. If $\delta(G) \geq 3$, we have

$$|E(G)| \geq \frac{3}{2}n > \frac{dn-2d-1}{d-1}.$$

Hence we may assume that $\delta(G)=2$. If $\Delta(G)=2$, G is a cycle and $\operatorname{diam}(G)=\lfloor n/2 \rfloor \leq d$. This implies that

$$|E(G)|=n \geq \frac{dn-2d-1}{d-1}.$$

Equality holds only if $n=2d+1$, and then G is isomorphic to $G(1, 1; d)$. In the rest of the proof, we assume that $\Delta(G) \geq 3$. Let $P=(v_0, v_1, \cdots, v_l)$ be a longest ear, and we shall apply the results in Section 2 by setting $D_0 := V(P)$. Note that $l \geq 2$ since $\delta(G)=2$. Moreover, all the vertices in $D_{d-\lfloor l/2 \rfloor}$ are of type M when l is odd, and $D_{d+1-\lfloor l/2 \rfloor}=\varnothing$.

Case I. $l \geq d+1$. Let C be an end-component of D_r. If $r \leq d-\lfloor l/2 \rfloor-1$,

$$w(C) \geq \frac{d|C|+\lfloor l/2 \rfloor-(d-\lfloor l/2 \rfloor-1)-1}{d-1} \geq \frac{d|C|+l-d-1}{d-1}$$

by Lemma 4(2). Suppose $r=d-\lfloor l/2 \rfloor$. If l is even,

$$w(C)\geq\frac{d|C|+l-d-1}{d-1}$$

by Lemma 4(2). If l is odd, all the vertices in C are of type M. Hence

$$w(C)\geq\frac{d|C|+\lfloor l/2\rfloor-(d-\lfloor l/2\rfloor)}{d-1}\geq\frac{d|C|+l-d-1}{d-1}$$

by Lemma 4(3). In every case, $w(C)\geq(d|C|+l-d-1)/(d-1)\geq d|C|/(d-1)$ for any end-component C. Let C_0 be an end-component. Then

$$|E(G)|\geq|E(P)|+\frac{d}{d-1}(n-|V(P)|-|C_0|)+w(C_0)$$

$$\geq l+\frac{d(n-l-1)}{d-1}+\frac{l-d-1}{d-1}=\frac{dn-2d-1}{d-1}.$$

If $|E(G)|=(dn-2d-1)/(d-1)$, then $l=d+1$ or C_0 is the unique end-component. First, suppose C_0 is the unique end-component. Then

$$w(C_0)=\frac{d|C_0|+l-d-1}{d-1}.$$

This is possible only if either (i) $r=d-1-\lfloor l/2\rfloor$, l is odd, $|C|=2$ and $\sum_{u\in C}t(u)=l-1$, (ii) $r=d-\lfloor l/2\rfloor$, l is even, $|C|=2$ and $t(u)=r$ for all $u\in C$, or (iii) $r=d-\lfloor l/2\rfloor$, l is odd, $|C|=1$, $|N(C)|=2$ and $t(x)=r-1$ for all $x\in N(C)$. In case (i),

$$l-1=\sum_{u\in C}t(u)\leq 2r=2\left(d-1-\frac{l-1}{2}\right)\leq l-3,$$

a contradiction. In case (ii) or (iii), the uniqueness of the end-component implies that $\deg(v_0)=\deg(v_l)=2$, which contradicts the definition of P.

Next, suppose $l=d+1$. Then

$$w(C)=\frac{d|C|+l-d-1}{d-1}=\frac{d}{d-1}|C|$$

for any end-component C. By Lemma 4, any such end-component is contained in $D_{d-1-\lfloor l/2\rfloor}\cup D_{d-\lfloor l/2\rfloor}$. Suppose an end-component C is contained in $D_{d-1-\lfloor l/2\rfloor}$. Then by Lemma 4(1), l is odd, $|C|=2$ and $\sum_{u\in C}t(u)=l-1$. However, $t(u)\leq d-1-\lfloor l/2\rfloor<(l-1)/2$. So, this cannot happen. Suppose an end-component C is contained in $D_{d-\lfloor l/2\rfloor}$. Then either (i) l is even, $|C|=2$ and $t(u)=r$ for all $u\in C$, or (ii) l is odd, $|C|=1$, $|N(C)|=2$ and $t(x)=r-1$ for all $x\in N(C)$. It is easily seen that G is isomorphic to $G(a, 1; d)$, where a is the number of end-components. □

In the rest of the proof, we assume that $l\leq d$. Suppose that the two end-vertices v_0 and v_l of P are adjacent. Then

$$H:=G-\{v_1,\cdots,v_{l-1}\}\in\mathscr{G}(n-l+1,d,n-l),$$

$$|E(G)|=l+|E(H)|\geq l+\frac{dn'-2d-1}{d-1}\geq\frac{dn-2d-1}{d-1}$$

by induction, where $n'=|V(H)|=n-l+1$. Furthermore, equality holds only if $l=d$ and

$$|E(H)|=\frac{dn'-2d-1}{d-1}.$$

By induction, H is isomorphic to some $G(a,b;d)$. However, it is easily seen that if we add an ear of length l to an edge of $G(a,b;d)$, the diameter of the resulting graph is greater than d. Hence we may assume that the two end-vertices of any ear of length l are nonadjacent.

If $w(C)\geq\frac{d}{d-1}|C|$ for any end-component C, we have

$$|E(G)|\geq l+\frac{d}{d-1}(n-l-1)>\frac{dn-2d-1}{d-1}.$$

Hence we may assume that some end-component C of D_r satisfies $w(C)<\frac{d}{d-1}|C|$. By Lemma 4(1), $r=d-\lfloor l/2\rfloor$. It is easily seen (by the proof of Lemma 3) that C must be a tree. Let $\alpha:=|C|$ and $\beta:=|E(D_{r-1},C)|-\alpha$. Then

$$w(C)\geq\alpha-1+(\alpha+\beta)\left(1-\min\left\{\frac{r-1}{d-1},\frac{r+l-2}{2(d-1)}\right\}\right),$$

which implies that

$$(\alpha+\beta)\lfloor l/2\rfloor\leq d+\alpha-2, \tag{3.1}$$

$$(\alpha+\beta)\lceil l/2\rceil\geq(\alpha+\beta-2)d-2\alpha+3. \tag{3.2}$$

Case II. $l=d$. By (3.1), we have

$$\beta\leq\frac{d+\alpha-2}{\lfloor l/2\rfloor}-\alpha<2.$$

Subcase II-1. l is odd. Suppose $\beta=0$. Since all the vertices in C are of type M,

$$w(C)\geq\alpha-1+\alpha\left(1-\frac{r-2}{d-1}\right).$$

This implies that $l-2\geq\alpha(l-1)/2$, a contradiction. Suppose $\beta=1$. Then $\alpha=1$. Let $C=\{u\}$ and $N(u)=\{x_1,x_2\}$. Then

$$\frac{d}{d-1}>w(C)\geq\sum_{i=1}^{2}\left(1-\frac{r+t(x_i)-1}{2(d-1)}\right),$$

which implies that $t(x_1)+t(x_2)\geq l-2$. On the other hand, $t(x_i)\leq r-1=(l-1)/2$. Hence we may assume that $t(x_1)=(l-3)/2$ and $t(x_2)=(l-1)/2$. This means that $s(x_1)\in D_1$ and

$s(x_2)\in D_0$. Let $s(C)$ be the connected component of D_1 that contains $s(x_1)$. If $|s(C)|>1$, then $w(x_1, u)\le(r-2)/(d-1)$, which implies that $w(C)\ge d/(d-1)$. Hence we may assume that $|s(C)|=1$. let $\{C_1, \cdots, C_a, C'_1, \cdots, C'_b\}$ be the set of end-components such that $w(C_i)<\frac{d}{d-1}|C_i|$ for $1\le i\le a$ and $w(C'_j)\ge\frac{d}{d-1}|C'_j|$ for $1\le j\le b$, and set $C_i=\{u_i\}$ for $1\le i\le a$. Since $d(u_i, u_j)\le d=l<2r$, we have $s(C_i)=s(C_j)$ for all i and j $(1\le i, j\le a)$. Let k be the degree of the vertex in $s(C_i)$. Then $k\ge a+1$ and

$$w(C_i)=\frac{d}{d-1}-\frac{1}{(k-1)(d-1)}\,.$$

Hence

$$\begin{aligned}|E(G)|&\ge l+\frac{d}{d-1}(n-l-1-a)+\sum_{i=1}^{a} w(C_i)\\&\ge l+\frac{d}{d-1}(n-l-1)-\frac{1}{d-1}=\frac{dn-2d-1}{d-1}\,.\end{aligned}$$

Equality holds only if $k=a+1$ and $w(C'_j)=\frac{d}{d-1}|C'_j|$ for $1\le j\le b$. By Lemma 4, C'_j is contained in $D_{d-\lfloor l/2\rfloor-1}\cup D_{d-\lfloor l/2\rfloor}$. Suppose $v\in D_{d-\lfloor l/2\rfloor}-\bigcup_{i=1}^{a} C_i$. Then $d(v, u_i)\ge 2(d-\lfloor l/2\rfloor)\ge d+1$, a contradiction. Hence C'_j is contained in $D_{d-\lfloor l/2\rfloor-1}$. By Lemma 4(1), $|C'_j|=2$ and $\sum_{u\in C'_j} t(u)=l-1=2(d-\lfloor l/2\rfloor-1)$. This means that G is isomorphic to $G(a, b+1; d)$. □

Subcase II-2. l is even. In this case, we have $(\alpha+\beta-2)d\le 2\alpha-4$ by (3.1). This implies $\beta=0$ and $\alpha=2$. Let $C=\{u_1, u_2\}$ and $N(u_i)\cap D_{r-1}=\{x_i\}$. Then

$$w(C)\ge 1+\sum_{i=1}^{2}\left(1-\frac{r-1+t(x_i)}{2(d-1)}\right),$$

which implies that

$$t(u_1)+t(u_2)=t(x_1)+t(x_2)+2\ge 2d-2r-1=l-1\,.$$

On the other hand, $t(u_1)+t(u_2)\le l-1$ and $t(u_i)\le r=l/2$. Hence we may assume that $t(u_1)=l/2-1$ and $t(u_2)=l/2$. Note that $s(u_1)\in D_1$ and $s(u_2)\in D_0$. Since $s(u_1)$ and $s(u_2)$ are joined by an ear of length l, $s(u_1)$ and $s(u_2)$ are not adjacent. Since $w(x_1, u_1)>(r-2)/(d-1)$, $\{s(u_1)\}$ is a connected component of D_1. Let $\{C_1, \cdots, C_a, C'_1, \cdots, C'_b\}$ be the set of end-components such that $w(C_i)<\frac{d}{d-1}|C_i|$, for $1\le i\le a$, $w(C'_j)\ge\frac{d}{d-1}|C'_j|$ for $1\le j\le b$. We can conclude that G is isomorphic to $G(a, b+1; d)$ by the same way as in Subcase II-1. □

Case III. $l\le d-1$. We shall show that some end-vertex of an ear of length l is of degree 3.

Subcase III-1. l is even. By (3.1) and (3.2),

$$d+\alpha-2\geq(\alpha+\beta-2)d-2\alpha+3\,,$$

which implies $\beta\leq 1$.

Suppose $\beta=1$. Then $\alpha\leq 2$. Suppose $\alpha=2$, and let $C=\{u, v\}$ with $\deg(u)=3$ and $\deg(v)=2$. If $t(v)=l-1$, then the end-vertex u of an ear of length l is of degree 3. Hence we may assume that $t(v)<l-1$. Then

$$w(C)\geq 1+2\left(1-\frac{r+l-2}{2(d-1)}\right)+\left(1-\frac{r+l-4}{2(d-1)}\right).$$

From this, we get $d<\frac{3}{2}l$ instead of (3.2). This contradicts (3.1). Suppose $\alpha=1$, $C=\{u\}$ and $N(u)=\{x_1, x_2\}$. Then

$$w(C)\geq\sum_{i=1}^{2}\left(1-\frac{r-1+t(x_i)}{2(d-1)}\right),$$

which implies that $t(x_1)+t(x_2)\geq l-1$, a contradiction.

Suppose $\beta=0$. If $\alpha\geq 2$, at least two vertices of C are of degree 2. Hence

$$d+\alpha-2\geq\alpha\lfloor l/2\rfloor=\alpha\lceil l/2\rceil\geq(\alpha-2)d-2\alpha+5\,.$$

This implies that $\alpha\leq 4$, but for $\alpha=4$, there is no integral solution. Suppose $\alpha=3$ and $C=\{u_1, u_2, v\}$ with $\deg(v)=3$. Then we may assume that $t(x)<l-1$ for all $x\in N(v)$. Then we get $d\leq\frac{3}{2}l-4$ instead of (3.2). This contradicts (3.1). Suppose $\alpha=2$ and $C=\{u_1, u_2\}$. Then by the same argument as in Subcase II-2, we get $t(u_1)+t(u_2)=l-1$. We may assume that $\deg(s(u_i))\geq 4$ for $i=1, 2$. Then

$$w(C)\geq 1+\sum_{i=1}^{2}\left(1-\max\left\{\frac{r-t(u_i)}{3(d-1)}, \frac{r-1-t(u_i)}{2(d-1)}\right\}-\frac{t(u_i)-1}{d-1}\right)$$

by Lemma 3(3). Let

$$S:=\sum_{i=1}^{2}\max\left\{\frac{r-t(u_i)}{3(d-1)}, \frac{r-1-t(u_i)}{2(d-1)}\right\}.$$

If

$$S=\frac{r-t(u_1)}{3(d-1)}+\frac{r-t(u_2)}{3(d-1)}=\frac{2r-l+1}{3(d-1)}\,,$$

we have

$$d-2l+2\lfloor l/2\rfloor-1<0\,,$$

which contradicts the assumption that $l\leq d-1$. If

$$S=\frac{r-1-t(u_1)}{2(d-1)}+\frac{r-1-t(u_2)}{2(d-1)}=\frac{2r-l-1}{2(d-1)}\,,$$

we have $2\lfloor l/2\rfloor < l-1$, a contradiction. Suppose

$$S=\frac{r-t(u_1)}{3(d-1)}+\frac{r-1-t(u_2)}{2(d-1)}.$$

Then we have

$$5r-6d+6l-3>2t(u_1)+3t(u_2)\geq 3l-3-r,$$

a contradiction. □

Subcase III-2. l is odd. By (3.1) and (3.2), we have $\beta\leq 2$. Suppose $\beta=2$. Then there are at least $\alpha-2$ vertices u in C such that $|N(u)\cap D_{r-1}|=1$. Since all the vertices in C are of type M,

$$w(C)\geq \alpha-1+4\left(1-\min\left\{\frac{r-1}{d-1},\frac{r+l-2}{2(d-1)}\right\}\right)$$
$$+(\alpha-2)\left(1-\min\left\{\frac{r-2}{d-1},\frac{r+l-3}{2(d-1)}\right\}\right),$$

which implies that $\alpha\leq 1+3/(d-2)<3$. Suppose $\alpha=2$. Then $d=5$ and $l=7/2$, a contradiction.

Suppose $\alpha=1$ and $C=\{u\}$. Then we may assume that $t(x)<l-1$ for all $x\in N(u)$. Hence

$$w(C)\geq 3\left(1-\frac{r+l-3}{2(d-1)}\right),$$

which implies that $3d\geq 2d+4$. This contradicts (3.1).

Next, suppose $\beta=1$. If $\alpha\geq 2$, there are $\alpha-1$ vertices u in C satisfying $|N(u)\cap D_{r-1}|=1$, one of which is of degree 2. Since all the vertices in C are of type M,

$$w(C)\geq \alpha+2\left(1-\min\left\{\frac{r-1}{d-1},\frac{r+l-2}{2(d-1)}\right\}\right)$$
$$+(\alpha-2)\left(1-\min\left\{\frac{r-2}{d-1},\frac{r+l-3}{2(d-1)}\right\}\right)$$
$$+\left(1-\min\left\{\frac{r-2}{d-1},\frac{r+l-4}{2(d-1)}\right\}\right),$$

which implies that

$$d-1\geq(\alpha+1)\lfloor l/2\rfloor=(\alpha+1)(\lceil l/2\rceil-1)\geq(\alpha-1)d-2\alpha+2.$$

Hence

$$\alpha \leq 2 + \frac{1}{d-2} < 3\,.$$

Suppose $\alpha=2$ and $C=\{u, v\}$ with $\deg(u)=2$ and $\deg(v)=3$. Then we may assume that $t(x)<l-1$ for all $x \in N(v)$. Hence

$$w(C) \geq 4 - \min\left\{\frac{3r-4}{d-1}, \frac{3r+3l-11}{2(d-1)}\right\},$$

which is impossible. Suppose $\alpha=1$, $C=\{u\}$, and $N(u)=\{x_1, x_2\}$. By the same argument as in Subcase II-1, we have $t(x_1)+t(x_2)=l-2$, and we may assume that $\deg(s(x_i)) \geq 4$ for $i=1, 2$. Then

$$w(C) \geq \sum_{i=1}^{2}\left(1 - \max\left\{\frac{r-1-t(x_i)}{3(d-1)}, \frac{r-2-t(x_i)}{2(d-1)}\right\} - \frac{t(x_i)}{d-1}\right)$$

by Lemma 3(3). It is easily verified that this leads to a contradiction as in Subcase III-1.

Finally, suppose $\beta=0$. Then for all $x \in C$, x is of type M and $|N(x) \cap D_{r-1}|=1$. Furthermore, at least two vertices in C are of degree 2. Hence

$$w(C) \geq \alpha - 1 + (\alpha-2)\left(1 - \min\left\{\frac{r-2}{d-1}, \frac{r+l-3}{2(d-1)}\right\}\right)$$
$$+ 2\left(1 - \min\left\{\frac{r-2}{d-1}, \frac{r+l-4}{2(d-1)}\right\}\right),$$

which implies that $\alpha \leq (3d-7)/(d-2) < 3$. Suppose $\alpha=2$ and $C=\{u_1, u_2\}$. Then

$$w(C) \geq 1 + \sum_{i=1}^{2}\left(1 - \frac{r-3+t(u_i)}{2(d-1)}\right),$$

which implies that

$$t(u_1)+t(u_2) > 2d-2r = l-1\,,$$

a contradiction. □

We have proved that there exists an ear of length l, one of whose end vertices is of degree 3. Hence we may assume that $N(v_l)=\{v_{l-1}, v_{l+1}, v'_{l+1}\}$. Set $D_0 := V(P) \cup N(v_l)$, and apply the results in Section 2. If l is odd, $D_{d-\lfloor l/2 \rfloor}=\emptyset$. If l is even, all the vertices in $D_{d-\lfloor l/2 \rfloor}$ are of type M. By Lemma 4(1) and (4), $w(C) \geq \frac{d}{d-1}|C|$ for any end-component C. Hence

$$|E(G)| \geq l+2+\frac{d}{d-1}(n-l-3) \geq \frac{dn-2d-1}{d-1}\,.$$

Equality holds only if $l=d-1$ and $w(C)=\frac{d}{d-1}|C|$ for any end-component C. Suppose l is odd, and C is an end-component. By Lemma 4(1), C is contained in $D_{d-1-\lfloor l/2 \rfloor}$,

$|C|=2$, and $\sum_{u\in C} t(u)=l-1$. It is easily verified that all the vertices in C are of type U. Since $\deg(v_{l+1})\geq 2$ and $\deg(v'_{l+1})\geq 2$, there are end-components $C_1=\{u_1, u_2\}$ and $C_2=\{u'_1, u'_2\}$ such that $d(u_1, v_{l+1})=d(u'_1, v'_{l+1})=d-1-\lfloor l/2 \rfloor$. Then $d(u_1, u'_1)=2(d-1-\lfloor l/2 \rfloor)+2=d+2$, a contradiction. Suppose l is even. By Lemma 4(4), any end-component is contained in $D_{d-\lfloor l/2 \rfloor}$, and consists of a single vertex. In this case, there are end-components $C_1=\{u_1\}$ and $C_2=\{u'_1\}$ such that $d(u_1, v_{l+1})=d(u'_1, v'_{l+1})=d-\lfloor l/2 \rfloor$. Since all the vertices in $N(u_1)\cup N(u'_1)$ are of type U, $d(u_1, u'_1)=2(d-\lfloor l/2 \rfloor)=d+1$, a contradiction. □

This completes the proof of Theorem 2. □

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