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Strong Solutions of Two Dimensional Heat Convection Equations with Dissipating Terms

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1. Introduction.

In this paper, we consider the existence of the strong solutions for the initial boundary value problems and the periodic problems of heat convection equations with dissipative terms in time dependent domains. The "dissipative terms" represent the friction of the fluid.

Let $Q(t)$ be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\Gamma(t)$ for each $t\in[0, T]$, T be any positive number.

We consider the following heat convection equations in the noncylindrical domain $Q=\bigcup_{0\le t\le T} Q(t)\times\{t\}$ with lateral boundary $\Gamma=\bigcup_{0\le t\le T}\Gamma(t)\times\{t\}$.

$$\mathbf{u}_t-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla p+\{1-\eta(\theta-d)\}\mathbf{g}+\mathbf{f}_1 \qquad (x, t)\in Q\,, \tag{1.1}$$

$$\operatorname{div}\mathbf{u}=0 \qquad (x, t)\in Q\,, \tag{1.2}$$

$$\theta_t-\kappa\Delta\theta+(\mathbf{u}\cdot\nabla)\theta+\frac{\nu}{2}D[\mathbf{u}]=f_2 \qquad (x, t)\in Q\,, \tag{1.3}$$

$$\mathbf{u}(x, t)=\mathbf{a}(x, t)\,,\quad \theta(x, t)=b(x, t) \qquad (x, t)\in\Gamma\,, \tag{1.4}$$

$$\mathbf{u}(x, 0)=\mathbf{u}_0(x)\,,\quad \theta(x, 0)=\theta_0(x) \qquad x\in Q(0)\,, \tag{1.5}$$

$$\mathbf{u}(x, 0)=\mathbf{a}(x, T)\,,\quad \theta(x, 0)=b(x, T) \qquad x\in Q(0)=Q(T)\,, \tag{1.6}$$

where

$$(\mathbf{u}\cdot\nabla)=\sum_{j=1}^{N}u^j\frac{\partial}{\partial x_j} \quad\text{and}\quad D[\mathbf{u}]=\sum_{i,j=1}^{N}\left(\frac{\partial u^i}{\partial x_j}+\frac{\partial u^j}{\partial x_i}\right)^2.$$

Unknown functions $\mathbf{u}=(u^1, u^2, \cdots, u^N)$, p and θ are the solenoidal velocity, pressure

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and temperature of the fluid which occupies Q respectively; $\mathbf{a}$, b, u_0, θ_0 are given data and $\mathbf{g}$ is the body force field (say gravity); $\mathbf{f}_1$ and f_2 are external forces; constants ν, ρ, κ, η, d represent kinematic viscosity, density, thermal conductivity, volume expansion coefficient and some datum point of the temperature of the fluid respectively, (see Joseph [13], Landau-Lipschitz [18]).

The case of $D[\mathbf{u}]=0$ *i.e.* Oberbeck-Boussinesq equations which give the first order approximation of the heat equations in fluid in some sense, have been studied by several authors (see the references in Inoue-Ôtani [11]).

Our equation is derived from the second order approximation, and Bui An Ton and Lukaszewicz [4] showed the existence of the weak solutions of the initial boundary problems, (I.B.P), (1.1)–(1.5) for the case $N=3$ and Kagei ([14]) studied the attractor of weak solutions with periodic boundary conditions for the case $N=2$. However, it seems that the study for the existence of strong solutions of (I.B.P) and for the periodic problems (P.P), (1.1)–(1.4) and (1.6), and not fully pursued.

The main purpose of this paper is to give the existence of the strong solutions for (I.B.P) and (P.P) for the two space dimensional case, $N=2$. Our main tools here is based on the perturbation method for time dependent subdifferential operators developed in Ōtani ([21], [22]). Inoue-Ōtani ([11]). Main results are stated in the next section, and their proofs are given in §3.

2. Main Results.

2.1. Notations and some function spaces. To formulate our results, we prepare some notations. We denote by $|\cdot|$ the norm of Lebesgue space $L^2(\Omega)$, and adopt the following function spaces:

$\mathrm{H}^s(\Omega)=$ the Sobolev space of order s in $\mathrm{L}^2(\Omega)$ with norm $|\cdot|_{\mathrm{H}^s}$,

$\mathrm{H}_0^1(\Omega)=$ the completion of $\mathrm{C}_0^\infty(\Omega)$ under the $\mathrm{H}^1(\Omega)$-norm,

$\mathbf{L}^2(\Omega)=(\mathrm{L}^2(\Omega))^2$ with norm $\|\cdot\|$,

$\mathbf{H}^s(\Omega)=(\mathrm{H}^s(\Omega))^2$ with norm $\|\cdot\|_{\mathbf{H}^s}$,

$\mathbf{H}_0^1(\Omega)=(\mathrm{H}_0^1(\Omega))^2$,

$\mathbf{C}_\sigma^\infty(\Omega)=\{\mathbf{u}=(u^1, u^2); u^j\in \mathrm{C}_0^\infty(\Omega), j=1, 2, \operatorname{div}\mathbf{u}=0\}$,

$\mathbf{L}_\sigma^2(\Omega)=$ the completion of $\mathbf{C}_\sigma^\infty(\Omega)$ under the $\mathbf{L}^2(\Omega)$-norm,

$\mathbf{H}_\sigma^1(\Omega)=\mathbf{H}^1(\Omega)\cap \mathbf{L}_\sigma^2(\Omega)$,

$\mathrm{P}_\Omega=$ the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}_\sigma^2(\Omega)$,

Define two operators by:

$\mathrm{A}_0(\Omega)=-\Delta$ with domain $\mathrm{D}(\mathrm{A}_0(\Omega))=\mathrm{H}^2(\Omega)\cap H_0^1(\Omega)$,

$A(\Omega)=-P_\Omega\Delta$: Stokes operator with domain $D(A(\Omega))=\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_\sigma(\Omega)$,

$A_0^\alpha(\Omega)$, $A^\alpha(\Omega)$: the fractional power of $A_0(\Omega)$, $A(\Omega)$ of order α, whose domain $D(A_0^\alpha(\Omega))$ or $D(A^\alpha(\Omega))$ is characterized by Fujiwara [6] and Fujita and Morimoto [8]. Especially, for the case $\alpha=0$, we set $D(A_0^0(\Omega))=L^2(\Omega)$.

We also use the notations:

$$\begin{cases} \|\mathbf{u}\|_p=\|\mathbf{u}\|_{(L^p)^N}, \\ \|\mathbf{u}\|_{\infty,T}=\sup\limits_{0\le t\le T}\|\mathbf{u}(t)\|, \\ \|\mathbf{u}\|^2_{\mathbf{H}^s,M,T}=\begin{cases} \sup\limits_{1\le t\le T}\displaystyle\int_{t-1}^t\|\mathbf{u}(\tau)\|^2_{\mathbf{H}^s}d\tau, & \text{for } T\ge 1, \\ \displaystyle\int_0^T\|\mathbf{u}\|(\tau)\|^2_{\mathbf{H}^s}d\tau, & \text{for } 0<T\le 1, \end{cases} \end{cases}$$

$$\begin{cases} |\theta|_p=|\theta|_{L^p}, \\ |\theta|_{\infty,T}=\sup\limits_{0\le t\le T}|\theta(t)|, \\ |\theta|^2_{H^s,M,T}=\begin{cases} \sup\limits_{1\le t\le T}\displaystyle\int_{t-1}^t|\theta(\tau)|^2_{H^s}d\tau, & \text{for } T\ge 1, \\ \displaystyle\int_0^T|\theta(\tau)|^2_{H^s}d\tau, & \text{for } 0<T\le 1, \end{cases} \end{cases}$$

and in the case of the periodic problems for $0<T\le 1$,

$$\|\mathbf{u}\|^2_{\mathbf{H}^s,M,T}=\frac{1}{T}\int_0^T\|\mathbf{u}(\tau)\|^2_{\mathbf{H}^s}d\tau,\ |\theta|^2_{H^s,M,T}=\frac{1}{T}\int_0^T|\theta(\tau)|^2_{H^s}d\tau.$$

For simplicity we designate $\|\mathbf{u}\|_{\mathbf{H}^0,M,T}$ and $|\theta|_{H^0,M,T}$ by $\|\mathbf{u}\|_{M,T}$ and $|\theta|_{M,T}$ respectively.

Let B be a bounded domain in $\mathbf{R}^2$ such that $B\times[0,T]$ contains the closure of Q. For a function v defined on Q, we denote by $[v]^\wedge$ or simply by $\hat{v}$ the zero-extension of v to $B\times[0,T]$, *i.e.* $[v]^\wedge(x,t)=v(x,t)$ for $(x,t)\in Q$ and $[v]^\wedge(x,t)=0$ for $(x,t)\in B\times[0,T]\backslash Q$. We denote by $C([0,T];X(Q(t)))$ the set of all functions v defined on Q such that $v(\cdot,t)$ belongs to $X(Q(t))$ for all $t\in[0,T]$ and the zero extension $\hat{v}$ of v to $B\times[0,T]$ is an $X(B)$-valued continuous function on $[0,T]$ and $X(\Omega)$ is a function space defined on Ω such as $\mathbf{H}^1_\sigma(\Omega)$, $H^1_0(\Omega)$, etc. For the periodic problem, we prepare the function space $C_\pi([0,T];X)\overset{\text{def}}{=}\{f\in C([0,T];X(Q(t)));\ f(0)=f(T)\}$.

We put

$$\lambda = \min_{0\le t\le T} \lambda_1(t)\,, \qquad \mu = \min_{0\le t\le T} \mu_1(t)\,,$$

where $\lambda_1(t)$ (resp. $\mu_1(t)$) is the first eigenvalue of the Stokes operator $A(Q(t))$ (resp. $-\Delta$) with domain $D(A(Q(t)))=\mathbf{H}^2(Q(t))\cap \mathbf{H}^1_\sigma(Q(T))$, (resp. $H^2(Q(t))\cap H^1_0(Q(t))$).

2.2. Main results. We assume that Q is smooth and $\mathbf{a}$ and b have nice extensions $\bar{\mathbf{u}}$ and $\bar\theta$ to Q in the following sense:

(A.Q) There exists a level preserving C^3-diffeomorphism $\mathscr{G}$ from Q onto $Q_0\times[0, T]$ for some bounded domain in Q_0 in $\mathbf{R}^2$.

$(\mathrm{A.Q})_\pi$ Q satisfies (A.Q) and the periodic condition $Q(0)=Q(T)$.

(A.a) There exists a vector function $\bar{\mathbf{u}}$ in $\mathbf{C}^1(Q)$ such that

$$\bar{\mathbf{u}}\in L^\infty(0, T; \mathbf{H}^1(Q(t)))\cap L^2(0, T; \mathbf{H}^2(Q(t)))\,,$$

$$\bar{\mathbf{u}}_t\in L^2(0, T; \mathbf{L}^2(Q(t)))\,, \quad \mathrm{div}\,\bar{\mathbf{u}}=0 \text{ in } Q \text{ and } \bar{\mathbf{u}}=\mathbf{a} \text{ on } \Gamma\,.$$

$(\mathrm{A.}a)_\pi$ (A.a) is satisfied and $\bar{\mathbf{u}}(x, 0)=\bar{\mathbf{u}}(x, T)$.

(A.b) There exists a function $\bar\theta$ in $C^1(Q)$ such that

$$\bar\theta\in L^\infty(0, T; H^1(Q(t)))\cap L^2(0, T; H^2(Q(t)))$$

$$\bar\theta_t\in L^2(Q) \text{ and } \bar\theta=b \text{ on } \Gamma\,.$$

$(\mathrm{A.}b)_\pi$ (A.b) is satisfied and $\bar\theta(x, 0)=\bar\theta(x, T)$.

(A.f) $\mathbf{f}_1\in L^2(0, T; L^2(Q(t)))$ and $f_2\in L^2(Q)$.

(A.g) $\mathbf{g}$ has the potential $G\in L^\infty(0, T; W^{1,\infty}(Q(t)))$ *i.e.*, $\mathbf{g}=\nabla G$.

(When $\mathbf{g}$ is the gravity, this condition is always satisfied.)

Now our main results are stated as follows:

THEOREM I (Time Local existence for (I.B.P.)). *Let* (A.Q), (A.a), (A.b), (A.f) *and* (A.g) *be satisfied, and let* $\mathbf{u}_0-\bar{\mathbf{u}}(\cdot, 0)\in D(A^\alpha(Q(0)))$ *with* $\alpha\in[\frac14, \frac12]$ *and* $\theta_0-\bar\theta(\cdot, 0)\in D(A_0^\beta(Q(0)))$ *with* $\beta\in[0, \frac12]$. *Then there exist* $S\in(0, T]$ *such that (I.B.P) has a unique solution* $(\mathbf{u}, \theta)$ *satisfying*

$$(S1)\quad \begin{cases} \mathbf{u}-\bar{\mathbf{u}}\in C([0, S]; \mathbf{L}^2_\sigma(Q(t)))\cap C((0, S]; \mathbf{H}^1_\sigma(Q(t)))\,, \\ t^{1/2-\alpha}\|([\mathbf{u}-\bar{\mathbf{u}}]^\wedge)_t\|\,, \quad t^{1/2-\alpha}\|\Delta\hat{\mathbf{u}}\|\in L^2(0, S)\,, \\ t^{-\alpha}\|[\mathbf{u}-\bar{\mathbf{u}}]^\wedge\|\,, \quad t^{1/2-\alpha}\|\nabla([u-\bar{u}]^\wedge)\|\in L^q_*(0, S) \qquad \textit{for all} \quad q\in[2, \infty]\,, \end{cases}$$

$$(S2)\quad \begin{cases} \theta-\bar\theta\in C([0, S]; L^2(Q(t)))\cap C((0, S]; H^1_0(Q(t)))\,, \\ t^{1/2-\beta}|([\theta-\bar\theta])^\wedge)_t|\,, \quad t^{1/2-\beta}|\Delta\hat\theta|\in L^2(0, S)\,, \\ t^{-\beta}|[\theta-\bar\theta]^\wedge|\,, \quad t^{1/2-\beta}|\nabla([\theta-\bar\theta]^\wedge)|\in L^q_*(0, S) \qquad \textit{for all} \quad q\in[2, \infty]\,, \end{cases}$$

where

$$L^q_*(0, S)=\left\{f;\ \int_0^S \frac{|f|^q}{t}\,dt<+\infty\right\} \quad \textit{and} \quad L^\infty_*(0, S)=L^\infty(0, S)\,.$$

THEOREM II (Global existence for (I.B.P.)). *Let* (A.Q), (A.*a*), (A.*b*), (A.*f*) *and* (A.*g*) *be satisfied, and let* $\|\nabla\bar{\mathbf{u}}\|_{\mathbf{H}^1,\infty,T}\in L^2(0,T)$, $\mathbf{u}_0-\bar{\mathbf{u}}(\cdot,0)\in D(A^{1/2}(Q(0)))$, $\theta_0-\bar{\theta}(\cdot,0)\in D(A^{\beta}(Q(0)))$ *with* $\beta\in[0,\frac{1}{2}]$ *and* $\|\mathbf{u}_0\|_{\mathbf{H}^1}$, $\|\bar{\mathbf{u}}_t\|_{M,T}$, $\|\bar{\mathbf{u}}\|_{\mathbf{H}^1,\infty,T}$, $\|\bar{\mathbf{u}}\|_{\mathbf{H}^2,M,T}$, $\|\eta\mathbf{g}\|_{\mathbf{L}^\infty(Q)}$, $\|\eta\mathbf{g}\bar{\theta}\|_{M,T}$, $\|\mathbf{f}_1\|_{M,T}$ *be sufficiently small, then (I.B.P.) has a unique solution for any T satisfying* (*S*1) *and* (*S*2) *with* $\alpha=1/2$.

THEOREM III (Existence for (P.P.)). *Let* $(A.Q)_\pi$, $(A.a)_\pi$, $(A.b)_\pi$, (A.*f*) *and* (*A.g*) *be satisfied, and let* $\|\bar{\mathbf{u}}_t\|_{M,T}$, $\|\bar{\mathbf{u}}\|_{\mathbf{H}^1,\infty,T}$, $\|\bar{\mathbf{u}}\|_{\mathbf{H}^2,M,T}$, $|\bar{\theta}_t|_{M,T}$, $|\bar{\theta}|_{H^1,\infty,T}$, $|\bar{\theta}|_{H^2,M,T}$, $\|\mathbf{f}_1\|_{M,T}$, $|f_2|_{M,T}$ *be sufficiently small, then (P.P.) has a unique periodic solution satisfying*

$$(SP)\quad \begin{cases} \mathbf{u}-\bar{\mathbf{u}}\in C_\pi([0,T];\mathbf{H}^1_\sigma(Q(t)))\,,\quad \mathbf{u}_t,\,\Delta\mathbf{u},\,(\mathbf{u}\cdot\nabla)\mathbf{u}\in L^2(0,T;\mathbf{L}^2(Q(t)))\,,\\ \|\nabla\mathbf{u}(t)\|^2 \text{ is absolutely continuous on } [0,T]\,,\\ \theta-\bar{\theta}\in C_\pi([0,T];H^1_0(Q)))\,,\quad \theta_t,\,\Delta\theta,\,(\mathbf{u}\cdot\nabla)\theta D[\mathbf{u}]\in L^2(Q)\,,\\ |\nabla\theta(t)|^2 \text{ is absolutely continuous on } [0,T] \end{cases}$$

3. Proofs of theorems.

3.1. Some Abstract Results. To prove our theorems, we need some abstract results given in [21] and [22]. In this subsection, we collect them without their proofs.

Let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$, and $\Phi(H)$ be the family of all proper lower semicontinuous convex functions φ from H to $(-\infty,\infty]$, where "proper" means that the effective domain $D(\varphi)=\{u\in H;\ \varphi(u)<\infty\}$ of φ is not empty. We define the subdifferential $\partial\varphi$ of φ at u by

$$\partial\varphi(u)=\{f\in H;\ \varphi(v)-\varphi(u)\geq(f,v-u)_H \text{ for all } v\in H\}$$

with domain $D(\partial\varphi)=\{u\in H;\ \partial\varphi(u)\neq\varnothing\}$. Then it is well known that $\partial\varphi$ becomes a maximal monotone operator and is possibly multivalued. We designate by $\partial^o\varphi$ the minimal section of $\partial\varphi(u)$, *i.e.*, $\partial^o\varphi(u)$ is the unique element of least norm in $\partial\varphi(u)$, (see H. Brézis [2]).

For a maximal monotone operator A in H with domain D(A), the nonlinear interpolation class $\mathscr{B}_{\alpha,p}(A)$ between D(A) and $\overline{D(A)}$, is defined by

$$\mathscr{B}_{\alpha,p}(A)=\{u\in\overline{D(A)};\ t^{-\alpha}|u-J_t^Au|_H\in L^p_*(0,1)\}$$

where $\alpha\in(0,1)$, $p\in[1,\infty]$, $J_t^A=(I+tA)^{-1}$, $|f|_{L^p_*(0,1)}=\left(\int_0^1\frac{|f|^p}{t}\,dt\right)^{1/p}$ for $1\leq p<\infty$ and $|f|_{L^\infty_*(0,1)}=|f|_{L^\infty(0,1)}$. We also use the notation $|u|_{\alpha,p,A}=|t^{-\alpha}|u-J_t^Au|_H|_{L^p_*(0,1)}$, (see D. Brézis [1]).

Consider the following abstract Cauchy problem and periodic problem in a real separable Hilbert space H.

$$\text{(C.P.)}\qquad \begin{cases} \dfrac{du}{dt}(t)+\partial\varphi^t(u(t))+\mathrm{B}(t,u(t))\ni f(t)\,, & 0<t<T\,,\\ u(0)=u_0\,, \end{cases}$$

and

$$\text{(A.P.P.)}\qquad \begin{cases} \dfrac{du}{dt}(t)+\partial\varphi^t(u(t))+\mathrm{B}(t,u(t))\ni f(t)\,, & 0<t<T\,,\\ u(0)=u(T)\,, \end{cases}$$

where $\partial\varphi^t$ is the subdifferential of a time-dependent proper lower semicontinuous convex functions φ^t from H into $[0,\infty]$, and $\mathrm{B}(t,\cdot)$ is a nonlinear operator with $\mathrm{D}(\partial\varphi^t)\subset\mathrm{D}(\mathrm{B}(t,\cdot))$, regarded as a perturbation for $\partial\varphi^t$.

DEFINITION OF A STRONG SOLUTION. A function $u\in\mathrm{C}([0,S];\mathrm{H})$ is said to be a *strong solution* of (CP) on [0, S], if the following (i) and (ii) are satisfied.

(i) $u(t)$ is an H-valued absolutely continuous function on $[\delta,S]$ for all $\delta>0$ and $u(t)\to u_0$ as $t\downarrow 0$.

(ii) $u(t)\in\mathrm{D}(\partial\varphi^t)$ and $-du(t)/dt-\mathrm{B}(t,u(t))+f(t)\in\partial\varphi^t(u(t))$ hold for a.e. $t\in(0,S)$.

A function $u\in\mathrm{C}_\pi([0,T];\mathrm{H})$ is said to be a *strong solution* of (A.P.P.) on $[0,T]$, if the following (i) and (ii) are satisfied.

(i) $u(t)$ is an H-valued absolutely continuous function on $[0,T]$.

(ii) $u(t)\in\mathrm{D}(\partial\varphi^t)$ and $-du(t)/dt-\mathrm{B}(t,u(t))+f(t)\in\partial\varphi^t(u(t))$ hold for a.e. $t\in[0,T]$.

In order to assure the existence of strong solutions of (C.P.) and (A.P.P.), some smoothness conditions on φ^t with respect to t are required such as in (Kenmochi [15], Yamada [27], Ôtani [21], [22]). Here, we assume the following somewhat a simplified version.

(A.φ^t) There exist nonnegative constants m_1, m_2, m_3 such that for each $t_0\in[0,T]$ and $x_0\in\mathrm{D}(\varphi^{t_0})$, there exists a function $x(t)$ such that

$$|x(t)-x_0|_{\mathrm{H}}\le m_1|t-t_0|(\varphi^{t_0}(x_0)+m_2)^{1/2}\,,$$

and

$$\varphi^t(x(t))\le\varphi^{t_0}(x_0)+m_3|t-t_0|(\varphi^{t_0}(x_0)+m_2)$$

hold for all $t\in[0,T]$.

(A.φ^t)$_\pi$ The following (i)–(iv) hold.

(i) For each $t\in[0,T]$, $\varphi^t\in\Phi(\mathrm{H})$ and $\varphi^0(u)=\varphi^T(u)$ for all $u\in\mathrm{H}$,

(ii) There exist nonnegative constants m_1, m_2, m_3 such that for each $t_0\in[0,T]$ and $x_0\in\mathrm{D}(\varphi^{t_0})$, there exists an H-valued function $x(t)$ on $[0,T]$ satisfying

$$|x(t)-x_0|_H \le m_1 |t-t_0|(\varphi^{t_0}(x_0)+m_2)^{1/2}, \qquad \text{for all} \quad t\in[0, T],$$

$$\varphi^t(x(t)) \le \varphi^{t_0}(x_0)+m_3|t-t_0|(\varphi^{t_0}(x_0)+m_2), \qquad \text{for all} \quad t\in[0, T].$$

(iii) There exist constants $k_0>0$ and $p>1$ such that

$$k_0|u|_H^p \le \varphi^t(u), \qquad \text{for all} \quad u\in D(\varphi^t).$$

(iv) For each $t\in[0, T]$, $\partial\varphi^t$ is strictly monotone, *i.e.*, $(w_1-w_2, u_1-u_2)_H=0$ with $u_i\in D(\partial\varphi^t)$ and $w_i\in\partial\varphi^t(u_i)$ $(i=1, 2)$ implies $u_1=u_2$.

LEMMA 3.1. *Let* (A.φ^t) *or* (i) *and* (ii) *of* $(A.\varphi^t)_\pi$ *be satisfied and* $u(t)$ *be an* H-*valued absolutely continuous function on* $[0, T]$. *Put*

$$\mathscr{L}=\left\{t\in[0, T]; \frac{du(t)}{dt} \quad \textit{and} \quad \frac{d\varphi^t(u(t))}{dt}\ \textit{exist},\ u(t)\in D(\partial\varphi^t)\right\},$$

then

$$\left|\frac{d\varphi^t(u(t))}{dt}-\left(g, \frac{d}{dt}u(t)\right)_H\right| \le m_1|g(t)|_H(\varphi^t(u(t))+m_2)^{1/2}+m_3(\varphi^t(u(t))+m_2) \qquad (3.1)$$

holds for all $t\in\mathscr{L}$ *and all* $g(t)\in\partial\varphi^t(u(t))$.

PROOF. See [23] and [24].

We impose the following conditions on $\partial\varphi^t$ and $B(t, \cdot)$:

(A.1) For each $t\in[0, T]$ and $L\in(0, \infty)$, the set $\{u\in H; \varphi^t(u)+|u|_H^2\le L\}$ is compact in H.

(A.2) $B(t, \cdot)$ is measurable in the following sense:
For each interval $[a, b]$ in $[0, T]$, the following (1) and (2) hold.

(1) For each function $u(t)\in C([a, b]; H)$ such that $du(t)/dt\in L^2(a, b; H)$ and there exists a function $g(t)\in L^2(a, b; H)$ with $g(t)\in\partial\varphi^t(u(t))$ for a.e. $t\in[a, b]$, $B(t, u(t))$ is measurable in $t\in[a, b]$.

(2) If $u_n\to u$ in $C([a, b]; H)$, $g_n\rightharpoonup g$ weakly in $L^2(a, b; H)$ with $g_n(t)\in\partial\varphi^t(u_n(t))$, $g(t)\in\partial\varphi^t(u(t))$ for a.e. $t\in[a, b]$, and $B(t, u_n(t))\rightharpoonup b(t)$ weakly in $L^2(a, b; H)$, then $b(t)=B(t, u(t))$ for a.e. $t\in[a, b]$.

$(A.3)_\alpha$ For an exponent $\alpha\in(0, 1/2)$, there exist functions $l_0, l_1\in\mathscr{M}$ and $a_1(t)\in L^2(0, T)$,

$$a_2(t)\in X_T^\alpha \overset{\text{def}}{=} \{a(t)\colon [0, T]\to H;\ \int_0^T t^{1-2\alpha}|a(t)|_H^2 dt<+\infty\} \text{ such that}$$

$$|B(t, u)|_H \le l_0(|u|_H)\{\varepsilon|\partial^\circ\varphi^t(u(t))|_H + l_1(1/\varepsilon)(|\varphi^t(u)|^{(1-\alpha)/(1-2\alpha)}+|\varphi^t(u)|^{1/2}|a_1(t)|)+|a_2(t)|\}$$

for all $\varepsilon\in(0, 1)$, a.e. $t\in[0, T]$ and all $u\in D(\partial\varphi^t)$,
where $\mathscr{M}$ stands for the family of all positive monotone increasing functions on $[0, \infty)$.

(A.4) There exist functions $l_2 \in \mathcal{M}$, $a_3(t) \in L^1(0, T)$ and a constant $k \in [0, 1)$ such that

$$|B(u, t)|_H^2 \le k|\partial^\circ \varphi^t(u)|_H^2 + l_2(\varphi^t(u) + |u|_H) \cdot |a_3(t)|$$

for *a.e.* $t \in [0, T]$ and all $u \in D(\partial\varphi^t)$,

(A.5) There exist functions $l_3 \in \mathcal{M}$, $a_4(t) \in L^1(0, T)$ and a constant $k \in [0, 1)$ such that

$$|B(t, u)|_H^2 \le k|\partial^\circ \varphi^t(u)|_H^2 + l_3(|u|_H)\{|\varphi^t(u)|^2 + (\varphi^t(u) + 1)|a_4(t)|\},$$

for *a.e.* $t \in [0, T]$ and all $u \in D(\partial\varphi^t)$,

(A.5)$_\pi$ There exist functions $l_4 \in \mathcal{M}$, and a constant $k \in [0, 1)$ such that

$$|B(t, u)|_H^2 \le k|\partial^\circ \varphi^t(u)|_H^2 + l_4(|u|_H)(\varphi^t(u) + 1)^2,$$

for *a.e.* $t \in [0, T]$ and all $u \in D(\partial\varphi^t)$,

(A.6) There exist a positive constant α and a function $a_5(t) \in L^1(0, T)$ such that

$$(-g(t) - B(t, u), u)_H + \alpha\varphi^t(u) \le |a_5(t)|(|u|_H^2 + 1)$$

for *a.e.* $t \in [0, T]$ and all $u \in D(\partial\varphi^t)$, $g(t) \in \partial\varphi^t(u)$.

(A.6)$_\pi$ There exist positive constants α and d such that

$$(-g(t) - B(t, u), u)_H + \alpha\varphi^t(u) \le d$$

for *a.e.* $t \in [0, T]$ and all $u \in D(\partial\varphi^t)$, $g(t) \in \partial\varphi^t(u)$.

(A.7) There exists a constant c such that $\varphi^t(0) \le c$ for all $t \in [0, T]$.

Then the following local existence results hold.

THEOREM 3.2. *Let* (A.φ^t), (A.1), (A.2) *and* (A.3)$_\alpha$ *be satisfied. Then for any* $u_0 \in \mathcal{B}_{\alpha,2}(\partial\varphi^0)$ *and* $f \in X_T^\alpha$, *there exists a positive number* $T_0 \in (0, T]$ *which is a monotone decreasing function of* $|u_0|_H$ *and* $|u_0|_{\alpha,2,\partial\varphi^0}$ *such that* (*C.P.*) *has a strong solution* $u(t)$ *on* $[0, T_0]$ *satisfying*

$$t^{1/2-\alpha}du(t)/dt\,, \quad t^{1/2-\alpha}\partial\varphi^t(u(t))\,, \quad t^{1/2-\alpha}B(t, u(t)) \in L^2(0, T_0; H)\,, \tag{3.2}$$

$$t^{-\alpha}|u(t) - u_0|_H\,, \quad t^{1/2-\alpha}|\varphi^t(u(t))|^{1/2} \in L_*^q(0, T_0)\,, \qquad \text{for all} \quad q \in [2, \infty]\,. \tag{3.3}$$

PROOF. See [21].

THEOREM 3.3. *Let* (A.φ^t), (A.1), (A.2) *and* (A.4) *be satisfied. Then for any* $u_0 \in D(\varphi^0)$ *and* $f \in L^2(0, T; H)$, *there exists a positve number* $T_0 \in (0, T]$ *which is a monotone decreasing function of* $|u_0|_H$ *and* $\varphi^0(u_0)$ *such that* (*C.P.*) *has a strong solution satisfying* (3.2) *and* (3.3) *with* $\alpha = 1/2$. *Furthermore* $\varphi^t(u(t))$ *is absolutely continuous on* $[0, T_0]$.

PROOF. See the proof of Theorem II in [21].

As for the existence of global solutions, the following theorem is known.

THEOREM 3.4. *Let* $f \in L^2(0, T; H)$, (A.φ^t), (A.1), (A.2), (A.5) *and* (A.6) *be satisfied. Then every local strong solution of* (*C.P.*) *can be continued as a strong solution of* (*C.P.*) *to* $[0, T]$.

PROOF. See the proof for Theorem IV in [21].

As for the periodic problems, the following result is obtained.

THEOREM 3.5. *Let* $(A.\varphi^t)_\pi$ *be satisfied and* $f(t)\in L^2(0, T; H)$. *Then (A.P.P.) with* $B(t, \cdot)\equiv 0$ *has a unique periodic strong solution u satisfying*

$$\frac{du}{dt}\in L^2(0, T; H),$$

$$\varphi^t(u(t)) \text{ is absolutely continuous on } [0, T].$$

PROOF. See [27].

For the perturbation problem (A.P.P.) with $B(t, \cdot)\neq 0$, following result holds (see [22]).

THEOREM 3.6. *Let* $(A.\varphi^t)_\pi$, (A.1), (A.2), $(A.5)_\pi$ *and* $(A.6)_\pi$ *be satisfied and* $f(t)\in L^2(0, T; H)$. *Then (A.P.P.) has at least one periodic strong solution u satisfying*

$$\frac{du}{dt},\ B(t, u(t))\in L^2(0, T; H),$$

$$\varphi^t(u(t)) \text{ is absolutely continuous on } [0, T].$$

3.2. Reduction to Abstract Equations. In this subsection we are going to show that (I.B.P.) (resp. (P.P.)) can be reduced to the abstract equation (C.P.) (resp. (A.P.P.)) with $H=\mathbf{L}^2_\sigma(B)$ or $H=L^2(B)$ as in [21] and [24]. To this end, we put

$$\varphi_1^t(\mathbf{u})=\varphi_1(\mathbf{u})+I_1^t(\mathbf{u}), \qquad \mathbf{u}\in\mathbf{L}^2_\sigma(B),$$

$$\varphi_1(\mathbf{u})=\begin{cases} \dfrac{\nu}{2}\displaystyle\int_B |\nabla\mathbf{u}|^2dx, & \mathbf{u}\in\mathbf{H}^1_\sigma(B),\\ +\infty, & \mathbf{u}\in\mathbf{L}^2_\sigma(B)\setminus\mathbf{H}^1_\sigma(B),\end{cases}$$

$$K_1(t)=\{\mathbf{u}\in\mathbf{L}^2_\sigma(B);\quad \mathbf{u}=0,\ \text{ a.e. } x\in B\setminus Q(t)\},$$

$$I_1^t(\mathbf{u})=\begin{cases} 0, & \mathbf{u}\in K_1(t),\\ +\infty, & \mathbf{u}\in\mathbf{L}^2_\sigma(B)\setminus K_1(t),\end{cases}$$

$$\varphi_2^t(\theta)=\varphi_2(\theta)+I_2^t(\theta), \qquad \theta\in L^2(B),$$

$$\varphi_2(\theta)=\begin{cases} \dfrac{\kappa}{2}\displaystyle\int_B |\nabla\theta|^2dx, & \theta\in H^1_0(B),\\ +\infty, & \theta\in L^2(B)\setminus H^1_0(B),\end{cases}$$

$$K_2(t)=\{\theta\in L^2(B);\ \theta=0,\ \text{ a.e. } x\in B\setminus Q(t)\},$$

$$I_2^t(\theta)=\begin{cases} 0, & \theta\in K_2(t),\\ +\infty, & \theta\in L^2(B)\setminus K_2(t),\end{cases}$$

Then $\varphi_1^t \in \Phi(\mathbf{L}_\sigma^2(B))$, $\varphi_2^t \in \Phi(\mathrm{L}^2(B))$, and their subdifferentials are characterized as follows:

$$\partial\varphi_1^t(\mathbf{u}) = \{\mathbf{f} \in \mathbf{L}_\sigma^2(B);\ \mathbf{P}_{Q(t)}\mathbf{f}|_{Q(t)} = \nu A(Q(t))\mathbf{u}|_{Q(t)}\}$$

with domain

$$\mathrm{D}(\partial\varphi_1^t) = \{\mathbf{u} \in \mathbf{L}_\sigma^2(B);\ \mathbf{u}|_{Q(t)} \in \mathbf{H}^2(Q(t)) \cap \mathbf{H}_\sigma^1(Q(t)),\ \mathbf{u}|_{B\setminus Q(t)} = 0\}\,,$$

hence $\|\partial^o\varphi_1^t(\mathbf{u})\| = \|\nu A(Q(t))\mathbf{u}|_{Q(t)}\|$.

$$\partial\varphi_2^t(\theta) = \{h \in \mathrm{L}^2(B);\ h|_{Q(t)} = \kappa A_0(Q(t))\theta|_{Q(t)}\}$$

with domain

$$\mathrm{D}(\partial\varphi_2^t) = \{\theta \in \mathrm{L}^2(B);\ \theta|_{Q(t)} \in \mathrm{H}^2(Q(t)) \cap \mathrm{H}_0^1(Q(t)),\ \theta|_{B\setminus Q(t)} = 0\}\,,$$

hence $|\partial^o\varphi_2^t(\theta)| = |\kappa A_0(Q(t))\theta|_{Q(t)}|$.
Furthermore we put

$$A_1^t = \partial\varphi_1^t\,, \quad A_2^t = \partial\varphi_2^t\,,$$

$$\mathbf{B}_1^t(\mathbf{u}) = \mathbf{P}_B\{(\mathbf{u}\cdot\nabla)\mathbf{u} + ([\bar{\mathbf{u}}]^\wedge\cdot\nabla)\mathbf{u} + [(\mathbf{u}|_{Q(t)}\cdot\nabla)\bar{\mathbf{u}}]^\wedge\}\,,$$

$$\mathbf{B}_2^t(\mathbf{u}, \theta) = (\mathbf{u}\cdot\nabla)\theta + ([\bar{\mathbf{u}}]^\wedge\cdot\nabla)\theta\,,$$

$$\mathbf{F}_1(t) = \mathbf{P}_B[-\bar{\mathbf{u}}_t + \nu\Delta\bar{\mathbf{u}} - (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} - \eta\bar{\theta}\mathbf{g} + \mathbf{f}_1]^\wedge\,,$$

$$\mathrm{F}_2(t) = [-\bar{\theta}_t + \kappa\Delta\bar{\theta} - (\bar{\mathbf{u}}\cdot\nabla)\bar{\theta} + f_2]^\wedge\,,$$

$$\mathrm{F}_2(\mathbf{u}, t) = \mathrm{F}_2(t) - [(\mathbf{u}|_{Q(t)}\cdot\nabla)\bar{\theta}]^\wedge - \frac{\nu}{2}D[[\mathbf{u}|_{Q(t)} + \bar{\mathbf{u}}]^\wedge]\,,$$

and consider the following abstract Cauchy and periodic problems in $\mathbf{L}_\sigma^2(B)$ and $\mathrm{L}^2(\mathrm{B})$:

$$\begin{cases} \tilde{\mathbf{u}}_t + A_1^t\tilde{\mathbf{u}} + \mathbf{B}_1^t(\tilde{\mathbf{u}}) + \mathbf{P}_B\eta\tilde{\theta}\hat{\mathbf{g}} \ni \mathbf{F}_1(t)\,, \\ \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0 = [\mathbf{u}_0 - \bar{\mathbf{u}}(\cdot, 0)]^\wedge\,, \end{cases} \tag{3.4}$$

$$\begin{cases} \tilde{\theta}_t + A_2^t\tilde{\theta} + \mathbf{B}_2^t(\tilde{\mathbf{u}}, \tilde{\theta}) \ni \mathrm{F}_2(\tilde{\mathbf{u}}, t)\,, \\ \tilde{\theta}(0) = \tilde{\theta}_0 = [\theta_0 - \bar{\theta}(\cdot, 0)]^\wedge\,, \end{cases} \tag{3.5}$$

$$\begin{cases} \tilde{\mathbf{u}}_t + A_1^t\tilde{\mathbf{u}} + \mathbf{B}_1^t(\tilde{\mathbf{u}}) + \mathbf{P}_B\eta\tilde{\theta}\hat{\mathbf{g}} \ni \mathbf{F}_1(t)\,, \\ \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}(T)\,, \end{cases} \tag{3.6}$$

$$\begin{cases} \tilde{\theta}_t + A_2^t\tilde{\theta} + \mathbf{B}_2^t(\tilde{\mathbf{u}}, \tilde{\theta}) \ni \mathrm{F}_2(\tilde{\mathbf{u}}, t)\,, \\ \tilde{\theta}(0) = \tilde{\theta}(T)\,. \end{cases} \tag{3.7}$$

Then our original problems (I.B.P.) and (P.P.) can be reduced to (3.4)–(3.5) and (3.6)–(3.7) in the following sense, respectively.

LEMMA 3.7. *If* $(\tilde{\mathbf{u}}, \tilde{\theta})$ *is a strong solution of* (3.4)–(3.5) *on* $[0, S]$ *satisfying*

$$\begin{cases} \tilde{\mathbf{u}} \in C([0, S]; \mathbf{L}^2_\sigma(B)) \cap C((0, S]; \mathbf{H}^1_\sigma(B)), \\ t^{1/2-\alpha}\tilde{u}_t,\ t^{1/2-\alpha}\partial^o\varphi^t_1(\tilde{\mathbf{u}}) \in L^2(0, S; \mathbf{L}^2_\sigma(B)), \\ t^{-\alpha}|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_0|,\quad t^{1/2-\alpha}|\varphi^t_1(\tilde{\mathbf{u}})|^{1/2} \in L^q_*(0, S) \qquad \text{for all} \quad q \in [2, \infty], \end{cases} \tag{3.8}$$

$$\begin{cases} \tilde{\theta} \in C([0, S]; L^2(B)) \cap C((0, S]; H^1_0(B)), \\ t^{1/2-\mu}\tilde{\theta}_t,\quad t^{1/2-\mu}\partial^o\varphi^t_2(\tilde{\theta}) \in L^2(0, S; L^2(B)), \\ t^{-\mu}|\tilde{\theta}-\tilde{\theta}_0|,\quad t^{1/2-\mu}|\varphi^t_2(\tilde{\theta})|^{1/2} \in L^q_*(0, S) \qquad \text{for all} \quad q \in [2, \infty], \end{cases} \tag{3.9}$$

then $(\mathbf{u}, \theta) = (\tilde{\mathbf{u}}|_{Q(t)} + \bar{\mathbf{u}}, \tilde{\theta}|_{Q(t)} + \bar{\theta})$ *becomes a solution of* (1.1)–(1.5) *satisfying* (*S*1) *and* (*S*2) *with T replaced by S.*

PROOF. See [11].

LEMMA 3.8. *Suppose that* $(\tilde{\mathbf{u}}, \tilde{\theta})$ *is a strong solution of* (3.6)–(3.7) *satisfying*

$$\begin{cases} \tilde{\mathbf{u}} \in C_\pi([0, T]; \mathbf{L}^2_\sigma(B)), \quad \tilde{\mathbf{u}}_t, \Delta\tilde{\mathbf{u}}, (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} \in L^2(0, T; \mathbf{L}^2(B)), \\ \|\nabla\tilde{\mathbf{u}}\|^2 \text{ is absolutely continuous on } [0, T], \\ \tilde{\mathbf{u}}(t) \in D(\partial\varphi^t_1) \qquad \text{for} \quad \text{a.e. } t \in [0, T], \end{cases} \tag{3.10}$$

$$\begin{cases} \tilde{\theta} \in C_\pi([0, T]; L^2(B)), \quad \tilde{\theta}_t, \Delta\tilde{\theta}, (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\theta} \in L^2(0, T; L^2(B)), \\ |\nabla\tilde{\theta}|^2 \text{ is absolutely continuous on } [0, T], \\ \tilde{\theta}(t) \in D(\partial\varphi^t_2) \qquad \text{for} \quad \text{a.e. } t \in [0, T], \end{cases} \tag{3.11}$$

Then $(\mathbf{u}, \theta) = (\tilde{\mathbf{u}}|_{Q(t)} + \bar{\mathbf{u}}, \tilde{\theta}|_{Q(t)} + \bar{\theta})$ *gives a solution of* (*P.P.*) *satisfying* (*SP*).

PROOF. In view of $(A.a)_\pi$ and $(A.b)_\pi$, we easily see that $(\mathbf{u}, \theta)$ satisfies (1.2)–(1.4) and (1.6). By virtue of the boundedness of $\|\nabla\mathbf{u}(t)\|$, $|\nabla\theta(t)|$ and reflexivity of $\mathbf{H}^1_\sigma$, H^1_0, we can easily derive the weak continuity of $[\mathbf{u}(t)-\bar{\mathbf{u}}(t)]^\wedge$ and $[\theta(t)-\bar{\theta}(t)]^\wedge$ in $\mathbf{H}^1_\sigma(B)$ and $H^1_0(B)$. Hence the continuity in the strong topology is also assured by the continuity of $\|\nabla\mathbf{u}(t)\|$ and $|\nabla\theta(t)|$. Other properties in (SP) are direct consequences of (3.10)–(3.11).

In order to check (1.1), it suffices to use (A.*g*), Helmholtz's decomposition, (3.6) operated by $P_{Q(t)}$ and the fact that $P_{Q(t)}(P_B h)|_{Q(t)} = P_{Q(t)} h|_{Q(t)}$ for all $h \in L^2(B)$. Q.E.D.

3.3. Proof of theorems. In what follows, we denote $\tilde{\mathbf{u}}$, $\tilde{\theta}$, $\tilde{\mathbf{u}}_0$, $\tilde{\theta}_0$, and $\hat{\mathbf{g}}$ by $\mathbf{u}$, θ, $\mathbf{u}_0$, θ_0 and $\mathbf{g}$ for simplicity.

For the later use, we here prepare some basic results:

LEMMA 3.9. *Let* $f(t) \in L^1(0, T)$, *and* $y(t)$ *be a nonnegative absolutely continuous function on* $[0, T]$ *satisfying*

$$\frac{d}{dt}y(t) + \alpha_0 y(t) \le |f(t)|, \qquad 0 < \alpha_0, \quad \text{for a.e. } t \in [0, T].$$

Then we have

$$\sup_{0\le t\le T}|y(t)|\le y(0)+\left(1+\frac{1}{\alpha_0}\right)|f(t)|_{1,T},$$

where

$$|f(t)|_{1,T}=\begin{cases}\sup_{1\le t\le T}\int_{t-1}^{t}|f(s)|ds & \text{for}\quad 1\le T,\\ \int_0^T|f(s)|ds & \text{for}\quad 0<T\le 1.\end{cases}$$

PROOF. See the proof for lemma 4.3 in [21].

LEMMA 3.10. *Let $f(t)\in L^1(0,T)$, and $y(t)$ be a nonnegative absolutely continuous function on $[0,T]$ satisfying $y(0)=y(T)$ and*

$$\frac{d}{dt}y(t)+\alpha_0 y(t)\le|f(t)|,\qquad 0<\alpha_0,\qquad \text{for}\quad a.e.\ t\in[0,T].$$

Then we have

$$\sup_{0\le t\le T}|y(t)|\le\left(2+\frac{3}{\alpha_0}\right)|f(t)|_{1,T^*},$$

where $|f(t)|_{1,T^*}=|f(t)|_{1,T}$ for $1<T$, $|f(t)|_{1,T^*}=\frac{1}{T}|f(t)|_{1,T}$ for $0<T\le 1$.

PROOF. See lemma 3.4 in [12].

LEMMA 3.11. *The following inequalities hold*:

$$\int_{Q(t)}|(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}|dx\le\sqrt{2}\,\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}\|\nabla\mathbf{v}\|\,\|\mathbf{w}\|^{1/2}\|\nabla\mathbf{w}\|^{1/2},\tag{3.12}$$

for all $\mathbf{u},\mathbf{w}\in\mathbf{H}_0^1(Q(t))$ and $\mathbf{v}\in\mathbf{H}^1(Q(t))$.

$$\int_{Q(t)}|(\mathbf{u}\cdot\nabla)\eta\theta|dx\le\sqrt{2}\,\|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}|\nabla\eta|\,|\theta|^{1/2}|\nabla\theta|^{1/2},\tag{3.13}$$

for all $\mathbf{u}\in\mathbf{H}_0^1(Q(t))$, $\theta\in H_0^1(Q(t))$ and $\eta\in H^1(Q(t))$.

LEMMA 3.12. *Let (A.Q) or $(\text{A.Q})_\pi$ be satisfied, then there exist constants c_1 and c_2 independent of t such that*

$$\|(\mathbf{v}\cdot\nabla)\mathbf{w}\|^2\le c_1\|\mathbf{v}\|\,\|\mathbf{v}\|_{\mathbf{H}^1}\|\nabla\mathbf{w}\|\,\|\nabla\mathbf{w}\|_{\mathbf{H}^1},\tag{3.14}$$

for all $\mathbf{v}\in\mathbf{H}^1(Q(t))$ and $\mathbf{w}\in\mathbf{H}^2(Q(t))$,

$$|(\mathbf{v}\cdot\nabla)\eta|^2\le c_2\|\mathbf{v}\|\,\|\mathbf{v}\|_{\mathbf{H}^1}|\nabla\eta|\,|\nabla\eta|_{H^1},\tag{3.15}$$

for all $\mathbf{v}\in\mathbf{H}^1(Q(t))$ and $\eta\in H^2(Q(t))$. Furthermore, if $\mathbf{v}\in\mathbf{H}_0^1(Q(t))$, then the term $\|\mathbf{v}\|_{\mathbf{H}^1}$ in

(3.14) *and* (3.15) *can be replaced by* $\|\nabla \mathbf{v}\|$.

LEMMA 3.13. *For every* $\mathbf{u}, \mathbf{v} \in \mathrm{H}^1_\sigma(\Omega)$ *and* $\theta \in \mathrm{H}^1_0(\Omega)$, *we have*

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{v} dx = 0 \quad and \quad \int_\Omega (\mathbf{u} \cdot \nabla)\theta\, \theta dx = 0 .$$

Those lemmas are shown in [11].

3.3.1. Proof of Theorem I. For each $R>0$ and $S \in (0, T]$ set

$$\mathrm{K}_{R,S} = \{h \in \mathrm{C}([0, S]; \mathrm{L}^2(B)) ; |h|_{\infty,S} \leq R\} .$$

Then, for sufficiently small S, we can show the following facts which assure the existence of local solution $(\mathbf{u}, \theta)$ of (3.4)–(3.5) satisfying (3.8)–(3.9).

Fact I For any $\theta \in \mathrm{K}_{R,S}$, there exists a unique solution $\mathbf{u} = \mathbf{u}_\theta$ of (3.4) with $\tilde{\theta}$ replaced by θ satisfying (3.8).

Fact II There exists a unique solution $\tilde{\theta} = \tilde{\theta}_{\mathbf{u}_\theta}$ of (3.5) with $\tilde{\mathbf{u}}$ replaced by $\mathbf{u}_\theta$ satisfying (3.9).

So we can define the operator $\mathscr{F}$ by

$$\mathscr{F} : \theta \longmapsto \mathbf{u}_\theta \longmapsto \tilde{\theta}_{\mathbf{u}_\theta} .$$

Fact III $\mathscr{F}$ is a contraction from $\mathrm{K}_{R,S}$ into itself.

PROOF OF FACT I. Condition (A.Q) assures that there exists a constant K_1 independent of t so that the following elliptic estimate holds (see [17] and [26]):

$$\|\mathbf{u}\|_{\mathbf{H}^2(Q(t))} \leq \frac{K_1}{\nu} \|\partial^o \varphi^t_1(\mathbf{u})\| \qquad \text{for } a.e. \quad t \in [0, T] \quad \text{and all} \quad \mathbf{u} \in \mathrm{D}(\partial \varphi^t_1) . \tag{3.16}$$

Therefore, it is easy to see that (3.14) gives

$$\|\mathrm{B}^t_1(\mathbf{u})\|^2 \leq 3c_1 \left\{ \varepsilon \left(\frac{K_1}{\nu} \right)^2 \|\partial^o \varphi^t_1(\mathbf{u})\|^2 + \frac{1}{\varepsilon} \left(\frac{2}{\nu} \varphi^t_1(\mathbf{u}) + 1 \right)^2 \cdot (\|\mathbf{u}\|^2 + 1) \right.$$
$$\left. + \|\tilde{\mathbf{u}}\|^4 \|\tilde{\mathbf{u}}\|^4_{\mathbf{H}^1} + \|\tilde{\mathbf{u}}\|^2_{\mathbf{H}^1} \|\tilde{\mathbf{u}}\|^2_{\mathbf{H}^2} \right\} ,$$

for all $\varepsilon \in (0, 1]$ and $\mathbf{u} \in \mathrm{D}(\partial \varphi^t_1)$. Hence, by (A.*a*), conditions (A.4) and $(\mathrm{A.3})_\alpha$ are satisfied with $\varphi^t = \varphi^t_1$, $\mathrm{B}(t, \cdot) = \mathrm{B}^t_1(\cdot)$, $\alpha \in (0, \frac{1}{4}]$.

In view of (A.*a*), (A.*b*), (A.*f*) and (A.*g*), we also have $\mathbf{F}_1(t) - \mathrm{P}_B \eta \theta \mathbf{g} \in \mathrm{L}^2(0, T; \mathbf{L}^2_\sigma(B))$. Furthermore, as in [21], and [24], we can verify $(\mathrm{A}.\varphi^t)$, (A.1) and (A.2) with $m_1 = m_0/\sqrt{\nu}$, $m_2 = 0$, $m_3 = m_0$ (m_0 is a constant depending only on Q), $\varphi^t = \varphi^t_1$ and $\mathrm{B}(t, \cdot) = \mathrm{B}^t_1(\cdot)$. Thus, noting that $\mathbf{u}_0 \in \mathrm{D}(A^\alpha(Q(0)))$ if and only if $\mathbf{u}_0 \in \mathscr{B}_{\alpha,2}(\partial \varphi^0_1)$ for $0 < \alpha \leq \frac{1}{2}$ (see [1] and [8]), we can apply Theorems 3.2 to (3.4). Consequently there exists a number $T_0 \in [0, T_1]$ such that for any $S \in [0, T_0]$, (3.4) has a strong solution $\mathbf{u} = \mathbf{u}_\theta$ on $[0, S]$ satisfying (3.8) with $\alpha\in(\frac{1}{4},\frac{1}{2}]$.

The uniqueness can be proved by the standard argument as in Serrin [25].

Q.E.D.

PROOF OF FACT II. Let $\mathbf{u}$ be a solution of (3.4) with $\theta\in K_{R,S}$ and we consider (3.5). In view of (A.a), (A.b), (A,f) and (3.15), we first note that $F_2(t)\in L^2(0, T; L^2(B))$. By using (3.15) again and the elliptic estimate:

$$|\theta|_{H^2(Q(t))}\leq\frac{K_2}{\kappa}|\partial^\circ\varphi_2^t(\theta)|\qquad\text{for } a.e.\ t\in[0, T]\quad\text{and all }\ \theta\in D(\partial\varphi_2^t),$$

we also obtain, for all $\varepsilon\in(0, 1)$,

$$|B_2^t(\mathbf{u}, \theta)|^2\leq\varepsilon|\partial^\circ\varphi_2^t(\theta)|^2+\frac{4c_2^2K_2^2}{\varepsilon\kappa^3}\varphi_2^t(\theta)\cdot a(t)\qquad\text{for all}\quad\theta\in D(\partial\varphi_2^t),\tag{3.17}$$

with

$$a(t)=\|\mathbf{u}(t)\|^2\|\mathbf{u}(t)\|_{\mathbf{H}^1}^2+\|\bar{\mathbf{u}}(t)\|^2\|\bar{\mathbf{u}}(t)\|_{\mathbf{H}^1}^2.$$

By (3.8), it follows that $\|\mathbf{u}(t)\|\in L^\infty(0, S)$ and $\|\mathbf{u}(t)\|_{\mathbf{H}^1}\in L^2(0, S)$. Using this and (A.$a$) we see that $a(t)\in L^1(0, S)$. Then (A.5) is satisfied.

Furthermore, by the integration by parts and Lemma 3.13, we get $(g(t), \theta(t))_H=2\varphi_2^t(\theta(t))$ for all $g(t)\in\partial\varphi_2^t(\theta(t))$ and $(B_2^t(\mathbf{u}(t), \theta(t)), \theta(t))=0$, *i.e.*,

$$(g+B_2^t(\mathbf{u}, \theta), \theta)=2\varphi_2^t(\theta)\qquad\text{for all}\quad\theta\in D(\partial\varphi_2^t)\quad\text{and}\quad g\in\partial\varphi_2^t(\theta).$$

This shows that (A.6) holds.

Now for any $\varepsilon\in(0, 1)$, we consider the following equation:

$$\theta_t^\varepsilon+A_2^t\theta^\varepsilon+B_2^t(\mathbf{u}, \theta^\varepsilon)\in\chi_\varepsilon(t)F_2(\mathbf{u}, t)\quad 0<t\leq S,\quad\theta^\varepsilon(0)=\theta_0,\tag{3.18$_\varepsilon$}$$

where $\chi_\varepsilon(t)=0$ for $0\leq t\leq\varepsilon$ and $\chi_\varepsilon(t)=1$ for $t>\varepsilon$. By (3.15) and Hölder's inequality, we get

$$|(\mathbf{u}\cdot\nabla)\bar{\theta}|^2\leq\frac{1}{2}c_2\{\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2+|\nabla\bar{\theta}|^2|\bar{\theta}|_{H^2}^2\}.$$

Hence it is easy to see that $|(\mathbf{u}\cdot\nabla)\bar{\theta}|\in L^2(0, S)$. Using the inequality $|h|_{L^4(Q(t))}^4\leq c|h|_{L^2(Q(t))}^2|h|_{H^1(Q(t))}^2$ for any $h\in H^1(Q(t))$ (see [17]), we get

$$\begin{aligned}t^{2(1-2\alpha)}\frac{\nu^2}{4}|D[\mathbf{u}+\bar{\mathbf{u}}]|^2&\leq c(\nu)t^{2(1-2\alpha)}\{\|\nabla\mathbf{u}\|_{L^4(Q(t))}^4+\|\nabla\bar{\mathbf{u}}\|_{L^4(Q(t))}^4\}\\&\leq c(\nu)t^{2(1-2\alpha)}\{\|\nabla\mathbf{u}\|^2(\|\Delta\mathbf{u}\|^2+\|\nabla\mathbf{u}\|^2)+\|\nabla\bar{\mathbf{u}}\|^2\|\nabla\bar{\mathbf{u}}\|_{\mathbf{H}^1}^2\}\\&\leq c(\nu)\{t^{1-2\alpha}\|\nabla\mathbf{u}\|^2t^{1-2\alpha}(\|\Delta\mathbf{u}\|^2+\|\nabla\mathbf{u}\|^2)\\&\quad+t^{2(1-2\alpha)}\|\nabla\bar{\mathbf{u}}\|^2\|\nabla\bar{\mathbf{u}}\|_{\mathbf{H}^1}^2\},\end{aligned}$$

and

$$|D[\mathbf{u}+\bar{\mathbf{u}}]|\leq c(\nu)\{t^{-1+2\alpha}\|\nabla\mathbf{u}\|^2+t^{1-2\alpha}\|\Delta\mathbf{u}\|^2+\|\nabla\bar{\mathbf{u}}\|\,\|\nabla\bar{\mathbf{u}}\|_{\mathbf{H}^1}\}\,.$$

Since $t^{1-2\alpha}\|\nabla\mathbf{u}\|^2\in L^\infty(0, S)$, $t^{1-2\alpha}\|\Delta\mathbf{u}\|^2\in L^1(0, S)$, and $t^{2(1-2\alpha)}\|\nabla\mathbf{u}\|^4_{\mathbf{L}^4}\in L^1(0, S)$, by (3.8) and $t^{2(1-2\alpha)}\|\nabla\bar{\mathbf{u}}\|^2\|\nabla\bar{\mathbf{u}}\|^2_{\mathbf{H}^1}\in L^1(0, S)$, by (A.$a$), we find that $t^{2(1-2\alpha)}|D[\mathbf{u}+\bar{\mathbf{u}}]|^2\in L^1(0, S)$, whence $D[\mathbf{u}+\bar{\mathbf{u}}]\in L^2(\varepsilon, S; L^2(Q(t)))$ for all $\varepsilon>0$. Hence it follows that $t^{1-2\alpha}|F_2(\mathbf{u}, t)|\in L^2(0, S)$, therefore $\chi_\varepsilon(t)F_2(\mathbf{u}, t)\in L^2(Q)$. Thus theorem 3.3 and Theorem 3.4 assure that for any $\theta_0\in D(\varphi_2^0)$ and $\varepsilon\in(0, 1)$, there exists a strong solution θ^ε of $(3.18)_\varepsilon$ on $(0, S]$.

Note that $|D[\mathbf{u}+\bar{\mathbf{u}}]|\leq 2(\|\nabla\mathbf{u}\|^2+\|\nabla\bar{\mathbf{u}}\|^2)\in L^1(0, S)$, then $|F_2(\mathbf{u}, t)|\in L^1(0, S)$. Multiplying $(3.18)_\varepsilon$ by θ^ε and using (3.13) and Lemma 3.13, we have

$$\max_{0\leq t\leq S}|\theta^\varepsilon(t)|^2+2\kappa\int_0^S|\nabla\theta^\varepsilon(t)|^2dt\leq 2\left\{|\theta_0|+\int_0^S|F_2(\mathbf{u}, s)|ds\right\}^2\,. \tag{3.19}$$

Moreover, multiplication of $(3.18)_\varepsilon$ by $g^\varepsilon=\chi_\varepsilon(t)F_2(\mathbf{u}, t)-\theta_t^\varepsilon-B_2^t(\mathbf{u}, \theta^\varepsilon)\in A_2^t\theta^\varepsilon$ and using (3.1), we have

$$\begin{aligned}\frac{d}{dt}\varphi_2^t(\theta^\varepsilon(t))+|g^\varepsilon(t)|^2&\leq\sqrt{m_0}|g^\varepsilon(t)|\cdot\varphi_2^t(\theta^\varepsilon(t))^{1/2}+m_0\varphi_2^t(\theta^\varepsilon(t))\\&\quad+|g^\varepsilon(t)|(|B_2^t(\mathbf{u}, \theta^\varepsilon)|+|F_2(\mathbf{u}, t)|)\,,\end{aligned} \tag{3.20}$$

where we use the fact that (A.φ^t) holds with $\varphi^t=\varphi_2^t$, $m_1=\sqrt{m_0}$, $m_2=0$, $m_3=m_0$(m_0 is a constant depending only on Q).

Then, by (3.17) and (3.20), we get

$$\begin{aligned}&\frac{d}{dt}\{t\varphi_2^t(\theta^\varepsilon(t))\}+\frac{t}{4}|g^\varepsilon(t)|^2\\&\leq\varphi_2^t(\theta^\varepsilon(t))+t\varphi_2^t(\theta^\varepsilon(t))\left(2m_0+\frac{16K_1^2c_2^2}{\kappa^3}a(t)\right)+t|F_2(\mathbf{u}, t)|^2\,.\end{aligned} \tag{3.21}$$

Now application of Gronwall's inequality for (3.21) assures that there exists a constant c_0 which depends on $|\theta_0|$, $\int_0^S|F_2(\mathbf{u}, t)|dt$, $\int_0^S t|F_2(\mathbf{u}, t)|^2dt$, $\int_0^S a(t)dt$, m_0 and κ such that

$$\sup_{0\leq t\leq S}t\varphi_0^t(\theta^\varepsilon(t))+\int_0^S t|g^\varepsilon(t)|^2dt\leq c_0\,. \tag{3.22}$$

On the other hand, since $\omega=\theta^\varepsilon-\theta^\rho$ $(0<\varepsilon<\rho\leq 1)$ satisfies

$$\omega_t+A_2^t\omega+B_2^t(\mathbf{u}, \omega)\in(\chi_\varepsilon(t)-\chi_\rho(t))(F_2(\mathbf{u}, t))\,,\qquad \omega(0)=0\,.$$

The same verification as for (3.19) yields

$$\sup_{0\leq t\leq S}|\omega(t)|\leq\int_\varepsilon^\rho|F_2(\mathbf{u}, t)|dt$$

Hence, $\{\theta^\varepsilon\}$ forms a Cauchy sequence in $C([0, S]; L^2(B))$. Therefore, by (3.17) and (3.22) with the standard argument, we can extract a subsequence ε_n which tend to 0 as $n \to \infty$ such that

$$\theta^{\varepsilon_n} \longrightarrow \theta \qquad \text{strongly in} \quad C([0, S]; L^2(B)),$$

$$\sqrt{t}\,\frac{d\theta^{\varepsilon_n}}{dt} \longrightarrow \sqrt{t}\,\frac{d\theta}{dt} \qquad \text{weakly in} \quad L^2(0, S; L^2(B)),$$

$$\sqrt{t}\,g^{\varepsilon_n} \longrightarrow \sqrt{t}\,g \qquad \text{weakly in} \quad L^2(0, S; L^2(B)),$$

with $g(t) \in \partial\varphi_2^t(\theta(t))$ a.e. $t \in [0, S]$,

$$\sqrt{t}\,B_2^t(\mathbf{u}, \theta^{\varepsilon_n}) \longrightarrow \sqrt{t}\,B_2^t(\mathbf{u}, \theta) \qquad \text{weakly in} \quad L^2(0, S; L^2(B)).$$

This implies that θ is a solution of $(3.18)_{\varepsilon=0}$, *i.e.* $(3.18)_\varepsilon$ with χ_ε replaced by 1.

For any $\theta_0 \in \overline{D(\varphi_2^0)}$, take $\theta_0^n \in D(\varphi_2^0)$ such that $\theta_0^n \to \theta_0$ in $L^2(B)$ as $n \to \infty$ and let θ^n be the solution of $(3.18)_0$ with $\theta^n(0) = \theta_0^n$. Since $\omega = \theta^n - \theta^m$ satisfies

$$w_t + A_2^t w + B_2^t(\mathbf{u}, w) \in 0\,, \qquad w(0) = \theta_0^n - \theta_0^m\,.$$

By the same verification for (3.19), we have

$$\sup_{0 \le t \le S} |w(t)|^2 + \kappa \int_0^S |\nabla w(t)|^2 dt \le |\theta_0^n - \theta_0^m|^2\,. \tag{3.23}$$

Noting that θ^n satisfies the same a priori estimates as (3.19) and (3.22) with $F_2(\mathbf{u}, t) = 0$, we can again extract a subsequence θ^{n_k} as above. Thus its limit θ gives a solution for $(3.18)_0$ satisfying (3.9). The uniqueness follows from (3.23) at once.

Q.E.D.

PROOF OF FACT III Take a sufficiently large R satisfying

$$|\theta_0| + \int_0^S |F_2(\mathbf{u}, s)|\,ds < \frac{\sqrt{R}}{\sqrt{2}}\,.$$

Then by (3.19), the operator $\mathscr{F}$ maps $K_{R,S}$ into itself. For $\theta_i \in K_{R,S}$ $(i = 1, 2)$, let $\mathbf{u}_i$ be the solutions of (3.4) with $\tilde{\theta} = \theta_i$ and ψ_i be the solutions of (3.5) with $\tilde{\mathbf{u}} = \mathbf{u}_i$. Then $\Theta = \theta_1 - \theta_2$, $\mathbf{U} = \mathbf{u}_1 - \mathbf{u}_2$ and $\Psi = \psi_1 - \psi_2$ satisfy

$$\begin{cases} \mathbf{U}_t + A_1^t\mathbf{U} + P_B\{(\mathbf{u}_1 \cdot \nabla)\mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{u}_2 + (\bar{\mathbf{u}} \cdot \nabla)\mathbf{U} \\ \qquad\qquad + (\mathbf{U} \cdot \nabla)\bar{\mathbf{u}}\} + P_B\eta\mathbf{g}\Theta \ni 0\,, \\ \mathbf{U}(0) = 0\,. \end{cases} \tag{3.24}$$

$$\begin{cases} \Psi_t + A_2^t\Psi + (\mathbf{U} \cdot \nabla)\psi_1 + (\mathbf{u}_2 \cdot \nabla)\Psi + (\bar{\mathbf{u}} \cdot \nabla)\Psi \\ + (\mathbf{U} \cdot \nabla)\bar{\theta} + \frac{\nu}{2} D[\mathbf{u}_1 + \bar{\mathbf{u}}] - \frac{\nu}{2} D[\mathbf{u}_2 + \bar{\mathbf{u}}] \ni 0 \\ \Psi(0) = 0\,. \end{cases} \tag{3.25}$$

where we abbreviate the symbols of the restriction to Q and the extension to B.

Multiplying (3.24) by $\mathbf{U}$ and using Lemma 3.13 and (3.12), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{U}(t)\|^2+\frac{\nu}{2}\|\nabla\mathbf{U}(t)\|^2\leq\|\mathbf{U}(t)\|^2 b(t)+\frac{1}{4}\|\eta\mathbf{g}\|^2_{\mathbf{L}^\infty(Q)}|\Theta(t)|^2\,,\qquad t\in(0,S)\,,$$

where $b(t)=\frac{2}{\nu}(\|\nabla\mathbf{u}_2(t)\|^2+\|\nabla\bar{\mathbf{u}}(t)\|^2)+1$.

Therefore, using Gronwall's inequality, we get

$$\|\mathbf{U}\|^2_{\infty,S}+\nu\int_0^S\|\nabla\mathbf{U}(t)\|^2dt\leq \mathrm{D}_1(S)\cdot S\cdot|\Theta|^2_{\infty,S}\,, \tag{3.26}$$

where

$$\mathrm{D}_1(S)=\|\eta\mathbf{g}\|^2_{\mathbf{L}^\infty(Q)}\left(1+\int_0^S b(t)dt\exp\int_0^S b(t)dt\right).$$

Here we note that (A.a) and (3.8) assure that $b(t)\in\mathrm{L}^1(0,S)$ and $|b|_{\mathrm{L}^1(0,S)}$ is dominated by a monotone non decreasing function of R and $\|\mathbf{u}_0\|$. Furthermore, we multiply (3.24) by $\mathbf{G}_1=-\mathbf{U}_t-\mathrm{P}_B\{(\mathbf{u}_1\cdot\nabla)\mathbf{U}+(\mathbf{U}\cdot\nabla)\mathbf{u}_2+(\bar{\mathbf{u}}\cdot\nabla)\mathbf{U}+(\mathbf{U}\cdot\nabla)\bar{\mathbf{u}}\}-\mathrm{P}_B\eta\mathbf{g}\Theta\in A_1^t\mathbf{U}$ and use (3.1) with $m_1=\sqrt{m_0}$, $m_2=0$, and $m_3=m_0$ (m_0 is a constant depending only on Q) see [11]. Putting $d_1(t)=t^{-2\alpha}\|\nabla\mathbf{u}_2(t)\|^2+\|\nabla\bar{\mathbf{u}}(t)\|^2$, and

$$d_2(t)=3m_0+8\frac{c_1^2K_1^2}{\nu^4}(\|\mathbf{u}_1(t)\|^2\|\nabla\mathbf{u}_1(t)\|^2+\|\bar{\mathbf{u}}(t)\|^2\|\bar{\mathbf{u}}(t)\|^2_{\mathbf{H}^1})$$
$$+\frac{c_1^2}{2\nu}\left(\left(\frac{K_1}{\nu}\right)^2t^{2\alpha}\|\Delta\mathbf{u}_2(t)\|^2+\|\bar{\mathbf{u}}(t)\|^2_{\mathbf{H}^2}\right).$$

We get

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{U}(t)\|^2+\frac{1}{8}\|\mathbf{G}_1(t)\|^2\leq d_2(t)\frac{\nu}{2}\|\nabla\mathbf{U}\|^2+d_1(t)\|\mathbf{U}\|^2+2\|\eta\mathbf{g}\|^2_{\mathbf{L}^\infty(Q)}|\Theta(t)|^2\,.$$

Note that by (3.8) with $\alpha\in[1/4,1/2)$, $t^{1-2\alpha}\|\Delta\mathbf{u}\|^2\in\mathrm{L}^1(0,S)$ and $t^{1-2\alpha}\|\nabla\mathbf{u}\|^2/t\in\mathrm{L}^1(0,S)$ hold. This assure that $d_1(t)\in\mathrm{L}^1(0,S)$ and $d_2(t)\in\mathrm{L}^1(0,S)$. Thus Gronwall's inequality together with (3.26) yields

$$\frac{\nu}{2}\|\nabla\mathbf{U}\|^2_{\infty,S}\leq\{D_1(S)\int_0^S d_1(t)dt+2\|\eta\mathbf{g}\|^2_{\mathbf{L}^\infty(Q)}\}S|\Theta|^2_{\infty,S}\cdot\exp\int_0^S d_2(t)dt$$

Therefore we have

$$\|\nabla\mathbf{U}\|^2_{\infty,S}\leq c_0'|\Theta|^2_{\infty,S}S \tag{3.27}$$

where c_0' is a constant depending on R, $\|A^\alpha(Q(0))\mathbf{u}_0\|$ and $\bar{\mathbf{u}}$.

On the other hand, multiplying (3.25) by Ψ and using Lemma 3.13, (3.15), Young's inequality and the following inequalities:

$$\frac{\nu}{2}|D[\mathbf{u}_1+\bar{\mathbf{u}}]-D[\mathbf{u}_2+\bar{\mathbf{u}}]|\leq 4\nu\{|\nabla\mathbf{u}_1|+|\nabla\mathbf{u}_2|+|\nabla\bar{\mathbf{u}}|\}|\nabla\mathbf{U}|,$$

$$\int_B|\nabla\mathbf{u}_1||\nabla\mathbf{U}||\Psi|dx\leq\frac{\kappa}{32}|\nabla\Psi(t)|^2+\frac{1}{32}|\Psi(t)|^2+\frac{8K_1}{\sqrt{\kappa}}\|\nabla\mathbf{u}_1(t)\|\,\|\Delta\mathbf{u}_1(t)\|\,\|\nabla\mathbf{U}(t)\|^2,$$

we deduce

$$\frac{1}{2}\frac{d}{dt}|\Psi(t)|^2+\frac{\kappa}{4}|\nabla\Psi(t)|^2\leq\frac{3}{4}|\Psi(t)|^2+\frac{2}{\sqrt{\kappa}}(|\nabla\psi_1(t)|^2+|\nabla\bar{\theta}(t)|^2)\|\mathbf{U}(t)\|\,\|\nabla\mathbf{U}(t)\|$$

$$+\frac{8}{\sqrt{\kappa}}\{K_1(\|\nabla\mathbf{u}_1\|\,\|\Delta\mathbf{u}_1\|+\|\nabla\mathbf{u}_2\|\,\|\Delta\mathbf{u}_2\|)+\|\nabla\bar{\mathbf{u}}\|\,\|\bar{\mathbf{u}}\|_{\mathbf{H}^2}\}\|\nabla\mathbf{U}\|^2.$$

Now Gronwall's inequality assures that the following inequality holds:

$$|\Psi|^2_{\infty,S}\leq\frac{4}{\sqrt{\kappa}}\Bigg[\left\{\int_0^S(|\nabla\psi_1(\tau)|^2+|\nabla\bar{\theta}(\tau)|^2)d\tau\right\}\|\mathbf{U}\|_{\infty,S}\|\nabla\mathbf{U}\|_{\infty,S}$$

$$+4\left\{\int_0^S\{K_1(\|\nabla\mathbf{u}_1(\tau)\|\,\|\Delta\mathbf{u}_1(\tau)\|+\|\nabla\mathbf{u}_2(\tau)\|\,\|\Delta\mathbf{u}_2(\tau)\|)\right.$$

$$\left.+\|\nabla\bar{\mathbf{u}}(\tau)\|\,\|\bar{\mathbf{u}}\|_{\mathbf{H}^2}\}d\tau\right\}\|\nabla\mathbf{U}\|^2_{\infty,S}\Bigg]e^{(3/2)S}.$$

Hence, by (3.26) and (3.27), we find that there exists a constant c_0'' which depends on c_0', $\bar{\mathbf{u}}$ and $\bar{\theta}$ such that

$$|\Psi|^2_{\infty,S}\leq c_0''|\Theta|^2_{\infty,S}\cdot S.$$

Thus it is clear that $\mathscr{F}$ becomes a contraction mapping for a sufficiently small S.

Q.E.D.

PROOF OF THEOREM II. For $\bar{\mathbf{u}}$, $\bar{\theta}$, we assume

$$\|\nabla\bar{\mathbf{u}}\|_{\infty,T}<\frac{\nu\sqrt{\lambda}}{2\sqrt{2}},\qquad|\nabla\bar{\theta}|_{\infty,T}<\mu^{1/4}.$$

Let $(\mathbf{u},\theta)$ be the local solution of (3.4)–(3.5) on $[0,S]$ constructed in Theorem I. For the later use, we here introduce several constants independent of S:

$$\alpha_1=\sqrt{\lambda}(\nu\sqrt{\lambda}-\sqrt{2}\|\nabla\bar{\mathbf{u}}\|_{\infty,T}),$$

$$A_1=\frac{2\sqrt{\lambda}}{\nu\sqrt{\lambda}-2\sqrt{2}\|\nabla\bar{\mathbf{u}}\|_{\infty,T}}\left\{\frac{9}{2}+\frac{2}{\lambda\nu}\left(\frac{\alpha_1}{1+\alpha_1}\right)^2\right\},$$

$$A_2=\left(1+\frac{2}{\lambda\nu}\right)\frac{2^2 3^5 c_1^2 K_1^2}{\nu^3}A_1 .$$

Here we introduce another parameter ε satisfying $A_2\varepsilon^4<1$, and we put

$$A_3=\frac{1}{1-A_2\varepsilon^4}\left[1+\left(1+\frac{2}{\lambda\nu}\right)\left\{3m_0+\frac{2^2 3^3 c_1^2 K_1^2}{\nu^3}\|\bar{\mathbf{u}}\|_{\infty,T}^2\|\bar{\mathbf{u}}\|_{\mathbf{H}^1,\infty,T}^2\right.\right.$$

$$\left.\left.+\frac{4c_1}{\sqrt{\lambda}\,\nu}\|\nabla\bar{\mathbf{u}}\|_{\infty,T}\|\nabla\bar{\mathbf{u}}\|_{\mathbf{H}^1\infty,T}\right\}A_1+\frac{8}{\nu}\left(\frac{\alpha_1}{1+\alpha_1}\right)^2\right],$$

$$A_4=\frac{2}{\nu}\left\{A_3+\left(1+\frac{2}{\lambda\nu}\right)^{-1}(A_3-1)\right\},$$

$$B_1=\left(1+\frac{2}{\kappa\mu(1-\sqrt{\mu}\,|\nabla\bar{\theta}|_{\infty,T}^2)}\right)\frac{4}{\kappa\mu},$$

$$B_2=\sqrt{|\theta_0|^2+B_1\left\{3\sqrt{A_1}\,\varepsilon^2+\nu^2K_1^2A_3A_4\varepsilon^4+\nu^2\|\nabla\bar{\mathbf{u}}\|_{\infty,T}^2\|\bar{\mathbf{u}}\|_{\mathbf{H}^2,M,T}^2+\frac{1}{2}|\mathrm{F}_2|_{M,T}^2\right\}},$$

where c_1 and m_0 are constants depending only on Q which appear in (3.14) and the definition of $d_2(t)$.

We here claim that if we assure

$$\|\mathbf{u}_0\|+\|\nabla\mathbf{u}_0\|\le\varepsilon, \tag{3.28}$$

$$\|\eta\mathbf{g}\|_{L^\infty(Q)}<\frac{\alpha_1}{1+\alpha_1}\frac{\varepsilon}{B_2}, \tag{3.29}$$

$$\|\mathbf{F}_1\|_{M,T}<\frac{\alpha_1}{1+\alpha_1}\varepsilon, \tag{3.30}$$

then the following a priori estimates hold:

$$\|\mathbf{u}(t)\|+\|\nabla\mathbf{u}(t)\|<(3+\sqrt{A_3})\varepsilon \qquad \text{for all} \quad t\in[0,S],$$

$$|\theta(t)|<B_2 \qquad \text{for all} \quad t\in[0,S].$$

Suppose that these estimates do not hold. Then there exists a positive number T_1 such that

$$T_1=\min\{t:\ \|\mathbf{u}(t)\|+\|\nabla\mathbf{u}(t)\|=(3+\sqrt{A_3})\varepsilon \text{ or } |\theta(t)|=B_2\}.$$

Multiplying (3.4) by $\mathbf{u}$ and using (3.12) and Lemma 3.13, we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2+\nu\|\nabla\mathbf{u}\|^2\le\int_B|(\mathbf{u}\cdot\nabla)\bar{\mathbf{u}}\cdot\mathbf{u}|dx+\|\eta\mathbf{g}\theta\|\,\|\mathbf{u}\|+\|\mathbf{F}_1(t)\|\,\|\mathbf{u}\|. \tag{3.31}$$

Then much the same verification for (3.26) assures

$$\frac{d}{dt}\|\mathbf{u}\|+\alpha_1\|\mathbf{u}\|\le\|\eta\mathbf{g}\theta\|+\|\mathbf{F}_1(t)\|\,.$$

Hence, Lemma 3.9 with (3.28), (3.29) and (3.30) yields

$$\|\mathbf{u}\|_{\infty,T_1}\le\|\mathbf{u}_0\|+\left(1+\frac{1}{\alpha_1}\right)(\|\eta\mathbf{g}\|_{\mathbf{L}^\infty(Q)}|\theta|_{M,T_1}+\|\mathbf{F}_1\|_{M,T})<3\varepsilon\,. \tag{3.32}$$

Since $\int_B |(\mathbf{u}\cdot\nabla)\bar{\mathbf{u}}\cdot\mathbf{u}|dx\le\frac{\nu}{4}\|\nabla\mathbf{u}\|^2+\frac{2}{\nu}\|\nabla\bar{\mathbf{u}}\|\,\|\mathbf{u}\|^2$, by (3.12), we can also derive from (3.31)

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2+\left(\frac{\nu}{2}-\frac{\sqrt{2}\,\|\nabla\bar{\mathbf{u}}\|_{\infty,T}}{\sqrt{\lambda}}\right)\|\nabla\mathbf{u}\|^2\le\frac{1}{\lambda\nu}\{\|\eta\mathbf{g}\theta\|^2+\|\mathbf{F}_1(t)\|^2\}\,. \tag{3.33}$$

Integrate (3.33) over $[t-1, t]$, then we have

$$\begin{aligned}\|\nabla\mathbf{u}\|^2_{M,T_1}&\le\frac{2\sqrt{\lambda}}{\nu\sqrt{\lambda}-2\sqrt{2}\,\|\bar{\mathbf{u}}\|_{\infty,T}}\left\{\frac{1}{2}\|\mathbf{u}\|^2_{\infty,T_1}+\frac{1}{\lambda\nu}(\|\eta\mathbf{g}\|^2_{\mathbf{L}^\infty(Q)}|\theta|^2_{M,T_1}+\|\mathbf{F}_1\|^2_{M,T})\right\}\\&\le A_1\varepsilon^2\,.\end{aligned} \tag{3.34}$$

Multiplying (3.4) by $\mathbf{h}=-\mathbf{u}_t-\mathbf{B}_1^t(\mathbf{u})-\mathbf{P}_B\eta\mathbf{g}\theta+\mathbf{F}_1(t)\in A_1^t\mathbf{u}$ and using Lemma 3.1, we get

$$\begin{aligned}\frac{d}{dt}\varphi_1^t(\mathbf{u}(t))+\|\mathbf{h}(t)\|^2&\le\sqrt{m_0}\,\|\mathbf{h}(t)\|\cdot\varphi_1^t(\mathbf{u}(t))^{1/2}+m_0\varphi_1^t(\mathbf{u}(t))+\int_{Q(t)}|(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{h}(t)|dx\\&\quad+\int_{Q(t)}|(\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}\cdot\mathbf{h}(t)|dx+\int_{Q(t)}|(\mathbf{u}\cdot\nabla)\bar{\mathbf{u}}\cdot\mathbf{h}(t)|dx\\&\quad+\|\eta\mathbf{g}\theta\|\,\|\mathbf{h}(t)\|+\|\mathbf{F}_1(t)\|\,\|\mathbf{h}(t)\|\,.\end{aligned}$$

Hence (3.14), (3.16) and the characterization of A_1^φ give

$$\int_{Q(t)}|(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{h}|\,dx\le\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|\,\|\mathbf{h}\|\le\left\{\frac{1}{8}\|\mathbf{h}\|^2+\frac{2\cdot3^3c_1^2K_1^2}{\nu^2}\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^4\right\}.$$

Similarly, we further get

$$\int_{Q(t)}|(\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}\cdot\mathbf{h}|dx\le\|(\bar{\mathbf{u}}\cdot\nabla)\mathbf{u}\|\,\|\mathbf{h}\|\le\left\{\frac{1}{8}\|\mathbf{h}\|^2+\frac{2\cdot3^3c_1^2K_1^2}{\nu^2}\|\bar{\mathbf{u}}\|^2\|\bar{\mathbf{u}}\|^2_{\mathbf{H}^1}\|\nabla\mathbf{u}\|^2\right\},$$

$$\int_{Q(t)}|(\mathbf{u}\cdot\nabla)\bar{\mathbf{u}}\cdot\mathbf{h}|\,dx\le\|(\mathbf{u}\cdot\nabla)\bar{\mathbf{u}}\|\,\|\mathbf{h}\|\le\left\{\frac{1}{8}\|\mathbf{h}\|^2+\frac{2c_1}{\sqrt{\lambda}}\|\nabla\bar{\mathbf{u}}\|\,\|\nabla\bar{\mathbf{u}}\|_{\mathbf{H}^1}\|\nabla\mathbf{u}\|^2\right\}.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|^2+\frac{\nu}{4}\|\Delta \mathbf{u}\|^2\le\frac{2\cdot 3^3c_1^2K_1^2}{\nu^3}\|\mathbf{u}\|^2\|\nabla \mathbf{u}\|^4+\left\{\frac{3m_0}{2}+\frac{2\cdot 3^3c_1^2K_1^2}{\nu^3}\|\bar{\mathbf{u}}\|^2\|\bar{\mathbf{u}}\|^2_{\mathbf{H}^1}\right.$$

$$\left.+\frac{2c_1}{\sqrt{\lambda}\,\nu}\|\nabla\bar{\mathbf{u}}\|\,\|\nabla\bar{\mathbf{u}}\|_{\mathbf{H}^1}\right\}\|\nabla \mathbf{u}\|^2+\frac{2}{\nu}\{\|\eta \mathbf{g}\theta\|^2+\|\mathbf{F}_1(t)\|^2\}\,. \tag{3.35}$$

The fact that $\|\nabla \mathbf{u}\|\in L^\infty(0,T_1)$ and Lemma 3.9 assure

$$\|\nabla \mathbf{u}\|^2_{\infty,T_1}\le\|\nabla \mathbf{u}_0\|^2+\left(1+\frac{2}{\lambda\nu}\right)\left[\frac{2^23^3c_1^2K_1^2}{\nu^3}\|\mathbf{u}\|^2_{\infty,T_1}\|\nabla \mathbf{u}\|^2_{\infty,T_1}\|\nabla \mathbf{u}\|^2_{M,T_1}+2\left\{\frac{3m_0}{2}\right.\right.$$

$$\left.+\frac{2\cdot 3^3c_1^2K_1^2}{\nu^3}\|\bar{\mathbf{u}}\|^2_{\infty,T}\|\bar{\mathbf{u}}\|^2_{\mathbf{H}^1,\infty,T}+\frac{2c_1}{\sqrt{\lambda}\,\nu}\|\nabla\bar{\mathbf{u}}\|_{\infty,T}\|\nabla\bar{\mathbf{u}}\|_{\mathbf{H}^1,\infty,T}\right\}\|\nabla \mathbf{u}\|^2_{M,T_1}$$

$$\left.+\frac{4}{\nu}\{\|\eta \mathbf{g}\|^2_{L^\infty(Q)}|\theta|^2_{M,T_1}+\|\mathbf{F}_1(t)\|^2_{M,T}\}\right]$$

$$<\varepsilon^2+\left(1+\frac{2}{\lambda\nu}\right)\left[\frac{2^23^5c_1^2K_1^2}{\nu^3}A_1\varepsilon^4\|\nabla \mathbf{u}\|^2_{\infty,T}+2\left\{\frac{3m_0}{2}\right.\right.$$

$$\left.+\frac{2^2\cdot 3^3c_1^2K_1^2}{\nu^3}\|\bar{\mathbf{u}}\|^2_{\infty,T}\|\bar{\mathbf{u}}\|^2_{\mathbf{H}^1,\infty,T}+\frac{2c_1}{\sqrt{\lambda}\,\nu}\|\nabla\bar{\mathbf{u}}\|_{\infty,T}\|\nabla\bar{\mathbf{u}}\|_{\mathbf{H}^1,\infty,T}\right\}A_1\varepsilon^2$$

$$\left.+\frac{8}{\nu}\left(\frac{\alpha_1}{1+\alpha_1}\right)^2\varepsilon^2\right].$$

Then we have

$$\|\nabla \mathbf{u}\|^2_{\infty,T_1}<A_3\varepsilon^2\,. \tag{3.36}$$

Integrate (3.35) over $[t-1,t]$ and by the same argument as the verification for (3.34), we can obtain

$$\|\Delta \mathbf{u}\|^2_{M,T}<A_4\varepsilon^2\,.$$

Multiplying (3.5) by θ and use the argument similar to that for (3.33), we can derive the following inequality:

$$\frac{d}{dt}|\theta|^2+\frac{\kappa\mu}{4}(1-\sqrt{\mu}\,|\nabla\bar{\theta}|^2_{\infty,T})|\theta|^2\le\frac{2}{\kappa\mu}\{\|\mathbf{u}\|\,\|\nabla \mathbf{u}\|+\nu^2K_1^2\|\nabla \mathbf{u}\|^2\|\Delta \mathbf{u}\|^2$$

$$+\nu^2\|\nabla\bar{\mathbf{u}}\|^2\|\nabla\bar{\mathbf{u}}\|^2_{\mathbf{H}^1}+\frac{1}{2}|\mathrm{F}_2|^2\}\,.$$

Then Lemma 3.9 implies

$$|\theta|^2_{\infty,T_1} \le |\theta_0|^2 + B_1\Big\{\|\mathbf{u}\|_{\infty,T_1}\|\nabla\mathbf{u}\|_{M,T_1}$$

$$+\nu^2 K_1^2\|\nabla\mathbf{u}\|^2_{\infty,T_1}\|\Delta\mathbf{u}\|^2_{M,T_1} + \nu^2\|\nabla\bar{\mathbf{u}}\|^2_{\infty,T}\|\nabla\bar{\mathbf{u}}\|^2_{\mathbf{H}^1,M,T} + \frac{1}{2}|\mathrm{F}_2|^2_{M,T}\Big\}$$

$$< |\theta_0|^2 + B_1\{3\sqrt{A_1}\varepsilon^2 + \nu^2 K_1^2 A_3 A_4 \varepsilon^4 \tag{3.37}$$

$$+\nu^2\|\nabla\bar{\mathbf{u}}\|^2_{\infty,T}\|\nabla\bar{\mathbf{u}}\|^2_{\mathbf{H}^1,M,T} + \tfrac{1}{2}|\mathrm{F}_2|^2_{M,T}\}$$

$$= B_2^2\,.$$

Thus (3.32), (3.36) and (3.37) contradict the definition of T_1. This completes the proof of Theorem II. Q.E.D.

3.4. The Proof of Theorem III. To prove Theorem III, we rely on the Schauder's fixed point argument in $\mathrm{L}^2(0, T; \mathbf{L}^2_\sigma(\mathrm{B}))$. The close convex subset where we work is given by

$$K_R = \{\mathbf{h} \in \mathrm{L}^2(0, T; \mathbf{L}^2_\sigma(\mathrm{B}))\,;\ \|\mathbf{h}\|_{M,T} \le R\}\,.$$

For given $\mathbf{b}$ in K_R, we first solve

$$\begin{cases} \dfrac{d}{dt}\mathbf{v}_b + A_1^t\mathbf{v}_b(t) + \mathbf{b} \in \mathbf{F}_1(t)\,, \\ \mathbf{v}_b(0) = \mathbf{v}_b(T)\,. \end{cases} \tag{3.38}$$

Next, for each solution $\mathbf{v}_b$, we construct the solution $\theta_b = \theta\mathbf{v}_b$ of the equation:

$$\begin{cases} \dfrac{d}{dt}\theta_b(t) + A_2^t\theta_b(t) + \mathrm{B}_2(\mathbf{v}_b(t), \theta_b(t)) \in \mathrm{F}_2(\mathbf{v}_b(t), t)\,, \\ \theta_b(0) = \theta_b(T)\,. \end{cases} \tag{3.39}$$

Define the operator $\mathscr{B}$ by

$$\mathscr{B} : \mathbf{b} \longmapsto \mathbf{B}_1^t(\mathbf{v}_b) + \mathrm{P}_B\eta\mathbf{g}\theta_b\,.$$

Then it is easy to see that if $\mathscr{B}$ has a fixed point $\bar{\mathbf{b}}$, the $(\mathbf{v}_{\bar{b}}, \theta\mathbf{v}_{\bar{b}})$ gives a solution of (3.6)–(3.7). Therefore, in what follows, we are going to show that $\mathscr{B}$ has a fixed point $\bar{\mathbf{b}}$ in K_R for a suitably chosen R.

LEMMA 3.14. *Let* $(\mathrm{A.Q})_\pi$, $(\mathrm{A}.a)_\pi$, $(\mathrm{A}.b)_\pi$, $(\mathrm{A}.f)$ *and* (A, g) *be satisfied and let* $\mathbf{b} \in K_R$. *Then* (3.38) *has a unique solution* $\mathbf{v}_b$ *satisfying* (3.10).

LEMMA 3.15. *Let* $(\mathrm{A.Q})_\pi$, $(\mathrm{A}.a)_\pi$, $(\mathrm{A}.b)_\pi$, $(\mathrm{A}.f)$ *and* $(\mathrm{A}.g)$ *be satisfied and let* $\mathbf{v}_b$ be a solution of (3.38) *satisfying* (3.10). *Then* (3.39) *has a unique solution* θ_b *satisfying* (3.11).

PROOF OF LEMMA 3.14. Assumption $(\mathrm{A.Q})_\pi$ assures (i)–(ii) of $(\mathrm{A}.\varphi^t)_\pi$ with $\varphi^t = \varphi_1^t$ and it is easy to see that (iii)–(iv) of $(\mathrm{A}.\varphi^t)_\pi$ is satisfied with $\varphi^t = \varphi_1^t$. It follows from

$(A.a)_\pi$, $(A.b)_\pi$, $(A.f)$ and $(A.g)$ that $\mathbf{F}_1-\mathbf{b}$ belongs to $L^2(0, T; \mathbf{L}^2_\sigma(B))$. Therefore we can apply Theorem 3.5. Q.E.D.

PROOF OF LEMMA 3.15. *Existence*: It easily follows from $(A.a)_\pi$, $(A.b)_\pi$ and $(A.f)$ and (3.10) that $F_2(t)$ and $(\mathbf{v}_b\cdot\nabla)\bar{\theta}$ belong to $L^2(0, T; L^2(B))$ and that $\|\nabla\mathbf{v}_b\|\in L^\infty(0, T)$, $\|A_1^t\mathbf{v}_b\|\in L^2(0, T)$. Using the same argument as in the verification of Fact II of the proof of Theorem I, we get $|D[\mathbf{v}_b+\bar{\mathbf{u}}]|\in L^2(0, T)$. Hence $F_2(\mathbf{v}_b, t)\in L^2(Q)$. $(A.\varphi^t)_\pi$, (A.1) and (A.2) with $\varphi^t=\varphi_2^t$ and $B(t, \cdot)=B_2(\mathbf{v}_b(t), \cdot)$ can be verified in the same way as before. Lemma 3.13 assures that $(A.6)_\pi$ holds. Estimate (3.17) with replaced $\mathbf{u}$ by $\mathbf{v}_b$ and the fact that $a(t)\in L^\infty(0, T)$ assure that $(A.5)_\pi$ holds. Thus we can apply Theorem 3.6.

Uniqueness: By standard argument, we can easily show the uniqueness of the solution of (3.39). Q.E.D.

The next, we are going to show that $\mathscr{B}$ maps from K_R into itself. For this purpose, we need some a priori estimates given in the following lemma:

LEMMA 3.16. *Let* $\mathbf{b}\in K_R$ *and let* $\mathbf{v}_b$ *and* θ_b *be the solutions of* (3.38) *and* (3.39). *Then we have*

$$\|\mathbf{v}_b\|^2_{\infty,T}\le d_1(\lambda, \nu)(\|\mathbf{F}_1\|^2_{M,T}+\|\mathbf{b}\|^2_{M,T}), \tag{3.40}$$

$$\|\nabla\mathbf{v}_b\|^2_{M,T}\le d_2(\lambda, \nu)(\|\mathbf{F}_1\|^2_{M,T}+\|\mathbf{b}\|^2_{M,T}), \tag{3.41}$$

$$\|\nabla\mathbf{v}_b\|^2_{\infty,T}\le d_3(\lambda, \nu, m_0)(\|\mathbf{F}_1\|^2_{M,T}+\|\mathbf{b}\|^2_{M,T}), \tag{3.42}$$

$$\|\Delta\mathbf{v}_b\|^2_{M,T}\le d_4(\lambda, \nu, m_0)(\|\mathbf{F}_1\|^2_{M,T}+\|\mathbf{b}\|^2_{M,T}), \tag{3.43}$$

$$\begin{aligned}|\theta_b|^2_{\infty,T}\le 4\left(2+\frac{3}{\kappa\mu}\right)^2\{|F_2|^2_{M,T}&\\ &+2\nu^2K_1^2d_3(\lambda, \nu, m_0)d_4(\lambda, \nu, m_0)(\|\mathbf{F}_1\|^2_{M,T}+\|\mathbf{b}\|^2_{M,T})^2\\ &+c_2\sqrt{d_1(\lambda, \nu)}\sqrt{d_3(\lambda, \nu, m_0)}|\nabla\bar{\theta}|_{\infty,T}|\bar{\theta}|_{H^2,M,T}(\|\mathbf{F}_1\|^2_{M,T}+\|\mathbf{b}\|^2_{M,T})\\ &+2\nu^2\|\nabla\bar{\mathbf{u}}\|^2_{\infty,T}\|\bar{\mathbf{u}}\|^2_{\mathbf{H}^2,M,T}\},\end{aligned} \tag{3.44}$$

where $d_1(\lambda, \nu)=2\left(2+\dfrac{3}{\lambda\nu}\right)^2$, $d_2(\lambda, \nu)=\dfrac{3}{2\nu}d_1(\lambda, \nu)$, $d_3(\lambda, \nu, m_0)=\dfrac{\sqrt{d_1(\lambda, \nu)}}{\sqrt{2}}\left\{\dfrac{4}{\nu}+2m_0d_2(\lambda, \nu)\right\}$, $d_4(\lambda, \nu, m_0)=\dfrac{1}{\nu}\left\{\dfrac{4}{\nu}+2m_0d_2(\lambda, \nu)+d_3(\lambda, \nu, m_0)\right\}$.

PROOF. For the sake of brevity, we here denote $\mathbf{v}_b$ and θ_b by $\mathbf{v}$ and θ. Multiply (3.38) by $\mathbf{v}$ and integrate over B, then we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}(t)\|^2+\nu\|\nabla\mathbf{v}(t)\|^2\le(\|\mathbf{F}_1(t)\|+\|\mathbf{b}(t)\|)\|\mathbf{v}(t)\|. \tag{3.45}$$

Then we have

$$\frac{d}{dt}\|\mathbf{v}(t)\|+\lambda\nu\|\mathbf{v}(t)\|\le\|\mathbf{F}_1(t)\|+\|\mathbf{b}(t)\|\,.$$

By Lemma 3.10, we get (3.40).

Integrating of (3.45) over $[t-1, t]$ (or $[0, T]$) gives

$$\nu\|\nabla\mathbf{v}\|^2_{M,T}\le\|\mathbf{v}\|_{\infty,T}(\|\mathbf{F}_1\|_{M,T}+\|\mathbf{v}\|_{M,T})+\frac{1}{2}\|\mathbf{v}(t)\|^2_{\infty,T}\,,$$

which implies (3.41).

Multiply (3.38) by $\mathbf{g}_1(t)=\mathbf{F}_1(t)-\mathbf{b}-d\mathbf{v}_b/dt\in A_1^t\mathbf{v}(t)$ and integrate over B. Then recalling the fact that φ_1^t satisfies (ii) of $(\mathrm{A}.\varphi^t)_\pi$ with $m_1=\sqrt{m_0}$, $m_2=0$ and $m_3=m_0$ (see [24], [21]) and using (3.1), we obtain

$$\begin{aligned}\frac{d}{dt}\varphi_1^t(\mathbf{v}(t))+\|\mathbf{g}_1(t)\|^2&\le(\|\mathbf{F}_1(t)\|+\|\mathbf{b}(t)\|)\|\mathbf{g}_1(t)\|+\sqrt{m_0}\|\mathbf{g}_1(t)\|\cdot\varphi_1^t(\mathbf{v}(t))^{1/2}+m_0\varphi_1^t(\mathbf{v}(t))\\&\le\frac{1}{2}\|\mathbf{g}_1(t)\|^2+2(\|\mathbf{F}_1\|^2+\|\mathbf{b}\|^2)+2m_0\varphi_1^t(\mathbf{v}(t))\,.\end{aligned}$$

Since $\|\mathbf{g}_1\|\ge\nu\|\Delta\mathbf{v}\|$, we have

$$\frac{d}{dt}\|\nabla\mathbf{v}\|^2+\lambda\nu\|\nabla\mathbf{v}\|^2\le\frac{d}{dt}\|\nabla\mathbf{v}\|^2+\nu\|\Delta\mathbf{v}\|^2\le\frac{4}{\nu}(\|\mathbf{F}_1\|^2+\|\mathbf{b}\|^2)+2m_0\|\nabla\mathbf{v}\|^2\,. \tag{3.46}$$

Using Lemma 3.10 again and integrating (3.46) with respect to t over $[t-1, t]$, we can derive (3.42) and (3.43).

To show (3.44), multiply (3.39) by θ and integrate over B, then we get

$$\frac{1}{2}\frac{d}{dt}|\theta(t)|^2+\kappa|\nabla\theta(t)|^2\le|\mathrm{F}_2(t)||\theta|+\left|\int_{Q(t)}(\mathbf{v}\cdot\nabla)\bar{\theta}\theta dx\right|+\frac{\nu}{2}|D[\mathbf{v}+\bar{\mathbf{u}}]|\theta|\,.$$

By (3.15), we can reduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\theta(t)|^2+\kappa\mu|\theta(t)|^2\le\{&|\mathrm{F}_2(t)|+c_2^{1/2}\|\mathbf{v}\|^{1/2}\|\nabla\mathbf{v}\|^{1/2}|\nabla\bar{\theta}|^{1/2}|\bar{\theta}|^{1/2}_{\mathbf{H}^2}\\&+\sqrt{2}\,\nu(K_1\|\nabla\mathbf{v}\|\,\|\Delta\mathbf{v}\|+\|\nabla\bar{\mathbf{u}}\|\,\|\bar{\mathbf{u}}\|_{\mathbf{H}^2}\}|\theta|\,.\end{aligned}$$

Lemma 3.10 assures

$$\begin{aligned}|\theta|_{\infty,T}\le\left(2+\frac{3}{\kappa\mu}\right)\{&|\mathrm{F}_2(t)|_{M,T}+c_2^{1/2}\|\mathbf{v}\|^{1/2}_{\infty,T}\|\nabla\mathbf{v}\|^{1/2}_{\infty,T}|\nabla\bar{\theta}|^{1/2}_{\infty,T}|\bar{\theta}|^{1/2}_{\mathbf{H}^2,M,T}\\&+\sqrt{2}\,\nu(K_1\|\nabla\mathbf{v}\|_{\infty,T}\|\Delta\mathbf{v}\|_{M,T}+\|\nabla\bar{\mathbf{u}}\|_{\infty,T}\|\bar{\mathbf{u}}\|_{\mathbf{H}^2,M,T}\}\,.\end{aligned}$$

By (3.40)–(3.43), we have (3.44). Q.E.D.

PROOF OF THEOREM III. *Step I*: Let $\|\mathbf{F}_1\|_{M,T} \leq R$, and we put $I(\mathbf{v}, \mathbf{w}) = \|(\mathbf{v} \cdot \nabla)\mathbf{w}\|^2_{M,T}$. Then

$$\|\mathscr{B}(\mathbf{b})\|^2_{M,T} \leq 4(\mathrm{I}(\mathbf{v}_b, \mathbf{v}_b) + \mathrm{I}(\bar{\mathbf{u}}, \mathbf{v}_b) + \mathrm{I}(\mathbf{v}_b, \bar{\mathbf{u}}) + \|\eta \mathbf{g}\theta_b\|^2_{M,T}) .$$

Recalling (3.14) and (3.40)–(3.43), we can obtain

$$I(\mathbf{v}_b, \mathbf{v}_b) \leq c_1 K_1 \|\mathbf{v}_b\|_{\infty,T} \|\nabla \mathbf{v}_b\|^2_{\infty,T} \|\Delta \mathbf{v}_b\|_{M,T} \leq 4 c_1 K_1 d_5(\lambda, \nu, m_0) R^4 ,$$

where $d_5(\lambda, \nu, m_0) = \sqrt{d_1(\lambda, \nu)} d_3(\lambda, \nu, m_0) \sqrt{d_4(\lambda, \nu, m_0)}$ and c_1 is the constant given in (3.14);

$$\begin{aligned} I(\bar{\mathbf{u}}, \mathbf{v}_b) &\leq c_1 K_1 \|\bar{\mathbf{u}}\|_{\infty,T} \|\bar{\mathbf{u}}\|_{\mathbf{H}^1,\infty,T} \|\nabla \mathbf{v}_b\|_{\infty,T} \|\Delta \mathbf{v}_b\|_{M,T} \\ &\leq 2 c_1 K_1 d_6(\lambda, \nu, m_0) \|\bar{\mathbf{u}}\|_{\infty,T} \|\bar{\mathbf{u}}\|_{\mathbf{H}^1,\infty,T} R^2 , \end{aligned}$$

where $d_6(\lambda, \nu, m_0) = \sqrt{d_3(\lambda, \nu, m_0)} \sqrt{d_4(\lambda, \nu, m_0)}$;

$$\begin{aligned} I(\mathbf{v}_b, \bar{\mathbf{u}}) &\leq c_1 \|\mathbf{v}_b\|_{\infty,T} \|\nabla \mathbf{v}_b\|_{\infty,T} \|\nabla \bar{\mathbf{u}}\|_{\infty,T} \|\bar{\mathbf{u}}\|_{\mathbf{H}^2,M,T} \\ &\leq 2 c_1 d_7(\lambda, \nu, m_0) \|\nabla \bar{\mathbf{u}}\|_{\infty,T} \|\bar{\mathbf{u}}\|_{\mathbf{H}^2,M,T} R^2 , \end{aligned}$$

where $d_7(\lambda, \nu, m_0) = \sqrt{d_1(\lambda, \nu)} \sqrt{d_3(\lambda, \nu, m_0)}$.

Furthermore using (3.44), we get

$$\begin{aligned} \|\eta \mathbf{g}\theta_b\|^2_{M,T} &\leq \|\eta \mathbf{g}\|^2_{\mathbf{L}^\infty(Q)} |\theta|^2_{\infty,T} \\ &\leq 4 \|\eta \mathbf{g}\|^2_{\mathbf{L}^\infty(Q)} \left(2 + \frac{3}{\kappa\mu}\right)^2 \{|\mathrm{F}_2|^2_{M,t} + 8\nu^2 K_1^2 (d_6(\lambda, \nu, m_0))^2 R^4 \\ &\quad + 4 c_2 d_7(\lambda, \nu, m_0) |\nabla \bar{\theta}|_{\infty,T} |\bar{\theta}|_{\mathrm{H}^2,M,T} R^2 + 2\nu^2 \|\bar{\mathbf{u}}\|^2_{\infty,T} \|\bar{\mathbf{u}}\|^2_{\mathbf{H}^2,M,T}\} . \end{aligned}$$

Now we fix R as follows

$$R^2 = \frac{1}{2^7 c_1 K_1 d_5(\lambda, \nu, m_0)} ,$$

and assume that $\bar{\mathbf{u}}$ and $\eta\mathbf{g}$ satisfy

$$max\{c_1 K_1 d_6(\lambda, \nu, m_0) \|\bar{\mathbf{u}}\|_{\infty,T} \|\bar{\mathbf{u}}\|_{\mathrm{H}^1,\infty,T} , \quad c_1 d_7(\lambda, \nu, m_0) \|\nabla \bar{\mathbf{u}}\|_{\infty,T} \|\bar{\mathbf{u}}\|_{\mathbf{H}^2,M,T}\} < \frac{2}{2^6} ,$$

$$\|\eta \mathbf{g}\|^2_{\mathbf{L}^\infty(Q)} \left(2 + \frac{3}{\kappa\mu}\right)^2 |\mathrm{F}_2|^2_{M,T} < \frac{1}{2^7} R^2 ,$$

$$\|\eta \mathbf{g}\|^2_{\mathbf{L}^\infty(Q)} \left(2 + \frac{3}{\kappa\mu}\right)^2 \nu^2 K_1^2 (d_6(\lambda, \nu, m_0))^2 < \frac{1}{2^{10} R^2} ,$$

$$\|\eta \mathbf{g}\|^2_{\mathbf{L}^\infty(Q)} \left(2 + \frac{3}{\kappa\mu}\right)^2 2 c_2 d_7(\lambda, \nu, m_0) |\nabla \bar{\theta}|_{\infty,T} |\bar{\theta}|_{\mathrm{H}^2,M,T} < \frac{1}{2^9} ,$$

$$\|\eta \mathbf{g}\|^2_{\mathbf{L}^\infty(Q)} \left(2 + \frac{3}{\kappa\mu}\right)^2 \nu^2 \|\nabla \bar{\mathbf{u}}\|^2_{\infty,T} \|\bar{\mathbf{u}}\|^2_{\mathbf{H}^2,M,T} < \frac{1}{2^8} R^2 .$$

Then it is easy to see $\|\mathscr{B}(\mathbf{v}_b)\|^2_{M,T}=\|\mathbf{B}_1^t(\mathbf{v}_b)+\mathrm{P}_B\eta\mathbf{g}\theta\|^2_{M,T}<R^2$. This shows that $\mathscr{B}$ maps K_R into itself.

Step II: Let $\mathscr{H}_W$ be $\mathrm{L}^2(0, T; \mathbf{L}^2_\sigma(\mathrm{B}))$ endowed with the weak topology. For applying Schauder-Tychonoff's fixed point theorem, we need to show that $\mathscr{B}$ is continuous from $\mathscr{H}_W$ into $\mathscr{H}_W$.

Let $\mathbf{b}_n\in K_R$ and $\mathbf{b}_n\rightharpoonup\mathbf{b}$ weakly in $\mathrm{L}^2(0, T; \mathbf{L}^2_\sigma(\mathrm{B}))$, and let $\mathbf{v}_n=\mathbf{v}_{b_n}$ and $\theta_n=\theta_{v_{b_n}}$ be the solutions of (3.38) and (3.39) with $\mathbf{b}=\mathbf{b}_n$ and $\mathbf{v}_b=\mathbf{v}_{b_n}$, respectively. By (3.40)–(3.43), we can extract a subsequence $\{\mathbf{v}_{n_k}\}$ of $\{\mathbf{v}_n\}$ such that

$$\begin{cases} \mathbf{v}_{n_k} \to \mathbf{v} & \text{strongly in } \mathrm{C}([0, T]; \mathbf{L}^2_\sigma(\mathrm{B}))\,, \\ d\mathbf{v}_{n_k}/dt \rightharpoonup d\mathbf{v}/dt & \text{weakly in } \mathrm{L}^2(0, T; \mathbf{L}^2_\sigma(\mathrm{B}))\,, \\ \mathbf{g}_1^{n_k} \rightharpoonup \mathbf{g}_1 & \text{weakly in } \mathrm{L}^2(0, T; \mathbf{L}^2_\sigma(\mathrm{B}))\,, \end{cases} \tag{3.47}$$

where $\mathbf{g}_1^n=\mathbf{F}_1(t)-\mathbf{b}_n(t)-d\mathbf{v}_{b_n}/dt\in\partial\varphi_1^t(\mathbf{v}_{b_n}(t))$. Since $\partial\varphi_1^t(\cdot)$ is demiclosed in $\mathrm{L}^2(0, T; \mathbf{L}^2_\sigma(\mathrm{B}))$, we can easily see that $\mathbf{g}_1(t)\in\partial\varphi_1^t(\mathbf{v}(t))$ for *a.e.* $t\in[0, T]$, which means that $\mathbf{v}$ is a solution of (3.38). Moreover, the above argument does not depend on the choice of subsequences, therefore we find that (3.47) holds with $\mathbf{v}_{n_k}=\mathbf{v}_n$ and $\mathbf{g}^{n_k}=\mathbf{g}^n$.

Furthermore, noting $\|\nabla(\mathbf{v}_n-\mathbf{v})\|^2=(\mathbf{v}_n-\mathbf{v}, \mathbf{g}_1^n-\mathbf{g})$, we easily find that $|\nabla\mathbf{v}_n|\to|\nabla\mathbf{v}|$ in $\mathrm{L}^2(0, T; \mathrm{L}^2(B))$, which together with (3.47) assure that

$$\bar{D}[\mathbf{v}_n]\longrightarrow\bar{D}[\mathbf{v}] \quad \text{strongly in} \quad \mathrm{L}^1(0, T; \mathrm{L}^2(B))\,, \tag{3.48}$$

where $\bar{D}[\mathbf{v}_n]=D[\mathbf{v}_n+\bar{\mathbf{u}}]$.

On the other hand, (3.42) and (3.43) imply that $\bar{D}[\mathbf{v}_n]$ is bounded in $\mathrm{L}^2(0, T; \mathrm{L}^2(B))$. Then, by virtue of (3.48), we can reduce

$$\bar{D}[\mathbf{v}_n]\rightharpoonup\bar{D}[\mathbf{v}] \quad \text{weakly in} \quad \mathrm{L}^2(0, T; \mathrm{L}^2(B))\,.$$

It is clear that (3.47) implies that $(\mathbf{v}_n\cdot\nabla)\bar{\theta}\to(\mathbf{v}\cdot\nabla)\bar{\theta}$ strongly in $\mathrm{L}^2(0, T; \mathrm{L}^2(B))$. Thus it is shown that $\mathrm{F}_2(\mathbf{v}_n, t)$ converges to $\mathrm{F}_2(\mathbf{v}, t)$ weakly in $\mathrm{L}^2(0, T; \mathrm{L}^2(B))$.

Multiplying (3.39) by $g_2^n(t)=\mathrm{F}_2(\mathbf{v}_n, t)-\mathrm{B}(\mathbf{v}_n, \theta_n)-d\theta_n/dt\in\partial\varphi_2^t(\theta_n)$, we can get the relation (3.20) with $\mathbf{u}$, θ^ε and g^ε replaced by $\mathbf{v}_n$, θ_n and g_2^n. Hence, in parallel with (3.21), we get

$$\frac{d}{dt}\varphi_2^t(\theta_n(t))+\frac{1}{4}|g_2^n(t)|^2\le\varphi_2^t(\theta_n(t))\left(2m_0+\frac{16K_1^2c_2^2}{\kappa^3}a(t)\right)+|\mathrm{F}_2(\mathbf{v}_n, t)|^2\,.$$

Then, recalling that $\mathrm{F}_2(\mathbf{v}_n, t)$ is bounded in $\mathrm{L}^2(0, T; \mathrm{L}^2(B))$ and applying Lemma 3.10, we can easily establish a priori bounds for $|\nabla\theta_n|_{\infty,T}$, $|\Delta\theta_n|_{M,T}$ and $|\partial\theta_n/\partial t|_{M,T}$. Consequently, by the same argument as above, we find

$$\begin{cases} \theta_{n_k} \to \theta & \text{strongly in} \quad C([0, T]; L^2(B)), \\ d\theta_{n_k}/dt \rightharpoonup d\theta/dt & \text{weakly in} \quad L^2(0, T; L^2(B)), \\ g_2^{n_k} \rightharpoonup g_2 \in \partial\varphi_2^t(\theta) & \text{weakly in} \quad L^2(0, T; L^2(B)), \\ B_2(\mathbf{v}_{n_k}, \theta_{n_k}) \rightharpoonup b_2 & \text{weakly in} \quad L^2(0, T; L^2(B)). \end{cases}$$

Thus θ is shown to be the solution of (3.39).

Now, it is easy to see that $\mathscr{B}(\mathbf{b}_n) = B_1^t(\mathbf{v}_n) + P_B\eta\mathbf{g}\theta_n$ converges to $\mathscr{B}(\mathbf{b}) = B_1^t(v) + P_B\eta\mathbf{g}\theta$ weakly in $L^2(0, T; L_\sigma^2(B))$ as $n \to \infty$. Thus $\mathscr{B}$ is shown to be a continuous mapping from $\mathscr{H}_W$ into itself. Q.E.D.

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