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On the Joint Distribution of the First Hitting Time and the First Hitting Place to the Space-Time Wedge Domain of a Biharmonic Pseudo Process

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Abstract. We consider the equation

$$\frac{\partial u}{\partial t}(t, x) = -\Delta^2 u(t, x)$$

for the biharmonic operator $-\Delta^2$. We define the pseudo process corresponding to this equation as Nishioka's sense. We obtain the Laplace-Fourier transform of the joint distribution of the first hitting time $\tau(\omega) = \inf\{t > 0 : \omega(t) < \alpha t - a\}$ ($a > 0$, $\alpha \in \mathbf{R}$) and the first hitting place $\omega(\tau)$, where each path $\omega(t)$ starts from 0 at $t = 0$.

1. Introduction.

We consider the partial differential equation

$$\frac{\partial u}{\partial t}(t, x) = -\Delta^2 u(t, x) \qquad t > s, \quad x \in \mathbf{R} \tag{1.1}$$

$$u(s, x) = \delta_x . \tag{1.2}$$

The fundamental solution of this equation can be expressed as

$$p(t-s, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \exp\{-ix\xi - \xi^4(t-s)\} . \tag{1.3}$$

This $p(t, x)$ has the following property. For $t > 0$,

(1.4) $\quad p(t, x)$ is in the Schwartz class $\mathcal{S}$ on $\mathbf{R}$ and even function in x,

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$$(1.5)\qquad \int_{-\infty}^{\infty} p(t, x)dx = 1\,,$$

$$(1.6)\qquad \int_{-\infty}^{\infty} p(t, x-y)p(s, y)dy = p(t+s, x)\,,$$

$$(1.7)\qquad p(t, x) = t^{-1/4}p(1, x/t^{1/4})\,.$$

As shown by Hochberg [7], $p(t, x)$ is not positive valued. In fact, for sufficiently large $|x|$, he obtained

$$p(1, |x|) = a|x|^{-1/3}\exp\{-b|x|^{4/3}\}\cos c|x|^{4/3} + \text{lower order}\,,$$

where a, b and c are positive constants. Thus $p(t, x)$ takes both signs and by (1.7) we obtain

$$(1.8)\qquad \int_{-\infty}^{\infty} |p(t, x)|dx = \int_{-\infty}^{\infty} |p(1, x)|dx \equiv V > 1\,.$$

However, because of (1.4)–(1.7) some authors have discussed how to apply probabilistic method to it ([4], [5], [7], [8] and [10]).

Using the composition of two independent Brownian motions some solutions of (1.1) and (1.2) are represented by Funaki [4].

Krylov [8], later Hochberg [7] and Nishioka [10] considered a signed finitely additive measure on $C[0, 1]$ (Nishioka considered on $D[0, \infty)$) which may be viewed as the distribution of a process corresponding to (1.1). In particular, Nishioka [10] obtained the Laplace-Fourier transform of joint distribution of the first hitting time and the first hitting place to $D' = \{(t, x) \in [0, \infty) \times (-\infty, 0)\}$ in his sense.

It should be mentioned that there exists completely different probabilistic approach to the $-\Delta^2$ problem (see [5]).

In this paper, we extend Nishioka's argument to $D = \{(t, x) : x < \alpha t - a\}$ ($\alpha \in \mathbf{R}$, $t > 0$, $a > 0$) and compute the Laplace-Fourier transform of its joint distribution. The main result of this paper is Theorem 3.4. In section 2, we shall define the expectation in Nishioka's sense [10]. In section 3, we obtain the Laplace-Fourier transform of the joint distribution of the first hitting time and the first hitting place to D in Nishioka's sense [10].

2. Notations and preliminary results.

In this section we will define the expectation to associate with (1.3) in Nishioka's sense [10].

We work on the path space $\Omega \equiv D[0, \infty)$, which is the space of all right continuous functions on $[0, \infty)$ which have left hand limits. We define a finitely additive measure on it.

DEFINITION 2.1. A subset $\Gamma\subset\Omega$ is said to be of finite observations if it has the representation

$$\Gamma\equiv\{\omega\in\Omega : \omega(t_1)\in B_1, \cdots, \omega(t_n)\in B_n\} \tag{2.1}$$

for a finite set $0\le t_1<\cdots<t_n$ and a cylinder set $B_1\times\cdots\times B_n$, where B_j is a Borel set in $\mathbf{R}$.

$\mathscr{C}(\Omega)$ is a finitely additive algebra consisting of all finite unions of sets of finite observations and $\mathscr{B}(\Omega)$ is a σ-algebra generated by $\mathscr{C}(\Omega)$.

$\mathscr{B}_t(\Omega)$ is a σ-algebra generated by $\{\omega(t_1), \cdots, \omega(t_n) : 0\le t_1<\cdots<t_n\le t\}$ for $t>0$ fixed. Clearly $\mathscr{B}_t(\Omega)\subset\mathscr{B}(\Omega)$.

First we define a signed measure adjoining (1.1) on $\mathscr{C}$. For a set Γ of the form (2.1), we set

$$P_x(\Gamma)\equiv\int_{B_1}dy_1\cdots\int_{\mathrm{B}_n}dy_n p(t_1, y_1-x)p(t_2-t_1, y_2-y_1)\cdots \times p(t_n-t_{n-1}, y_n-y_{n-1}), \tag{2.2}$$

where we use the convention:

$$p(0, y_1-x)dy_1=\delta_x(dy_1).$$

However we cannot apply Kolmogorov's extention theorem to this P_x, because its total variation is greater than one. Thus we can not extend (2.2) to a countably additive singned measure. But we have defined the expectation by P_x for sets of finite observations. Hence, we shall extend this expectation to functions of discrete observations and finally to functions of continuous observations.

Now, we set

$$\mathbf{T}_\Delta^k\equiv\{j\Delta=j/2^n : j=0, 1, \cdots, k\}, \qquad \mathbf{T}_\Delta=\{j\Delta=j/2^n : j=0, 1, \cdots\}$$

for any fixed $n, k\in\mathbf{N}$.

DEFINITION 2.2. A function $f: \Omega\to\mathbf{R}$ is called tame, if it is a Borel function of finite observations included in $\mathbf{T}_\Delta$. That is,

$$f(\omega)=g(\omega(0), \omega(\Delta), \cdots, \omega(k\Delta)), \tag{2.3}$$

where g is a Borel function defined on $\mathbf{R}^{k+1}$.

Let $\mathscr{T}(\mathbf{T}_\Delta^k)$ denote the class of all tame functions as in (2.3). Naturally we can define the expectation of $f\in\mathscr{T}(\mathbf{T}_\Delta^k)$ by the formula similar to (2.2) and we write

$$\begin{aligned} E_x[f]&<\int f(\omega)P_x(d\omega) \\ &=\int_{-\infty}^{\infty}dy_1\cdots\int_{-\infty}^{\infty}dy_k g(x, y_1, \cdots, y_k) \\ &\quad\times p(\Delta, y_1-x)p(\Delta, y_2-y_1)\cdots p(\Delta, y_k-y_{k-1}), \end{aligned} \tag{2.4}$$

if this multiple integration exists.

PROPOSITION 2.3 ([10] K. Nishioka). *If* $f \in \mathscr{T}(\mathbf{T}_\Delta^k)$, *then we have* $|E_x[f(\omega)]| \leq V^k \sup_\omega |f|$, *where* V *is given by* (1.8).

DEFINITION 2.4. Let $\{f_k : k=1, 2, \cdots\}$ be a sequence of complex-valued functions on Ω such that

(i) for each k, $f_k \in \mathscr{T}(\mathbf{T}_\Delta^k)$,

(ii) for every ω, $\sum_{k=1}^\infty f_k(\omega)$ exists,

(iii) for each x, $\sum_{k=1}^\infty |E_x[f_k]| < \infty$.

Then we say

$$F(\omega) = \sum_{k=1}^\infty f_k(\omega) \tag{2.5}$$

is a function of discrete observations, and $\mathscr{T}(\mathbf{T}_\Delta)$ denotes the family of all such functions. Moreover for a function $F \in \mathscr{T}(\mathbf{T}_\Delta)$, we define the expectation by

$$E_x[F] = \sum_{k=1}^\infty E_x[f_k]. \tag{2.6}$$

PROPOSITION 2.5 ([10] K. Nishioka). *The expectation* E_x *is uniquely determined on* $\mathscr{T}(\mathbf{T}_\Delta)$ *and is a linear functional.*

For each $\omega \in \Omega$, we set

$$\omega_\Delta(t) \equiv \omega(k\Delta) \quad \text{if} \quad k\Delta \leq t < (k+1)\Delta, \ k=0, 1, \cdots. \tag{2.7}$$

This ω_Δ is a right continuous step function and $\omega_\Delta \in \Omega$.

DEFINITION 2.6. Let F be a complex-valued function on Ω such that

(i) for each ω, $F(\omega_\Delta)$ converges to $F(\omega)$ as n tends to ∞,

(ii) for each Δ, $F(\omega_\Delta) \in \mathscr{T}(\mathbf{T}_\Delta)$,

(iii) for every x, $\{E_x[F(\omega_\Delta)] : \Delta\}$ converges.

Then we say that the function F is admissible, and $\mathscr{K}$ denotes the set of all admissible functions. Moreover we define its expectation by

$$E_x[F] \equiv \lim_{\Delta \to 0} E_x[F(\omega_\Delta)]. \tag{2.8}$$

REMARK 2.7. (i) $E_x[F]$ is unique for $F \in \mathscr{K}$ since the sequence $\{E_x[F(\omega_\Delta)] : \Delta\}$ is specified by (2.7).

(ii) If F is a bounded Borel function of finite observations, then we have $f \in \mathscr{K}$ and (2.8) coincides with (2.4).

PROPOSITION 2.8 ([10] K. Nishioka). $\mathscr{K}$ *is a subspace of* $\mathscr{B}(\Omega)$*-measurable function. The expectation* E_x *is determined uniquely on* $\mathscr{K}$ *and is a linear functional.*

DEFINITION 2.9. Suppose that a function $W(t, \omega) : [0, \infty) \times \Omega \to \mathbf{C}$ satisfies the following conditions:
(i) For ω_Δ of (2.7),

$$W(t, \omega_\Delta) = W(k\Delta, \omega_\Delta) \text{ if } k\Delta \le t < (k+1)\Delta$$

and $W(t, \omega_\Delta)$ belongs to $\mathscr{T}(\mathbf{T}_\Delta)$.
(ii) For ω and $t \ge 0$, $\lim_{\Delta \to 0} W(t, \omega_\Delta) = W(t, \omega)$.
Then we call $W(t, \omega)$ a separable function. The set of all separable functions will be denoted by $\mathscr{L}$.

We consider the following $U, V \in \mathscr{L}$.

$$U : [0, \infty) \times \Omega \to \mathbf{R}$$
$$V : [0, \infty) \times \Omega \to \{0, 1\} .$$

For $u \in \mathbf{C}^+ = \{u : Re(u) > 0\}$ and $\lambda \in \mathbf{R}$, we set

$$F(u, \lambda; \omega) \equiv \int_0^\infty dt e^{-ut} e^{i\lambda U(t,\omega)} V(t, \omega) .$$

We shall find the expectation $E_x[F]$ by means of (2.8).

3. The Laplace-Fourier transform of the joint distribution of the first hitting time and the first hitting place in D.

Let $\alpha \in \mathbf{R}$ and $u \in \mathbf{C}^+$. For $\omega \in \Omega$ with $\omega(0) = 0$ and any $a \ge 0$, we set

$$\tau_a(\omega) = \begin{cases} \inf\{t > 0 : \omega(t) < \alpha t - a\} & \\ \infty & \text{if the above set is empty} , \end{cases}$$

$$F_a(u, \lambda) = F_a(u, \lambda : \omega) = \begin{cases} \exp\{i\lambda\omega(\tau_a) - u\tau_a(\omega)\} & \text{if } \tau_a < \infty \\ 0 & \text{if } \tau_a = \infty . \end{cases}$$

THEOREM 3.1. *If $Re(u) > 0$, then $F_0(u, \lambda)$ is admissible and*

$$E_0[F_0(u, \lambda)] = 1$$

PROOF. Let $\sigma_k = \omega(k\Delta) - \alpha k\Delta + a$ and $Re(u) > \eta > 0$. We set

$$\tau_a^\Delta = \tau_a(\omega_\Delta) = \begin{cases} k\Delta & \text{if } \sigma_0, \cdots, \sigma_{k-1} \ge 0 \text{ and } \sigma_k < 0 \\ \infty & \text{otherwise} . \end{cases}$$

We set

$$F_a^\Delta(u, \lambda) = F_a(u, \lambda : \omega_\Delta) .$$

Since ω is right continuous with left hand limits and τ_a is the first hitting time to the open set $D = \{(t, x) : x < \alpha t - a\}$, we get

$$\lim_{\Delta\to 0} \tau_a(\omega_\Delta) = \tau_a(\omega) .$$

Since $\omega_\Delta(\tau_a(\omega_\Delta)) = \omega(\tau_a(\omega_\Delta))$, we get

$$\lim_{\Delta\to 0} \omega_\Delta(\tau_a(\omega_\Delta)) = \omega(\tau_a) , \qquad \lim_{\Delta\to 0} F_a^\Delta(u, \lambda) = F_a(u, \lambda) .$$

We set

$$h_{k\Delta}^a(u, \lambda : \omega) \equiv \exp\{-\mathrm{uk}\Delta + i\lambda\omega(k\Delta)\} I_{\{\sigma_0, \cdots, \sigma_{k-1} \geq 0\}}(\omega_\Delta) I_{\{\sigma_k < 0\}}(\omega_\Delta) ,$$

where $I_A(\omega)$ denotes the defining function of the set $A \in \mathscr{B}(\Omega)$. Then we obtain

$$h_{k\Delta}^a \in \mathscr{T}(\mathbf{T}_\Delta^k) \quad \text{and} \quad |h_{k\Delta}^a| \leq e^{-Re(u)k\Delta} .$$

If u satisfies $\exp\{-Re(u)\Delta\} V < 1$, then we have

$$\left| \sum_{k=1}^{\infty} e^{-uk\Delta} E_0[e^{i\lambda\omega(k\Delta)} I_{\{\sigma_0, \cdots, \sigma_{k-1} \geq 0\}} I_{\{\sigma_k < 0\}}] \right| \leq \sum_{k=1}^{\infty} e^{-Re(u)k\Delta} V^k$$

by Proposition 2.3. Thus, the series $\sum_{k=1}^{\infty} E_0[h_{k\Delta}^a]$ is absolutely convergent and

$$F_a^\Delta(u, \lambda) = \sum_{k=1}^{\infty} h_{k\Delta}^a(u, \lambda) .$$

Therefore, if u satisfies $\exp\{-Re(u)\Delta\} V < 1$, then $F_a^\Delta(u, \lambda) \in \mathscr{T}(\mathbf{T}_\Delta)$.

In the following, we set $a = 0$. We shall show $F_0(u, \lambda)$ is admissible for u satisfying $Re(u) > 0$. We set

$$\begin{aligned} \chi_0^\Delta(u, \lambda) &= E_0[F_0^\Delta(u + i\lambda\alpha, \lambda)] \\ &= E_0[\exp\{-u\tau_0^\Delta + i\lambda(\omega(\tau_0^\Delta) - \alpha\tau_0^\Delta)\}] . \end{aligned}$$

We state the combinatorial theorem by W. Feller [3] in our notations:

LEMMA 3.2. *If u satisfies* $\exp\{-Re(u)\Delta\} V < 1$, *then*

$$\log \frac{1}{1 - \chi_0^\Delta(u, \lambda)} = \sum_{k=1}^{\infty} \frac{e^{-uk\Delta}}{k} \int_{-\infty}^{0} e^{i\lambda x} p(k\Delta, x + \alpha\Delta k) dx . \tag{3.1}$$

Then, we have

$$\begin{aligned} -\log(1 - \chi_0^\Delta(u, \lambda)) &= \sum_{k=1}^{\infty} \frac{e^{-uk\Delta}}{k} \int_{-\infty}^{\alpha k\Delta} dx e^{i\lambda(x - \alpha k\Delta)} p(k\Delta, x) \\ &= \sum_{k=1}^{\infty} \frac{e^{-uk\Delta}}{k} \left(\int_{-\infty}^{0} dx p(k\Delta, x) \{\cos(x - \alpha k\Delta)\lambda \right. \\ &\qquad \left. + i \sin(x - \alpha k\Delta)\lambda\} + \int_{0}^{\alpha k\Delta} dx p(k\Delta, x) e^{i\lambda(x - \alpha k\Delta)} \right) \end{aligned}$$

and we set

$$=A_1+iA_2+A_3\,.$$

Noting

$$\int_{-\infty}^{0} p(t,x)\cos\lambda x dx=\frac{1}{2}e^{-\lambda^4 t}\,,$$

and

$$-\log(1-x)=\sum_{k=1}^{\infty}\frac{x^k}{k}\qquad\text{for}\quad |x|<1\,,$$

we have

$$\begin{aligned}A_1=&\sum_{k=1}^{\infty}\frac{e^{-(\lambda^4+u)k\Delta}}{2k}\cos\lambda\alpha k\Delta\\&+\sum_{k=1}^{\infty}\frac{e^{-uk\Delta}}{k}\sin\lambda\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\lambda x\\=&-\tfrac{1}{4}\log(1-e^{-(\lambda^4-i\alpha\lambda+u)\Delta})-\tfrac{1}{4}\log(1-e^{-(\lambda^4+i\alpha\lambda+u)\Delta})\\&+\sum_{k=1}^{\infty}\frac{e^{-uk\Delta}}{k}\sin\lambda\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\lambda x\,.\end{aligned}$$

Similarly, we have

$$\begin{aligned}iA_2=&-i\sum_{k=1}^{\infty}\frac{e^{-(\lambda^4+u)k\Delta}}{2k}\sin\lambda\alpha k\Delta\\&+i\sum_{k=1}^{\infty}\frac{e^{-uk\Delta}}{k}\cos\lambda\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\lambda x\\=&\tfrac{1}{4}\log(1-e^{-(\lambda^4-i\alpha\lambda+u)\Delta})-\tfrac{1}{4}\log(1-e^{-(\lambda^4+i\alpha\lambda+u)\Delta})\\&+i\sum_{k=1}^{\infty}\frac{e^{-uk\Delta}}{k}\cos\lambda\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\lambda x\,.\end{aligned}$$

Then we get

$$\begin{aligned}(3.2)\qquad 1-\chi_0^{\Delta}(u,\lambda)=&(1-e^{-(\lambda^4+i\alpha\lambda+u)\Delta})^{1/2}\\&\times\exp\Bigg\{-i\sum_{k=1}^{\infty}\frac{1}{k}e^{-uk\Delta}e^{-i\lambda\alpha k\Delta}\int_{-\infty}^{0}dxp(k\Delta,x)\sin\lambda x\\&-\sum_{k=1}^{\infty}\frac{1}{k}e^{-uk\Delta}\int_{0}^{\alpha k\Delta}dxp(k\Delta,x)e^{i\lambda(x-\alpha k\Delta)}\Bigg\}\end{aligned}$$

and set

$$=(1-e^{-(\lambda^4+i\alpha\lambda+u)\Delta})^{1/2}\exp\{iI_1+I_2\}.$$

Now, we will estimate iI_1 and I_2. We write $u=u_r+iu_i$. Take any $\eta>0$ and suppose that $u_r>\eta$.

First, we estimate I_2:

$$\begin{aligned}|I_2|&\le\sum_{k=1}^{\infty}\frac{1}{k}e^{-\eta k\Delta}\int_0^{\alpha k\Delta}dx|p(k\Delta,x)|\\&=\sum_{k=1}^{\infty}\frac{1}{k}e^{-\eta k\Delta}\int_0^{\alpha k\Delta}dx(k\Delta)^{-1/4}|p(1,x(k\Delta)^{-1/4})|\\&=\sum_{k=1}^{\infty}\frac{1}{k}e^{-\eta k\Delta}\int_0^{\alpha(k\Delta)^{3/4}}dx|p(1,x)|.\end{aligned}$$

We set $M=\sup_x|p(1,x)|$, then we get

$$|I_2|\le|\alpha|\Delta^{3/4}M\sum_{k=1}^{\infty}\frac{1}{k^{1/4}}e^{-\eta k\Delta}.$$

Now we notice

$$\sum_{k=1}^{\infty}\frac{1}{k^p}e^{-k\Delta}=\sum_{k=1}^{\infty}\frac{1}{(k\Delta)^p}e^{-k\Delta}\Delta\Delta^{p-1}\quad(0<p<1),$$

$$\Gamma(1-p)=\int_0^{\infty}\frac{e^{-x}}{x^p}dx\ge\sum_{k=1}^{\infty}\frac{e^{-k\Delta}}{(k\Delta)^p}\Delta.$$

Therefore we get

$$|I_2|\le|\alpha|\eta^{-3/4}M\Gamma(3/4).$$

Next, we estimate $Re(iI_1)$:

$$\begin{aligned}Re(iI_1)=&-\sum_{k=1}^{\infty}\frac{1}{k}e^{-u_rk\Delta}\cos u_ik\Delta\sin\lambda\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\lambda x\\&-\sum_{k=1}^{\infty}\frac{1}{k}e^{-u_rk\Delta}\sin u_ik\Delta\cos\lambda\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\lambda x\end{aligned}$$

and we set

$$=I_1^1+I_1^2.$$

First we estimate I_1^1. Noting

$$\left|\int_{-\infty}^{0}p(k\Delta,x)\sin\lambda xdx\right|\le\frac{2}{|\lambda|(k\Delta)^{1/4}}\int_{-\infty}^{0}|p'(1,x)|dx=\frac{1}{|\lambda|(k\Delta)^{1/4}}V',$$

where $V'=2\int_{-\infty}^{0}|\partial_x p(1,x)|dx=\int_{-\infty}^{\infty}|p'(1,x)|dx$ and $|\sin x|\le|x|$, we get

$$|I_1^1|\le V'|\alpha|\sum_{k=1}^{\infty}\frac{1}{k}e^{-\eta k\Delta}|\lambda k\Delta|\frac{1}{|\lambda|(k\Delta)^{1/4}}\le V'|\alpha|\eta^{-3/4}\Gamma(3/4)\,.$$

Next we estimate I_1^2. We notice the following evaluation:

$$\left|\int_{-\infty}^{0}p(k\Delta,x)\sin\lambda x dx\right|\le|\lambda|(k\Delta)^{1/4}\int_{-\infty}^{\infty}|xp(1,x)|dx=|\lambda|(k\Delta)^{1/4}V_1\,,$$

where $V_1=\int_{-\infty}^{\infty}|xp(1,x)|dx$. Let $N=\sup\{n\in\mathbf{N}:\Delta^{-1}|\lambda|^{-4}\ge n\}$. Then,

$$\begin{aligned}|I_1^2|&\le\sum_{k=1}^{N}\frac{1}{k}e^{-\eta k\Delta}|\lambda|(k\Delta)^{1/4}V_1+\sum_{k=N+1}^{\infty}\frac{1}{k}e^{-\eta k\Delta}\frac{V'}{|\lambda|(k\Delta)^{1/4}}\\&\le V_1|\lambda|\int_0^{N\Delta}\frac{1}{x^{3/4}}dx+\frac{V'}{(N+1)|\lambda|((N+1)\Delta)^{1/4}}+\frac{V'}{|\lambda|}\int_{(N+1)\Delta}^{\infty}\frac{1}{x^{5/4}}dx\\&\le 4V_1+5V'\,.\end{aligned}$$

Hence we get

$$|1-\chi_0^\Delta(u,\lambda)|\le C|(1-e^{-(\lambda^4+i\alpha\lambda+u)\Delta})^{1/2}|\,, \tag{3.3}$$

where C is a positive constant which only depends on η and α.

And in a way similar to the above argument we can prove the imaginary part of the right hand side of (3.1) is bounded for $Re(u)>\eta>0$.

Let $z=e^{-u\Delta}$. Then the above estimate implies that $\log(1-\chi_0^\Delta(u,\lambda))$, given by (3.1), is an analytic function of z and converges absolutely for $|z|<1$. Therefore, using the next lemma, we see that $\chi_0^\Delta(u,\lambda)$ is also analytic for $|z|<1$. Since $|e^{-(u-i\lambda\alpha)\Delta}|=|e^{-u\Delta}|$, we get $\chi_0^\Delta(u-i\lambda\alpha,\lambda)$ is absolutely convergent for $Re(u)>0$. Therefore, substituting $u-i\lambda\alpha$ for u, we conclude $F_0^\Delta(\alpha,\lambda)$ is admissible for $Re(u)>0$. By (3.3) we get

$$\lim_{\Delta\to 0}\chi_0^\Delta(u,\lambda)=1$$

and also

$$\lim_{\Delta\to 0}E_0[F_0^\Delta(u,\lambda)]=1\,.$$ □

LEMMA 3.3 [1, p. 22 Proposition 5.1]. *Let $S(z)$ and $T(z)$ be analytic functions with their expansions*:

$$S(z)=\sum_{n=0}^{\infty}a_nz^n \quad \text{and} \quad T(z)=\sum_{n=1}^{\infty}b_nz^n\,.$$

If their radii of convergence $\rho(S)$ and $\rho(T)\ne 0$, then $U=S\circ T$ has the radius of convergence $\rho(U)\ne 0$, too.

Moreover there exists $r>0$ *such that* $\sum_{n\geq 1}|b_n|r^n<\rho(S)$, *and then* $\rho(U)\geq r$ *and for all* z *satisfying* $|z|\leq r$, *we have*

$$|T(z)|<\rho(S) \quad and \quad S(T(z))=U(z)\,.$$

Next, we will show the main theorem in this paper. Let $a>0$.

THEOREM 3.4. *If* $Re(u)>0$, *then* $F_a(u, \lambda)$ *is admissible and*

$$E_0[F_a(u, \lambda)]=\frac{\zeta_2-\lambda}{\zeta_2-\zeta_1}e^{-ia(\lambda-\zeta_1)}+\frac{\zeta_1-\lambda}{\zeta_1-\zeta_2}e^{-ia(\lambda-\zeta_2)}\,.$$

Here ζ_1 *and* ζ_2 *are solutions of* $\xi^4+i\alpha\xi+u-i\lambda\alpha=0$ *whose imaginary parts are positive.*

PROOF. By the proof of Theorem 3.1 we know $F_a^{\Delta}(u, \lambda)\in\mathscr{T}(\mathbf{T}_{\Delta})$ if $u=u_r+iu_i$ satisfies $e^{-u_r\Delta}V<1$.

Then we will show $F_a(u, \lambda)$ is admissible for u satisfying $Re(u)>\eta>0$. We set

$$\chi_a^{\Delta}(u, \lambda)=E_0[F_a^{\Delta}(u+i\lambda\alpha, \lambda)]\,.$$

We state the combinatorial theorem by T. Nakajima [9] in our situation:

LEMMA 3.5. *If* u *satisfies* $\exp\{-Re(u)\Delta\}V<1$, *then*

$$\chi_a^{\Delta}(u, \lambda)=\chi_0^{\Delta}(u, \lambda)-\frac{1}{2\pi}\lim_{N\to\infty}\int_{-N}^{N}\frac{1-e^{-iav}}{iv}\,\frac{1-\chi_0^{\Delta}(u, \lambda)}{1-\chi_0^{\Delta}(u, \lambda-v)}\chi_0^{\Delta}(u, \lambda-v)dv\,. \tag{3.4}$$

Let K be a positive constant and we write $\mu=\lambda-v$. We take Δ satisfying

$$|\lambda|<K^{1/4}\Delta^{-1/4} \quad \text{and} \quad K\Delta^{-1}>u_r>\eta>0\,.$$

By (3.2) we get

$$\frac{1-\chi_0^{\Delta}(u, \lambda)}{1-\chi_0^{\Delta}(u, \mu)}=\left(\frac{1-e^{-(\lambda^4+i\alpha\lambda+u)\Delta}}{1-e^{-(\mu^4+i\alpha\mu+u)\Delta}}\right)^{1/2}$$

$$\times\exp\left\{i\sum_{k=1}^{\infty}\frac{e^{-uk\Delta}}{k}\int_{-\infty}^{0}dxp(k\Delta, x)(e^{i\mu\alpha k\Delta}\sin\mu x-e^{i\lambda\alpha k\Delta}\sin\lambda x)\right\}$$

$$\times\exp\left\{\sum_{k=1}^{\infty}\frac{e^{-uk\Delta}}{k}\int_{0}^{\alpha k\Delta}dxp(k\Delta, x)(e^{i\mu(x-\alpha k\Delta)}-e^{i\lambda(x-\alpha k\Delta)})\right\}$$

and set

$$=\left(\frac{1-e^{-(\lambda^4+i\alpha\lambda+u)\Delta}}{1-e^{-(\mu^4+i\alpha\mu+u)\Delta}}\right)^{1/2}\exp\{iJ_1+J_2\}\,.$$

Noting $|e^{ix}-e^{iy}|\leq 2$ $(x, y\in\mathbf{R})$, we can estimate J_2 in a similar way as for I_2 of Theorem 3.1 and get

$$|J_2|\leq C_1\,,$$

where C_1 is a positive constant which depends on α and η.

On the othr hand, we consider

$$Re(iJ_1)=\sum_{k=1}^{\infty}\frac{1}{k}e^{-u_r k\Delta}\left(\cos u_i k\Delta \sin\lambda\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\lambda x - \sin\mu\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\mu x\right)$$
$$+\sum_{k=1}^{\infty}\frac{1}{k}e^{-u_r k\Delta}\left(\sin u_i k\Delta\cos\mu\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\mu x - \cos\lambda\alpha k\Delta\int_{-\infty}^{0}dxp(k\Delta,x)\sin\lambda x\right).$$

By an argument similar to the proof of Theorem 3.1, we get

$$|Re(iJ_1)|\le C_2,$$

where C_2 is a positive constant which depends on α and η.

Step 1: We consider the case of $|v|\ge 2K^{1/4}\Delta^{-1/4}$. Then we have

$$K^{1/4}\Delta^{-1/4}\le|v|-|\lambda|\le|\mu|$$

Noting that for $x>0$

$$|1-e^{-x+iy}|\le|x-iy| \quad\text{and}\quad |1-e^{-x+iy}|\ge|1-e^{-x}|,$$

we get

$$\left|\frac{1-\chi_0^\Delta(u,\lambda)}{1-\chi_0^\Delta(u,\mu)}\right|\le C_1C_2\left|\frac{(|\lambda|^4+|u|+|\alpha\lambda|)\Delta}{1-e^{-(\mu^4+u_r)\Delta}}\right|^{1/2} \tag{3.5}$$
$$\le C_1C_2\left|\frac{(|\lambda|^4+|u|+|\alpha\lambda|)\Delta}{1-e^{-(K+\eta\Delta)}}\right|^{1/2}$$

Next we estimate $\chi_0^\Delta(u,\mu)$. By (3.1) and integration by parts we get

$$\log\frac{1}{1-\chi_0^\Delta(u,\mu)}=-\sum_{k=1}^{\infty}\frac{e^{-uk\Delta}}{k}\int_{-\infty}^{\alpha(k\Delta)^{3/4}}p'(1,y)\frac{e^{i\mu((k\Delta)^{1/4}y-\alpha k\Delta)}-1}{i\mu(k\Delta)^{1/4}}dy$$

and we set

$$=-J_3\,.$$

Then we estimate J_3. Noting $|e^{ix}-1|\le 2$ $(x\in\mathbf{R})$,

$$|J_3|\le\frac{V'}{|\mu|\Delta^{1/4}}\sum_{k=1}^{\infty}\frac{2}{k^{5/4}}\le\frac{C_3}{|\mu|\Delta^{1/4}},$$

where C_3 is a positive constant.

On the other hand, by $1-\chi_0^\Delta(u,\mu)=e^{-J_3}$, we get

$$|\chi_0^\Delta(u,\mu)|\le|1-e^{-J_3}|$$
$$\le|J_3|e^{|J_3|}\le\frac{C_3}{|\mu|\Delta^{1/4}}\exp\left\{\frac{C_3}{|\mu|\Delta^{1/4}}\right\}.$$

By $|\mu|=|\nu-\lambda|\ge|\nu|/2\ge K^{1/4}\Delta^{-1/4}$ we get

$$|\chi_0^\Delta(u,\mu)|\le\frac{2C_3}{|\nu|\Delta^{1/4}}e^{C_3K^{-1/4}}.$$

Hence by $|1-e^{ix}|\le 2$ $(x\in\mathbf{R})$ we get

$$\left|\frac{1-e^{-ia\nu}}{i\nu}\,\frac{1-\chi_0^\Delta(u,\lambda)}{1-\chi_0^\Delta(u,\lambda-\nu)}\,\chi_0^\Delta(u,\lambda-\nu)\right| \tag{3.6}$$
$$\le\frac{2C_1C_2C_3\Delta^{1/4}}{|\nu|^2}\left|\frac{|\lambda|^4+|u|+|\alpha\lambda|}{1-e^{-(K+\eta\Delta)}}\right|^{1/2}e^{C_3K^{-1/4}}.$$

Step 2: We consider the case of $|\nu|\le 2K^{1/4}\Delta^{-1/4}$. We know that

$$\left|\frac{1-\chi_0^\Delta(u,\lambda)}{1-\chi_0^\Delta(u,\mu)}\right|\le C_1C_2\left|\frac{1-e^{-(\lambda^4+i\alpha\lambda+u)\Delta}}{1-e^{-(\mu^4+i\alpha\mu+u)\Delta}}\right|^{1/2}.$$

Since for $x>0$ and $y\in\mathbf{R}$

$$|1-e^{-x+iy}|\ge|1-e^{-x}|\ge xe^{-x}\quad\text{and}\quad|1-e^{-x+iy}|\le|x-iy|,$$

we get

$$\left|\frac{1-\chi_0^\Delta(u,\lambda)}{1-\chi_0^\Delta(u,\mu)}\right|\le C_1C_2\left|\frac{|\lambda|^4+|\alpha\lambda|+|u_r|+|u_i|}{\mu^4+u_r}\right|^{1/2}e^{1/2(\mu^4+u_r)\Delta}$$
$$\le C_1C_2\left|\frac{|\lambda|^4+|\alpha\lambda|+|u_r|+|u_i|}{\mu^4+\eta}\right|^{1/2}e^{1/2(\mu^4+u_r)\Delta}.$$

From the assumption we have $|\mu|\le 3K^{1/4}\Delta^{-1/4}$ and so

$$|\mu^4+u_r|\Delta\le 82K.$$

Thus we get

$$\left|\frac{1-\chi_0^\Delta(u,\lambda)}{1-\chi_0^\Delta(u,\mu)}\right|\le C_1C_2e^{41K}\left|\frac{|\lambda|^4+|\alpha\lambda|+|u_r|+|u_i|}{\mu^4+\eta}\right|^{1/2}.$$

Now by (3.3) and $|1-e^{-x+iy}|\le 2$ $(x>0)$, we get

$$|\chi_0^\Delta(u,\mu)|\le 2C_3+1.$$

Note that

$$\left|\frac{1-e^{-iav}}{iv}\right| \leq |a| .$$

Therefore we obtain

$$\text{(3.7)} \qquad \left|\frac{1-e^{-iav}}{iv}\frac{1-\chi_0^\Delta(u,\lambda)}{1-\chi_0^\Delta(u,\lambda-v)}\chi_0^\Delta(u,\lambda-v)\right|$$
$$\leq |\alpha| C_1 C_2(2C_3+1)e^{41K}\left|\frac{|\lambda|^4+|\alpha\lambda|+|u_r|+|u_i|}{|\lambda-v|^4+\eta}\right|^{1/2} .$$

Summing up these estimations, we get the following. Let

$$\varphi(\Delta, v)=\frac{1-e^{-iav}}{iv}\frac{1-\chi_0^\Delta(u,\lambda)}{1-\chi_0^\Delta(u,\lambda-v)}\chi_0^\Delta(u,\lambda-v) .$$

Then we have

$$|\varphi(\Delta, v)| \leq \frac{C_5}{C_4+v^2} ,$$

where C_4 and C_5 are constants which do not depend on Δ.

Next we will calculate $\chi_a(u, \lambda)$. By Lebesgue's theorem we get

$$\chi_a(u,\lambda)=1-\frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{\Delta\to 0}\frac{1-e^{-iav}}{iv}\frac{1-\chi_0^\Delta(u,\lambda)}{1-\chi_0^\Delta(u,\lambda-v)}\chi_0^\Delta(u,\lambda-v)dv .$$

By (3.1) and (1.3)

$$\lim_{\Delta\to 0}\log\frac{1-\chi_0^\Delta(u,\lambda)}{1-\chi_0^\Delta(u,\lambda-v)}$$
$$=\int_0^\infty ds\int_{-\infty}^0 dx\frac{e^{-us}}{s}(e^{ix(\lambda-v)}-e^{ix\lambda})p(s,x+\alpha s)$$
$$=\frac{1}{2\pi}\int_0^\infty ds\int_{-\infty}^0 dx\int_{-\infty}^\infty d\xi\frac{e^{-us}}{s}(e^{ix(\lambda-v-\xi)}-e^{ix(\lambda-\xi)})e^{-(\xi^4+i\alpha\xi)s}$$
$$=\frac{1}{2\pi}\int_0^\infty ds\int_{-\infty}^0 dx\int_{-\infty}^\infty d\xi\int_u^{\infty+iu_i} dpe^{-ps}(e^{ix(\lambda-v-\xi)}-e^{ix(\lambda-\xi)})e^{-(\xi^4+i\alpha\xi)s}$$
$$=\frac{1}{2\pi}\int_{-\infty}^0 dx\int_{-\infty}^\infty d\xi\int_u^{\infty+iu_i} dp\frac{e^{ix(\lambda-v-\xi)}-e^{ix(\lambda-\xi)}}{\xi^4+i\alpha\xi+p}$$
$$=\frac{1}{2\pi}\int_{-\infty}^0 dx\int_{-\infty}^\infty d\xi\int_u^{\infty+iu_i} dp\left(\frac{e^{-ix\xi}}{(\xi+(\lambda-v))^4+i\alpha(\xi+(\lambda-v))+p}\right.$$
$$\left.-\frac{e^{-ix\xi}}{(\xi+\lambda)^4+i\alpha(\xi+\lambda)+p}\right).$$

We apply the residue theorem to the above integral with respect to $d\xi$. Let ζ_1 and ζ_2 be solutions of $\xi^4 - i\alpha\xi + u = 0$ whose imaginary parts are positive. We can prove the existence of such solutions by Sturm's theorem [6]. And we calculate the integral with respect to dp and afterward with respect to dx, then we get

$$\frac{1-\chi_0(u,\lambda)}{1-\chi_0(u,\lambda-\nu)} = \frac{(\zeta_1-\lambda)(\zeta_2-\lambda)}{(\zeta_1-\lambda+\nu)(\zeta_2-\lambda+\nu)}.$$

Thus we have

$$\chi_a(u,\lambda) = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-e^{-ia\nu}}{i\nu} \frac{(\zeta_1-\lambda)(\zeta_2-\lambda)}{(\zeta_1-\lambda+\nu)(\zeta_2-\lambda+\nu)}\,d\nu$$

$$= \frac{\zeta_2-\lambda}{\zeta_2-\zeta_1} e^{-ia(\lambda-\zeta_1)} + \frac{\zeta_1-\lambda}{\zeta_1-\zeta_2} e^{-ia(\lambda-\zeta_2)}.$$

Recall that $\chi_a(u,\lambda) = E_0[F_a(u+i\lambda\alpha,\lambda)]$. We can easily see that the solutions of $\xi^4 + i\alpha\xi + p + iq = 0$ ($p > 0$ and $q \in \mathbf{R}$) depend on the parameter q continuously and do not cross the real axis. Hence, the equation $\xi^4 + i\alpha\xi + u - i\lambda\alpha = 0$ has two solutions whose imaginary parts are positive and whose multiplicity are at most one. Replacing u by $u - i\lambda\alpha$, we obtain

$$E_0[F_a(u,\lambda)] = \frac{\zeta_2-\lambda}{\zeta_2-\zeta_1} e^{-ia(\lambda-\zeta_1)} + \frac{\zeta_1-\lambda}{\zeta_1-\zeta_2} e^{-ia(\lambda-\zeta_2)},$$

where ζ_1 and ζ_2 are solutions of $\xi^4 + i\alpha\xi + u - i\lambda\alpha = 0$ whose imaginary parts are positive. □

Finally, we note the relation between our result and the partial differential equation. Let $D = \{(t,x) : \alpha t < x\}$. We set

$$w(t,x) = E_{(t,x)}[\exp\{-u\tau(\omega) + i\lambda\omega(\tau)\}]$$

$$= \left(\frac{\zeta_2-\lambda}{\zeta_2-\zeta_1} e^{-i(\lambda-\zeta_1)(x-\alpha t)} + \frac{\zeta_1-\lambda}{\zeta_1-\zeta_2} e^{-i(\lambda-\zeta_2)(x-\alpha t)}\right) e^{-ut+i\lambda x}.$$

It is a solution of the following partial differential equation.

$$\frac{\partial w}{\partial t}(t,x) = \Delta^2 w(t,x), \qquad (t,x) \in D$$

$$\left.\begin{aligned} w(s,x) &= f(s,x) \\ \frac{\partial w}{\partial x}(s,x) &= \frac{\partial f}{\partial x}(s,x) \end{aligned}\right\} \quad (s,x) \in \partial D,$$

where f is the function defined on D^c:

$$f(s, x) = e^{-us + i\lambda x} .$$

Note that the differential condition on the boundary appears here. This means that the distribution of the first hitting place includes the differential of δ-measure on the boundary. This fact was first found in Nishioka [10].

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