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BKW-Operators for Chebyshev Systems

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Abstract. This paper is concerned with Korovkin type approximation theorems. We characterize *BKW*-operators on the Banach space of real valued continuous functions on the unit interval for the test functions $\{1, t, t^2, t^3\}$. It is also investigated when subtraction of composition operators are *BKW*-operators for $\{1, t, t^2, t^3, t^4\}$.

1. Introduction.

Let $I=[0, 1]$ be the closed unit interval and let $C(I)$ be the Banach space of all real valued continuous functions on I. For $f\in C(I)$, let $\|f\|_\infty=\sup\{|f(t)|; t\in I\}$. In 1953, Korovkin [5] proved a well known approximation theorem; if $\{T_n\}_n$ is a sequence of positive operators on $C(I)$ such that $\|T_nt^j-t^j\|_\infty\to 0$ as $n\to\infty$ for $j=0, 1, 2$, then $\{T_n\}_n$ converges strongly to the identity operator, see also [6, 11] and the recent monograph [1].

As a generalization of the Korovkin theorem, Takahasi [8, 9, 10] has been studying Korovkin type approximation theorems for other operators. In his works (see also [2]), a bounded linear operator T on $C(I)$ is called a *BKW*-operator for the Chebyshev system $S_k=\{1, t, \cdots, t^k\}$ if $\{T_n\}_n$ is a sequence of bounded linear operators on $C(I)$ such that $\|T_n\|\to\|T\|$ and $\|T_nh-Th\|_\infty\to 0$ for each $h\in S_k$, then $\|T_ng-Tg\|_\infty\to 0$ for every $g\in C(I)$, that is, $\{T_n\}_n$ converges strongly to T. We denote by $BKW(C(I); S_k)$ the set of *BKW*-operators on $C(I)$ for the test functions S_k. We note that if $T\in BKW(C(I); S_k)$ then $aT\in BKW(C(I); S_k)$ for every $a\in\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers.

In [7], Micchelli gave a characterization of positive operators in $BKW(C(I); S_k)$, see also [4, 8]. Non-positive *BKW*-operators are more difficult to describe. In [10], Takahasi gives a characterization of $BKW(C(I); S_1)$. In [3], the second author and Takahasi give a complete characterization of $BKW(C(I); S_2)$.

Let $\tilde{\mathbf{S}}_k$ be the linear span of S_k in $C(I)$. By the Riesz representation theorem, the dual space of $C(I)$ coincides with $M(I)$ the Banach space of bounded real Borel measures on I with the total variation norm. Since $M(I)$ is a dual space, we can consider the

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weak*-topology on $M(I)$. We denote by $M_1(I)$ the set of $\mu \in M(I)$ with $\|\mu\| \leq 1$. For $t \in I$, let δ_t be the unit point mass at t.

In [4], the second author and Takahasi prove the following theorem.

THEOREM A. *Let T be a norm one linear operator on $C(I)$. Then $T \in BKW(C(I); S_k)$ if and only if T has the form*

$$(Tf)(t) = \sum_{j=1}^{k+1} a_j(t) f(x_j(t)), \qquad t \in I,$$

where $\sum_{j=1}^{k+1} |a_j(t)| = 1$, $\sum_{j=1}^{k+1} a_j(t)\delta_{x_j(t)}$ moves continuously in $M_1(I)$ with respect to the weak-topology, and*

(α) *for each $t \in I$, there exists a non-constant function $f_t \in \tilde{S}_k$ such that $\|f_t\|_\infty = 1$ and $\sum_{j=1}^{k+1} a_j(t) f_t(x_j(t)) = 1$.*

To describe $BKW(C(I); S_k)$ completely, we need to give precise conditions on $a_1, a_2, \cdots, a_{k+1}$ and $x_1, x_2, \cdots, x_{k+1}$ satisfying the following condition:

(β) $\sum_{j=1}^{k+1} |a_j| = 1$ and $\sum_{j=1}^{k+1} a_j f(x_j) = 1$ for some non-constant function f in $\tilde{S}_k$ with $\|f\|_\infty = 1$.

When $a_j > 0$ for every j, it is given a characterization of $x_1, x_2, \cdots, x_{k+1}$ satisfying condition (β) in [4, 7].

In Section 2, we give a complete characterization of $BKW(C(I); S_3)$. For $k \geq 4$, the most interesting problem is; for continuous functions $x_i(t)$ with $x_i(I) \subset I$, $i = 1, 2$, when does the operator

$$(T_0 f)(t) = f(x_1(t)) - f(x_2(t)), \qquad f \in C(I)$$

belong to $BKW(C(I); S_k)$? In Section 3, we answer this problem for $k = 4$. The proof is fairly complicated. It seems difficult to give a complete answer for $k \geq 5$.

2. *BKW*-operators for S_3.

For $a \in \mathbf{R}$ with $a \neq 0$, let $\operatorname{sgn} a = 1$ if $a > 0$ and $\operatorname{sgn} a = -1$ if $a < 0$. In this section, we give a complete characterization of operators in $BKW(C(I); S_3)$.

By Theorem A, $T \in BKW(C(I); S_3)$ and $\|T\| = 1$ if and only if T has the form as

$$(\sharp) \qquad (Tf)(t) = \sum_{j=1}^{k(t)} a_j(t) f(x_j(t)), \qquad f \in C(I),$$

where

(a) $1 \leq k(t) \leq 4$, $t \in I$,

(b) $\sum_{j=1}^{k(t)} |a_j(t)| = 1$ and $a_j(t) \neq 0$ for $1 \leq j \leq k(t)$, $t \in I$,

(c) $\mu_t = \sum_{j=1}^{k(t)} a_j(t)\delta_{x_j(t)}$, $t \in I$, moves weak*-continuously in $M_1(I)$,

(d) for each $t \in I$, there exists a non-constant function $f_t \in \tilde{S}_3$ such that $\|f_t\|_\infty = 1$ and $\int_I f_t \, d\mu_t = 1$.

To describe all operators in $BKW(C(I); S_3)$ explicitly, we need to give precise conditions on μ_t satisfying (a)–(d). Let

$$f_0(t)=32t^3-48t^2+18t-1 .$$

Then $\|f_0\|_\infty=1$,

$$f_0(0)=f_0(3/4)=-1 \quad \text{and} \quad f_0(1/4)=f_0(1)=1 . \tag{2.1}$$

LEMMA 2.1. *Let $\mu=\sum_{j=1}^4 a_j\delta_{x_j}$ with $\sum_{j=1}^4|a_j|=1$, $a_j\neq 0$ for every $1\leq j\leq 4$, and $0\leq x_1<x_2<x_3<x_4\leq 1$. Then there exists a non-constant f in $\tilde{S}_3$ such that $\|f\|_\infty=1$ and $\int_I f d\mu=1$ if and only if* $\operatorname{sgn} a_1=\operatorname{sgn} a_3\neq\operatorname{sgn} a_2=\operatorname{sgn} a_4$ *and* $(x_1, x_2, x_3, x_4)=(0, 1/4, 3/4, 1)$.

PROOF. Let $f\in\tilde{S}_3$ be non-constant and $\|f\|_\infty=1$. Then $\{t\in I; |f(t)|=1\}$ contains distinct four points if and only if $f(t)=\pm f_0(t)$. By (2.1), we have our assertion. □

Let

$$D^3_{+-+}=\{(0, x, 3x); 1/4\leq x\leq 1/3\}\cup\{(x, (x+2)/3, 1); 0\leq x\leq 1/4\} .$$

Then we have the following.

LEMMA 2.2. *Let $\mu=\sum_{j=1}^3 a_j\delta_{x_j}$ with $\sum_{j=1}^3|a_j|=1$, $a_j\neq 0$ for every $1\leq j\leq 3$, and $0\leq x_1<x_2<x_3\leq 1$. Then there exists a non-constant f in $\tilde{S}_3$ such that $\|f\|_\infty=1$ and $\int_I f d\mu=1$ if and only if one of the following conditions holds.*

i) $\operatorname{sgn} a_1=\operatorname{sgn} a_3\neq\operatorname{sgn} a_2$ *and* $(x_1, x_2, x_3)\in D^3_{+-+}$,

ii) $\operatorname{sgn} a_1=\operatorname{sgn} a_2\neq\operatorname{sgn} a_3$ *and* $(x_1, x_2, x_3)=(0, 3/4, 1)$,

iii) $\operatorname{sgn} a_1\neq\operatorname{sgn} a_2=\operatorname{sgn} a_3$ *and* $(x_1, x_2, x_3)=(0, 1/4, 1)$.

PROOF. Suppose that $\operatorname{sgn} a_1=\operatorname{sgn} a_3\neq\operatorname{sgn} a_2$. Then $f\in\tilde{S}_3$ satisfies $\|f\|_\infty=1$ and $\int_I f d\mu=1$ if and only if

$$f(t)=\pm f_0(at) \quad \text{or} \quad f(t)=\pm f_0(a(t-1)+1) \qquad \text{for some} \quad 1\leq a\leq 3/4 .$$

By (2.1), we have condition i).

Suppose that $\operatorname{sgn} a_1=\operatorname{sgn} a_2\neq\operatorname{sgn} a_3$. Then $g\in\tilde{S}_3$ satisfies $\|g\|_\infty=1$ and $\int_I g d\mu=1$ if and only if $g(t)=\pm f_0(t)$ and $(x_1, x_2, x_3)=(0, 3/4, 1)$. Hence we have condition ii).

In the same way, we can prove iii). □

Let $D^3_{++}=\{(0, y); 3/4\leq y\leq 1\}\cup\{(x, 1); 0\leq x\leq 1/4\}$ and

$$D^3_{+-}=\{(0, y); 1/4\leq y\leq 1\}\cup\{(x, y)\in I^2; y\geq 3x, 3y\geq x+2\}\cup\{(x, 1); 0\leq x\leq 3/4\} .$$

Then we have the following.

LEMMA 2.3. *Let $\mu=a_1\delta_{x_1}+a_2\delta_{x_2}$ with $|a_1|+|a_2|=1$, $a_j\neq 0$ for $1\leq j\leq 2$, and $0\leq x_1<x_2\leq 1$. Then there exists a non-constant f in $\tilde{S}_3$ such that $\|f\|_\infty=1$ and $\int_I f d\mu=1$ if and only if one of the following conditions holds.*

i) $\operatorname{sgn} a_1=\operatorname{sgn} a_2$ *and* $(x_1, x_2)\in D^3_{++}$,

ii) $\operatorname{sgn} a_1\neq\operatorname{sgn} a_2$ *and* $(x_1, x_2)\in D^3_{+-}$.

PROOF. We prove only the case that $\operatorname{sgn} a_1 \neq \operatorname{sgn} a_2$ and $0<x_1<x_2<1$. In this case by (2.1), a function $f \in \tilde{S}_3$ satisfies $\|f\|_\infty = 1$ and $\int_I f d\mu = 1$ if and only if

$$f(t)=f_0(a(t-x_1)+1/4),$$

$$\frac{1}{4x_1} \leq a, \quad \frac{1}{2(1-x_1)} \leq a \leq \frac{3}{4(1-x_1)}, \tag{2.2}$$

$$a(x_2-x_1)+1/4=3/4. \tag{2.3}$$

By (2.2) and (2.3), we have $x_2 \geq 3x_1$ and $3x_2 \geq x_1+2$. □

LEMMA 2.4. *Let $\mu = a\delta_x$ with $|a|=1$ and $0 \leq x \leq 1$. Then there exists a non-constant f in $\tilde{S}_3$ such that $\|f\|_\infty = 1$ and $\int_I f d\mu = 1$.*

Summing up these results, we have the following theorem.

THEOREM 2.1. *Let T be a bounded linear operator on $C(I)$ with $\|T\|=1$. Then $T \in BKW(C(I); S_3)$ if and only if T has the form (#) with* (a), (b), *and* (c), *and the following conditions are satisfied; for $t \in I$ with $0 \leq x_1(t) < \cdots < x_k(t) \leq 1$,*

i) *If $k(t)=4$, then $\operatorname{sgn} a_1(t) = \operatorname{sgn} a_3(t) \neq \operatorname{sgn} a_2(t) = \operatorname{sgn} a_4(t)$ and $(x_1(t), x_2(t), x_3(t), x_4(t)) = (0, 1/4, 3/4, 1)$.*

ii) *If $k(t)=3$, then one of the following conditions holds.*

1) $\operatorname{sgn} a_1(t) = \operatorname{sgn} a_3(t) \neq \operatorname{sgn} a_2(t)$ *and* $(x_1(t), x_2(t), x_3(t)) \in D^3_{+-+}$,

2) $\operatorname{sgn} a_1(t) = \operatorname{sgn} a_2(t) \neq \operatorname{sgn} a_3(t)$ *and* $(x_1(t), x_2(t), x_3(t)) = (0, 3/4, 1)$,

3) $\operatorname{sgn} a_1(t) \neq \operatorname{sgn} a_2(t) = \operatorname{sgn} a_3(t)$ *and* $(x_1(t), x_2(t), x_3(t)) = (0, 1/4, 1)$.

iii) *If $k(t)=2$, then one of the following conditions holds.*

4) $\operatorname{sgn} a_1(t) = \operatorname{sgn} a_2(t)$ *and* $(x_1(t), x_2(t)) \in D^3_{++}$,

5) $\operatorname{sgn} a_1(t) \neq \operatorname{sgn} a_2(t)$ *and* $(x_1(t), x_2(t)) \in D^3_{+-}$.

Now we study a special type of operators. Let

$$(Tf)(t) = a_1(t) f(x_1(t)) + a_2(t) f(x_2(t)), \qquad f \in C(I), \tag{2.4}$$

where

(2.5) $a_j(t)$ and $x_j(t)$ are continuous functions with $x_j(I) \subset I$ for $j=1, 2$,

$$|a_1(t)| + |a_2(t)| = 1, \qquad t \in I. \tag{2.6}$$

In this case, $a_1(t)\delta_{x_1(t)} + a_2(t)\delta_{x_2(t)}$, $0 \leq t \leq 1$, moves weak*-continuously in $M(I)$. When $a_j(t)$, $j=1, 2$, are constant functions and $\operatorname{sgn} a_1(t) \neq \operatorname{sgn} a_2(t)$, Takahasi [9, Theorem 4] gave a sufficient condition for which $T \in BKW(C(I); S_3)$. For a subset E of I^2, let

$$\hat{E} = \{(x, y); (y, x) \in E\}.$$

Then E and $\hat{E}$ are symmetric with respect to the line $y=x$.

Let $(x, y) \in I^2$. Then by Theorem 2.1, $(x, y) \in D^3_{+-} \cup \hat{D}^3_{+-}$ if and only if there exists f in $\tilde{S}_3$ with $\|f\|_\infty = 1$ such that $|f(x)| = |f(y)| = 1$ and $\operatorname{sgn} f(x) \neq \operatorname{sgn} f(y)$. Also

$(x, y) \in D^3_{++} \cup \hat{D}^3_{++}$ if and only if there exists f in $\tilde{S}_3$ with $\|f\|_\infty = 1$ such that $|f(x)| = |f(y)| = 1$ and $\operatorname{sgn} f(x) = \operatorname{sgn} f(y)$. By this fact, we have the following corollary.

COROLLARY 2.1. *Let T be an operator satisfying* (2.4), (2.5) *and* (2.6). *Then $T \in BKW(C(I); S_3)$ and $\|T\| = 1$ if and only if the following condition holds; for $t \in I$ with $a_1(t) \neq 0$ and $a_2(t) \neq 0$, it holds that*

i) *if $\operatorname{sgn} a_1(t) = \operatorname{sgn} a_2(t)$, then $x_1(t) = x_2(t)$ or $(x_1(t), x_2(t)) \in D^3_{++} \cup \hat{D}^3_{++}$,*
ii) *if $\operatorname{sgn} a_1(t) \neq \operatorname{sgn} a_2(t)$, then $(x_1(t), x_2(t)) \in D^3_{+-} \cup \hat{D}^3_{+-}$.*

COROLLARY 2.2. *Let T be an operator satisfying* (2.4), (2.5) *and* (2.6). *Moreover, suppose that $a_1(0) = 1$, $(x_1(0), x_2(0)) \in D^3_{++} \cup D^3_{+-}$, $a_2(1) = -1$, and $(x_1(1), x_2(1)) \in \hat{D}^3_{++} \cup \hat{D}^3_{+-}$. If $T \in BKW(C(I); S_3)$ and $\|T\| = 1$, then either $a_1(t)$ or $a_2(t)$ vanishes on some non-empty open subset of I.*

PROOF. By our assumptions, $(x_1(t), x_2(t))$, $0 \leq t \leq 1$, is a continuous map and there exists a non-empty open subinterval J of I such that

$$\{(x_1(t), x_2(t));\, t \in J\} \cap (\{(x, x);\, 0 \leq x \leq 1\} \cup D^3_{++} \cup \hat{D}^3_{++} \cup D^3_{+-} \cup \hat{D}^3_{+-}) = \varnothing\,.$$

By Corollary 2.1, we have that $a_1(t)a_2(t) = 0$ on J. Since $a_1(t)$ and $a_2(t)$ are continuous, we have our assertion. □

3. *BKW*-operators for S_4.

Let $I = [0, 1]$ and $I_0 = (0, 1)$. Let $x_j(t), j = 1, 2$, be continuous functions on I with $x_j(I) \subset I$. For $f \in C(I)$, let

$$(T_0 f)(t) = f(x_1(t)) - f(x_2(t))\,, \qquad t \in I\,.$$

In this section, we investigate the conditions on x_1 and x_2 for which $T_0 \in BKW(C(I); S_4)$. We note that by [4, 7, 8], the operator

$$(Tf)(t) = f(x_1(t)) + f(x_2(t))\,, \qquad f \in C(I)$$

belongs to $BKW(C(I); S_4)$.

Let

(3.1) $\quad G = \{(x, y) \in I^2;\, x < y,\, f(x) = 1 \text{ and } f(y) = -1 \text{ for some } f \in \tilde{S}_4,\ \|f\|_\infty = 1\}\,.$

To study whether $T_0 \in BKW(C(I); S_4)$, as in Section 2 we need to describe the set G. The following is the description of G.

THEOREM 3.1. *G is the union of the following seven subsets.*

i) $\{(0, y) \in I^2;\, (2-\sqrt{2})/4 \leq y \leq 1\}$,
ii) $\{(x, 1) \in I^2;\, 0 \leq x \leq (2+\sqrt{2})/4\}$,
iii) $\left\{(x, y) \in I_0^2;\, y \geq \dfrac{2-\sqrt{2}}{2}(x-1)+1 \text{ and } y \geq \dfrac{2+\sqrt{2}}{2}x\right\}$,
iv) $\{(x, y) \in I_0^2;\, y \geq (2-\sqrt{2})(x-1)+1 \text{ and } y \geq (2+\sqrt{2})x\}$,

v) $\{(x, y) \in I_0^2;\ y > \frac{1}{2}(x-1)+1$ *and* $y > 3x\}$,
vi) $\{(x, y) \in I_0^2;\ y > \frac{1}{3}(x-1)+1$ *and* $y > 2x\}$,
vii) $\{(1/4, 3/4)\}$.

The proof owes to elementary calculation, but the situation is a little bit complicated. First, we give two lemmas.

LEMMA 3.1. *If* $(x_0, y_0) \in G$, *then* $(x_0, y) \in G$ *for* $y_0 \leq y \leq 1$, *and* $(x, y_0) \in G$ *for* $0 \leq x \leq x_0$.

PROOF. Since $(x_0, y_0) \in G$, by (3.1) there exists $f \in \tilde{S}_4$ such that $\|f\|_\infty = 1, f(x_0) = 1$, and $f(y_0) = -1$. For $y_0 \leq y \leq 1$, let

$$h(t) = \frac{y_0 - x_0}{y - x_0}(t - x_0) + x_0\,, \qquad t \in I\,,$$

$$F(t) = f(h(t))\,, \qquad t \in I\,.$$

Then $F \in \tilde{S}_4$, $F(x_0) = f(x_0) = 1$ and $F(y) = f(y_0) = -1$. Since $0 \leq h(0) \leq 1$ and $0 \leq h(1) \leq 1$, we have $\|F\|_\infty = 1$. Thus $(x_0, y) \in G$. In the same way, we have the second assertion. □

LEMMA 3.2. *If* $(x_0, y_0) \in G$, *then* $(1 - y_0, 1 - x_0) \in G$.

PROOF. Since $(x_0, y_0) \in G$, there exists $f \in \tilde{S}_4$ such that $\|f\|_\infty = 1$, $f(x_0) = 1$, and $f(y_0) = -1$. Let $F(t) = -f(1-t)$. Then $F \in \tilde{S}_4$, $\|F\|_\infty = 1$, $F(1 - y_0) = -f(y_0) = 1$, and $F(1 - x_0) = -f(x_0) = -1$. □

For a subset E of I^2, let $\tilde{E} = \{(x, y) \in I^2;\ (1-y, 1-x) \in E\}$. Then E and $\tilde{E}$ are symmetric with respect to the line $x + y = 1$. By Lemma 3.2, we have $G = \tilde{G}$. We note that the sets i) and ii), iii) and iv), v) and vi) in Theorem 3.1 are symmetric with respect to the line $x + y = 1$, respectively.

PROOF OF THEOREM 3.1. The proof is long, so we devide into five steps.

STEP 1. In this step, we describe the set $G \cap \partial I^2$. The following fact shows that $G \cap \partial I^2$ is given by i) and ii) in Theorem 3.1.

FACT 1. $\{y \in I;\ (0, y) \in G\} = \{y \in I;\ (2 - \sqrt{2})/4 \leq y \leq 1\}$ *and* $\{x \in I;\ (x, 1) \in G\} = \{x \in I;\ 0 \leq x \leq (2 + \sqrt{2})/4\}$.

PROOF. We shall prove that

$$\{y \in I;\ (0, y) \in G\} = \{y \in I;\ (2 - \sqrt{2})/4 \leq y \leq 1\}\,. \tag{3.2}$$

Since $G = \tilde{G}$, by (3.2) we get

$$\{x \in I;\ (x, 1) \in G\} = \{x \in I;\ 0 \leq x \leq (2 + \sqrt{2})/4\}\,.$$

Let

$$\alpha = \inf\{y \in I;\ f(0) = 1,\ f(y) = -1 \text{ for some } f \in \tilde{S}_4,\ \|f\|_\infty = 1\}\,. \tag{3.3}$$

By (3.1), to prove (3.2) it is sufficient to prove that "inf" in (3.3) is attained and

$$\alpha=(2-\sqrt{2})/4\,. \tag{3.4}$$

Let Γ be the set of $f\in\tilde{S}_4$ such that $f(-1)=1$, $f(0)=0$, and $f'(0)=0$. For $f\in\Gamma$, let

$$A_f=\max\{x;\,0\le f(t)\le 1 \text{ on } [-1,x]\}\,, \tag{3.5}$$

$$A=\sup\{A_f;\,f\in\Gamma\}\,. \tag{3.6}$$

Then it is not difficult to see that $\alpha=1/(1+A)$. Hence by (3.4), to prove (3.2) it is sufficient to prove that "sup" in (3.6) is attained and

$$A=3+2\sqrt{2}\,. \tag{3.7}$$

By (3.5) and (3.6), it is clear that $A>1$. In the rest, we shall calculate the value of A. Our strategy is to find smaller subsets of Γ still satisfying (3.6).

Let Γ_1 be the set of $f\in\Gamma$ such that $A_f>1$. Then we have

$$A=\sup\{A_f;\,f\in\Gamma_1\}\,. \tag{3.8}$$

Let $f\in\Gamma_1$. By the definition of Γ, we can write as

$$f(t)=t^2h_f(t)\,, \tag{3.9}$$

$$h_f(t)=a_2t^2+a_1t+a_0\,,\quad h_f(-1)=1\,. \tag{3.10}$$

By (3.5),

$$h_f(t)\ge 0 \quad \text{on} \quad [-1,A_f]\,. \tag{3.11}$$

Also by (3.5),

$$f(A_f)=0 \quad \text{or} \quad 1\,. \tag{3.12}$$

Let Γ_2 be the set of $g\in\Gamma_1$ such that $g(A_g)=1$. When $f\in\Gamma_1$ and $f(A_f)=0$, we shall prove the existence of $G_f\in\Gamma_2$ such that

$$A_f<A_{G_f}\,. \tag{3.13}$$

Since $f(A_f)=0$ and $A_f>1$, by (3.5) and (3.9),

$$h_f(A_f)=0 \quad \text{and} \quad h_f(A_f+\varepsilon)<0 \tag{3.14}$$

for every $\varepsilon>0$ sufficiently closed to 0. Put

$$g_f(t)=\left(\frac{t-A_f}{1+A_f}\right)^2 \quad \text{and} \quad G_f(t)=t^2g_f(t)\,. \tag{3.15}$$

Then by (3.10) and (3.14), $h_f(t)\ge g_f(t)$ on $[-1,A_f]$. Hence by (3.9) and (3.15), $0\le G_f(t)\le f(t)\le 1$ on $[-1,A_f]$. Since $G_f(A_f)=0$ and $G_f(t)\ge 0$ for every x, we get (3.13).

By (3.13) and (3.15), $G_f(A_{G_f})\ne 0$. Hence by (3.12), $G_f(A_{G_f})=1$, so that $G_f\in\Gamma_2$. Therefore by (3.8) and (3.13), we have

$$A = \sup\{A_g;\ g \in \Gamma_2\}\,. \tag{3.16}$$

Now let Γ_3 be the set of $f(t) = t^2 h_f(t) \in \Gamma_2$ such that $h_f(t_0) = 0$ for some $-1 < t_0 < A_f$. We shall prove that

$$A = \sup\{A_g;\ g \in \Gamma_3\}\,. \tag{3.17}$$

To prove this, let $f \in \Gamma_2$ such that $h_f(t) \neq 0$ for every $-1 \leq t \leq A_f$. Then by (3.11), $h_f(t) = a_2 t^2 + a_1 t + a_0 > 0$ on $[-1, A_f]$. Since $f \in \Gamma_1$, $A_f > 1$. Since $f \in \Gamma_2$, $f(A_f) = 1$, so that $0 < h_f(A_f) < 1$. Hence it is not difficult to find a function $\psi(t) = b_2 t^2 + b_1 t + b_0$ such that $\psi(-1) = 1$, $0 \leq \psi(t) < h_f(t)$ on $(-1, A_f]$, and $\psi(t_0) = 0$ for some t_0, $0 < t_0 < A_f$. Put $\Psi(t) = t^2 \psi(t)$. Then $0 \leq \Psi(t) \leq 1$ on $[-1, A_f]$ and $0 < \Psi(A_f) < 1$. Hence $A_\Psi > A_f$. Since $\Psi(A_\Psi) \neq 0$, by (3.12) $\Psi(A_\Psi) = 1$. Therefore $\Psi \in \Gamma_3$, and by (3.16) we have (3.17).

At the last stage, let $g \in \Gamma_3$. Then $g(A_g) = 1$ and $h_g(\sigma_1) = 0$ for some σ_1, $0 < \sigma_1 < A_g$. Then

$$g(t) = t^2 \left(\frac{t - \sigma_1}{1 + \sigma_1} \right)^2 . \tag{3.18}$$

It holds that $0 \leq g(t) \leq 1$ on $[0, \sigma_1]$. Here suppose that

$$0 \leq g(t) < 1 \quad \text{on} \quad [0, \sigma_1]\,. \tag{3.19}$$

For $\sigma > \sigma_1$, let

$$G(t) = t^2 \left(\frac{t - \sigma}{1 + \sigma} \right)^2 . \tag{3.20}$$

By (3.18) and (3.19), we may assume that

$$0 \leq G(t) \leq 1 \quad \text{on} \quad [0, \sigma]\,, \tag{3.21}$$

$$G(y) = 1 \qquad \text{for some} \quad y, 0 < y < \sigma\,. \tag{3.22}$$

Since $\sigma_1 < \sigma$,

$$\left(\frac{t - \sigma}{1 + \sigma} \right)^2 < \left(\frac{t - \sigma_1}{1 + \sigma_1} \right)^2 \quad \text{on} \quad [\sigma, \infty)\,.$$

Hence by (3.18) and (3.20), we have that

$$A_G > A_g \quad \text{and} \quad G \in \Gamma_3\,. \tag{3.23}$$

Here we note that σ satisfying (3.21) and (3.22) is unique. By calculation, we have $\sigma = 2 + \sqrt{2}$ and

$$G(t) = (17 - 12\sqrt{2})t^2(t - 2 - 2\sqrt{2})^2\,. \tag{3.24}$$

By (3.17) and (3.23), we have $A = A_G$. Hence by (3.24), we obtain (3.7). This completes the proof of Fact 1. $\square$

STEP 2. Now we shall study the domain $G \cap I_0^2$. Let $0 < x_0 < y_0 < 1$. Then by (3.1), $(x_0, y_0) \in G \cap I_0^2$ if and only if there exists $h \in \tilde{S}_4$ such that $\|h\|_\infty = 1$, $h(x_0) = 1$ and $h(y_0) = -1$.

Let

$$G_0 = \{(x, y) \in I_0^2;\ x < y,\ h(x) = 1,\ h(y) = -1 \text{ for some } h \in \tilde{S}_4 \backslash \tilde{S}_3,\ \|h\|_\infty = 1\},$$

$$G_1 = \{(x, y) \in I_0^2;\ x < y,\ h(x) = 1,\ h(y) = -1 \text{ for some } h \in \tilde{S}_3,\ \|h\|_\infty = 1\}.$$

Then in the same way as the proof of Lemma 3.2,

$$G_0 = \tilde{G}_0, \tag{3.25}$$

$G_1 = \tilde{G}_1$, and

$$G \cap I_0^2 = G_0 \cup G_1. \tag{3.26}$$

By Lemma 2.3, we have

$$G_1 = \{(x, y) \in I_0^2;\ y \geq 3x,\ 3y \geq x + 2\}. \tag{3.27}$$

Hence to complete our proof, we need to determine the domain G_0.

Let $f_\zeta(t) = t^2(t+1)(t-\zeta)$, $1 \leq \zeta < \infty$. Then every $h \in \tilde{S}_4 \backslash \tilde{S}_3$ has the following form

$$h(t) = cf_\zeta(at + b) + d, \quad a, b, c, d \in \mathbf{R}, \quad a \neq 0, \quad c \neq 0, \quad 1 \leq \zeta < \infty. \tag{3.28}$$

Let $H = \{h \in \tilde{S}_4 \backslash \tilde{S}_3;\ h$ has the form (3.28) and $\|h\|_\infty = 1\}$. Then $(x_0, y_0) \in G_0$ if and only if $h(x_0) = 1$ and $h(y_0) = -1$ for some $h \in H$. Let

$$H_0 = \{h \in H;\ h \text{ has the form (3.28) with } a > 0\},$$

$$\Omega = \{(x, y);\ 0 < x < y < 1,\ h(x) = 1,\ h(y) = - \text{ for some } h \in H_0\}.$$

Then $\Omega \subset G_0$ and we have the following.

FACT 2. $G_0 = \Omega \cup \tilde{\Omega}$.

PROOF. By (3.25), $G_0 = \tilde{G}_0$. Since $\Omega \subset G_0$, $\Omega \cup \tilde{\Omega} \subset G_0$. To prove the converse inclusion, let $(x_0, y_0) \in G_0$. Suppose that $h(x_0) = 1$ and $h(y_0) = -1$ for some $h \in H$. Let $h(t) = cf_\zeta(at + b) + d$. When $a > 0$, we have $(x_0, y_0) \in \Omega$.

Suppose that $a < 0$. Let $h_1(t) = -h(1 - t) = -cf_\zeta(-at + a + b) - d$. Then $h_1 \in H_0$, $h_1(1 - y_0) = 1$ and $h_1(1 - x_0) = -1$. Hence $(1 - y_0, 1 - x_0) \in \Omega$. Therefore $(x_0, y_0) \in \tilde{\Omega}$. □

STEP 3. By Fact 2, to describe G_0 we need to describe Ω. For each ζ with $1 \leq \zeta < \infty$, let

$$H_\zeta = \{h;\ h = cf_\zeta(at + b) + d,\ \|h\|_\infty = 1,\ a > 0,\ b,\ c,\ d \in \mathbf{R},\ c \neq 0\}, \tag{3.29}$$

$$\Omega_\zeta = \{(x, y);\ 0 < x < y < 1,\ h(x) = 1,\ h(y) = -1 \text{ for some } h \in H_\zeta\}. \tag{3.30}$$

In this step, we study Ω_ζ. By the definitions, $H_0 = \bigcup_{\zeta \geq 1} H_\zeta$,

$$\Omega=\bigcup_{\zeta\geq 1}\Omega_\zeta\,, \tag{3.31}$$

and Ω_ζ has the same property as G in Lemma 3.1. In the rest of this step, we fix ζ. When $(x_0, y)\in\Omega_\zeta$ for some y, there exists y_0 such that $(x_0, y_0)\in\Omega_\zeta$ and $(x_0, y')\notin\Omega_\zeta$ for every $y'<y_0$. Thus to describe Ω_ζ, it is sufficient to describe points $(x_0, y_0)\in\Omega_\zeta$ such that $(x_0, y)\notin\Omega$ for $y<y_0$.

Recall that $f_\zeta(t)=t^2(t+1)(t-\zeta)$. Hence there exist $A=A(\zeta)$ and $B=B(\zeta)$ such that $f'_\zeta(A)=f'_\zeta(B)=0$, $-1<A<0$, and $0<B<\zeta$. Then $(A, f_\zeta(A))$ and $(B, f_\zeta(B))$ are local minimal points in the graph of f_ζ, and $0>f_\zeta(A)\geq f_\zeta(B)$. Also there exists $C=C(\zeta)$ such that $0<C\leq B$ and $f_\zeta(A)=f_\zeta(C)$. We note that $A(\zeta)$, $B(\zeta)$, and $C(\zeta)$ are continuous functions in ζ, $1\leq\zeta<\infty$.

We study Ω_ζ mainly for $1<\zeta<\infty$. In this case,

$$C<B \quad\text{and}\quad f_\zeta(C)>f_\zeta(B)\,. \tag{3.32}$$

Let $(x_0, y_0)\in\Omega_\zeta$ such that $(x_0, y)\notin\Omega_\zeta$ for every $y<y_0$. By (3.30), there exists $h\in H_\zeta$ such that

$$h(x_0)=1 \quad\text{and}\quad h(y_0)=-1\,. \tag{3.33}$$

Then we have

$$|h(0)|=1 \quad\text{or}\quad |h(1)|=1\,. \tag{3.34}$$

To prove (3.34), suppose not. Then $|h(0)|<1$ and $|h(1)|<1$. For $\alpha>0$, let

$$g_\alpha(t)=h(\alpha(t-x_0)+x_0)\,,\qquad t\in I\,.$$

Then there exists $\alpha_0>1$ such that $\|g_{\alpha_0}\|_\infty=1$. Since $h\in H_\zeta$, by (3.29) $g_\alpha\in H_\zeta$. We have $g_{\alpha_0}(x_0)=h(x_0)=1$,

$$x_0<\frac{y_0-x_0}{\alpha_0}+x_0<y_0 \quad\text{and}\quad g_{\alpha_0}\left(\frac{y_0-x_0}{\alpha_0}+x_0\right)=h(y_0)=-1\,.$$

This contradicts that $(x_0, y)\notin\Omega_\zeta$ for every $y<y_0$. Hence we obtain (3.34).

Since $h\in H_\zeta$, h has the following form

$$h(t)=cf_\zeta(at+b)+d\,,\quad a>0\,,\quad b, c, d\in\mathbf{R}\,,\quad c\neq 0\,,\quad \|h\|_\infty=1\,. \tag{3.35}$$

When $1<\zeta<\infty$, by (3.32), (3.33), (3.34), (3.35) and the graph of f_ζ, the following four cases occur. Cases 1, 2, 3 and 4 correspond to cases $h(0)=1$, $h(1)=1$, $h(0)=-1$, and $h(1)=-1$, respectively.

Case 1. $h(t)=-2f_\zeta(at+b)/f_\zeta(B)+1$,

$$ax_0+b=0\,,\quad ay_0+b=B\,,\quad b=-1\,,\quad B<a+b\leq\zeta\,.$$

Case 2. $h(t)=-2f_\zeta(at+b)/f_\zeta(B)+1$,

$$ax_0+b=0\,,\quad ay_0+b=\mathrm{B}\,,\quad -1\le b<0\,,\quad a+b=\zeta\,.$$

Case 3. $h(t)=2f_\zeta(at+b)/f_\zeta(A)-1,$

$$ax_0+b=A\,,\quad ay_0+b=0\,,\quad b=-1\,,\quad 0<a+b\le C\,.$$

Case 4. $h(t)=2f_\zeta(at+b)/f_\zeta(A)-1,$

$$ax_0+b=A\,,\quad ay_0+b=0\,,\quad -1\le b<A\,,\quad a+b=C\,.$$

Here we give some remarks when $\zeta=1$. In this case, C is replaced by 1 in Cases 3 and 4. And (x_0, y_0) satisfies the conditions in Cases 1 and 2 if and only if $(1-y_0, 1-x_0)$ satisfies the conditions in Cases 4 and 3 with $C=1$, respectively. Since $G_0=\tilde{G}_0$, when $\zeta=1$ it is sufficient to consider only Cases 1 and 2. Hence we study Cases 1 and 2 for $\zeta\ge1$, and Cases 3 and 4 for $\zeta>1$.

The set of points (x_0, y_0) which satisfy the condition of each case coincides with the line segment in I_0^2 jointing the following two points in I^2, respectively.

Case 1. $\left(\dfrac{1}{\zeta+1}, \dfrac{B+1}{\zeta+1}\right)$ and $\left(\dfrac{1}{B+1}, 1\right)$.

Case 2. $\left(0, \dfrac{B}{\zeta}\right)$ and $\left(\dfrac{1}{\zeta+1}, \dfrac{B+1}{\zeta+1}\right)$.

Case 3. $\left(\dfrac{A+1}{C+1}, \dfrac{1}{C+1}\right)$ and $(A+1, 1)$.

Case 4. $\left(0, \dfrac{-A}{C-A}\right)$ and $\left(\dfrac{A+1}{C+1}, \dfrac{1}{C+1}\right)$.

To describe Ω_ζ more explicitly, let $p_\zeta(t)$ and $q_\zeta(t)$ be the functions representing the joint line segments obtained by Cases 1 and 2, and Cases 3 and 4, respectively. Then

$$p_\zeta(t)=\begin{cases}\dfrac{\zeta-B}{\zeta}(t-1)+1\,, & \text{if } 0<t\le\dfrac{1}{\zeta+1}\\[2ex] (B+1)t\,, & \text{if } \dfrac{1}{\zeta+1}\le t<\dfrac{1}{B+1}\end{cases} \tag{3.36}$$

and $\left(\frac{1}{\zeta+1}, \frac{B+1}{\zeta+1}\right)$ is its jointing point, and

$$q_\zeta(t)=\begin{cases}\dfrac{C}{C-A}(t-1)+1\,, & \text{if } 0<t\le\dfrac{A+1}{C+1}\\[2ex] \dfrac{1}{A+1}t\,, & \text{if } \dfrac{A+1}{C+1}\le t<A+1\end{cases} \tag{3.37}$$

and $\left(\frac{A+1}{C+1}, \frac{1}{C+1}\right)$ is its jointing point. Let

$$V_\zeta=\{(x, y)\in I_0^2;\, p_\zeta(x)\le y,\, 0<x<1/(B+1)\} \quad \text{for} \quad \zeta\ge 1\,, \tag{3.38}$$

$$W_\zeta=\{(x, y)\in I_0^2;\, q_\zeta(x)\le y,\, 0<x<A+1\} \quad \text{for} \quad \zeta>1\,. \tag{3.39}$$

Then by our argument, we obtain

$$\Omega_\zeta=V_\zeta\cup W_\zeta \quad \text{for} \quad \zeta>1\,, \tag{3.40}$$

$$\Omega_1=V_1\cup \tilde{V}_1\,. \tag{3.41}$$

STEP 4. In this step, we shall prove that

$$V_1=\bigcup_{\zeta\ge 1} V_\zeta\,. \tag{3.42}$$

Here we note that

$$p_1(t)=\begin{cases} \frac{2-\sqrt{2}}{2}(t-1)+1\,, & \text{if} \quad 0<t\le 1/2\,, \\ \frac{2+\sqrt{2}}{2}\,t, & \text{if} \quad 1/2\le t<2-\sqrt{2}\,, \end{cases} \tag{3.43}$$

$$V_1=\{(x, y)\in I_0^2;\, p_1(x)\le y,\, 0<x<2-\sqrt{2}\}\,. \tag{3.44}$$

By elementary calculation, we have

$$A(\zeta)=\frac{3\zeta-3-\sqrt{9\zeta^2+14\zeta+9}}{8}\,,$$

$$B(\zeta)=\frac{3\zeta-3+\sqrt{9\zeta^2+14\zeta+9}}{8}\,. \tag{3.45}$$

Hence $A(\zeta)$ and $B(\zeta)$ are increasing functions in $1\le\zeta<\infty$, and

$$A(1)=-1/\sqrt{2}\,, \quad \lim_{\zeta\to\infty} A(\zeta)=-2/3\,, \tag{3.46}$$

$$B(1)=1/\sqrt{2}\,, \quad \lim_{\zeta\to\infty} B(\zeta)=\infty\,. \tag{3.47}$$

Since $B(\zeta)$ is increasing function, by (3.47) we have $1/(B+1)\le 2-\sqrt{2}$ for $\zeta\ge 1$. By (3.45), we can also prove that

$$B/\zeta\ge 1/\sqrt{2} \quad \text{for} \quad \zeta\ge 1\,.$$

Hence by (3.36), (3.38), (3.43) and (3.44), we get (3.42).

By (3.26), (3.31), (3.40), (3.41), (3.42) and Fact 2, we obtain the sets iii) and iv) in Theorem 3.1. The sets iii) and iv) coincide with V_1 and $\tilde{V}_1$, respectively.

STEP 5. In this step, we study W_ζ, $1<\zeta<\infty$ and complete the proof of our theorem. Let q_∞ be the limit function of q_ζ as $\zeta\to\infty$. To write q_∞ explicitly, we need the value of $\lim_{\zeta\to\infty} C(\zeta)$. By elementary calculation, we have

$$C(\zeta)=\frac{\zeta-1-2A-\sqrt{(2A+1-\zeta)^2-4(3A^2+2(1-\zeta)A-\zeta)}}{2}. \tag{3.48}$$

By (3.46), we get $\lim_{\zeta\to\infty} C(\zeta)=2/3$. Hence

$$q_\infty(t)=\begin{cases}(t-1)/2+1, & \text{if } 0<t\le 1/5\\ 3t, & \text{if } 1/5\le t<1/3.\end{cases} \tag{3.49}$$

Let

$$W_\infty=\{(x,y)\in I_0^2;\, q_\infty(x)<y,\quad 0<x<1/3\}. \tag{3.50}$$

We note that the boundary of W_∞ is not contained in W_∞. We shall prove that

$$W_\infty=\bigcup_{\zeta>1} W_\zeta. \tag{3.51}$$

Let $\zeta>1$. Since the vertices of W_ζ converge to the vertices of W_∞, it is sufficient to prove

$$W_\infty\supset W_\zeta. \tag{3.52}$$

By (3.37), (3.39), (3.49), (3.50) and $A+1<1/3$, to prove (3.52) it is sufficient to show

$$\frac{1}{2}\left(\frac{A+1}{C+1}-1\right)+1<\frac{1}{C+1}.$$

This inequality is the same as $A+C<0$. By (3.48), this is equivalent to

$$2A^2+(1-\zeta)A-\zeta<0. \tag{3.53}$$

By (3.46), $-1/\sqrt{2}<A(\zeta)<-2/3$. Hence to prove (3.53) we need to show that

$$2(-1/\sqrt{2})^2+(1-\zeta)(-1/\sqrt{2})-\zeta<0 \quad\text{and}\quad -\zeta<0.$$

Since $\zeta>1$, the above holds. Threfore we get (3.51).

As the final step, we have

$$\begin{aligned} G\cap I_0^2 &= G_1\cup G_0 \quad \text{by (3.26)}\\ &= G_1\cup\Omega\cup\tilde{\Omega} \quad \text{by Fact 2}\\ &= G_1\cup\left(\bigcup_{\zeta\ge 1}(\Omega_\zeta\cup\tilde{\Omega}_\zeta)\right) \quad \text{by (3.31)}\\ &= G_1\cup(V_1\cup\tilde{V}_1)\cup(W_\infty\cup\tilde{W}_\infty) \quad \text{by (3.40), (3.41), (3.42) and (3.51)}. \end{aligned}$$

The sets v) and vi) coincide with W_∞ and $\tilde{W}_\infty$, respectively. Now by (3.27), it is easy to see that $G_1 \backslash (V_1 \cup \tilde{V}_1 \cup W_\infty \cup \tilde{W}_\infty) = \{(1/4, 3/4)\}$. This completes the proof of Theorem 3.1. □

By Theorems A and 3.1, and the definition of G, we have the following theorem.

THEOREM 3.2. *Let x_j, $j=1, 2$, be continuous functions on I with $x_j(I) \subset I$. For $f \in C(I)$, let*

$$(T_0 f)(t) = f(x_1(t)) - f(x_2(t)).$$

Then $T_0 \in BKW(C(I); S_4)$ if and only if one of the following conditions holds.
i) $(x_1(t), x_2(t)) \in G$ *for every* $t \in I$,
ii) $(x_2(t), x_1(t)) \in G$ *for every* $t \in I$.

THEOREM 3.3. *Let $a_j(t)$ and $x_j(t)$, $j=1, 2$, are continuous functions on I such that $|a_1(t)| + |a_2(t)| = 1$ for every $t \in I$ and $x_j(I) \subset I$. For $f \in C(I)$, let*

$$(Tf)(t) = a_1(t) f(x_1(t)) + a_2(t) f(x_2(t)).$$

Then $T \in BKW(C(I); S_4)$ if and only if the following condition holds; if $a_1(t)a_2(t) \neq 0$ and $\operatorname{sgn} a_1(t) \neq \operatorname{sgn} a_2(t)$, then either $(x_1(t), x_2(t))$ or $(x_2(t), x_1(t))$ is contained in G.

PROOF. By Theorem A, $T \in BKW(C(I); S_4)$ if and only if for each $t \in I$ there exists a non-constant $f_t \in \tilde{S}_4$ such that

$$\|f_t\|_\infty = 1 \quad \text{and} \quad a_1(t) f_t(x_1(t)) + a_2(t) f_t(x_2(t)) = 1. \tag{3.54}$$

If $a_1(t)a_2(t) = 0$, it is not difficult to see the existence of $f_t \in \tilde{S}_4$ satisfying (3.54). If $a_1(t)a_2(t) \neq 0$ and $\operatorname{sgn} a_1(t) = \operatorname{sgn} a_2(t)$, by [4, 7] we can find $f_t \in \tilde{S}_4$ satisfying (3.54). When $a_1(t)a_2(t) \neq 0$ and $\operatorname{sgn} a_1(t) \neq \operatorname{sgn} a_2(t)$, Theorem 3.1 implies that either $(x_1(t), x_2(t))$ or $(x_2(t), x_1(t))$ is contained in G if and only if there exists $f_t \in \tilde{S}_4$ satisfying (3.54). □

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