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Purely Periodic Points of Complex Pisot Expansions

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Abstract. Using a "complex Pisot number" $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, the numerical expansion $\sum_{j=-k}^{\infty} a_j/\lambda^j$ of a complex number, where each digit a_j is chosen from some finite set Γ of $\mathbb{Z}[\lambda]$, was established recently as an analogue of β-numeration system $\sum_{j=-k}^{\infty} b_j/\beta^j$ of a real number, where $b_j \in \{0, 1, \ldots, \lfloor\beta\rfloor\}$.

In this paper, we give a necessary and sufficient condition for a complex number to have eventually or purely periodic complex Pisot expansion.

1. Introduction

Let $\beta > 1$ be a real number. The β-transformation $T_\beta : [0, 1) \to [0, 1)$ is defined by

$$T_\beta(x) = \beta x \mod 1\,.$$

For every $x \in [0, 1)$, we define a sequence $\{a_j\}_{j\geq 1}$ of non-negative integers by

$$a_j = \lfloor \beta T_\beta^{j-1} x \rfloor\,.$$

We call the sequence $\{a_j\}_{j\geq 1}$ the β-expansion of x. It holds that

$$x = \sum_{j=1}^{\infty} \frac{a_j}{\beta^j}\,.$$

We say that x has a *periodic β-expansion* $\{a_j\}_{j\geq 1}$ if there exist $p \geq 1$ and $M \geq 1$ such that $a_{n+p} = a_p$ for all $n \geq M$. Especially, if $a_{n+p} = a_p$ for all $n \geq 1$, then we say that x has a *purely periodic* β-expansion. We set

$$\mathrm{Per}(\beta) = \{x \in [0, 1) \mid x \text{ has a periodic } \beta\text{-expansion.}\}$$

and

$$\mathrm{Pur}(\beta) = \{x \in [0, 1) \mid x \text{ has a purely periodic } \beta\text{-expansion.}\}\,.$$

K. Schmidt has shown the following theorem on the periodic β-expansions.

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THEOREM ([8]). *Let $\beta > 1$ be a Pisot number. Then, $x \in \mathrm{Per}(\beta)$ if and only if $x \in \mathbb{Q}(\beta) \cap [0, 1)$.*

Especially, on the purely periodic β-expansions, S. Ito and H. Rao have shown the following theorem.

THEOREM 1.1 ([3]). *Let $\beta > 1$ be a Pisot unit. Let $\rho : \mathbb{Q}(\beta) \to \mathbb{R}^d$, $z_0 + z_1\beta + \cdots + z_{d-1}\beta^{d-1} \mapsto \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{d-1} \end{pmatrix}$. Then the following conditions are mutually equivalent.*

(1) $x \in \mathrm{Pur}(\beta)$;

(2) *$x \in \mathbb{Q}(\beta) \cap [0, 1)$ and $\rho(x) \in \hat{X}$, where $\hat{X}$ is a natural extension of the β-shift.*

This statement is slightly modified from the original one; see [3] for details.

The purpose of this paper is to extend the above theorems to the case of complex Pisot expansions.

In Section 2, we recall definitions of a complex Pisot number and complex Pisot tiles. The complex Pisot expansion is introduced in Section 3. One of the results established in this paper is stated as follows.

THEOREM 3.1. *Let λ be a complex Pisot unit. Assume that the companion matrix of the minimal polynomial for λ has a complex Pisot numeration set X. Then it holds that*

$$\mathrm{Per}(\lambda) = \mathbb{Q}(\lambda) \cap \phi_e(X \setminus N),$$

where ϕ_e is a canonical linear map: $\mathcal{P}_e \to \mathbb{C}$.

(A definition of the set N can be found in Section 3.)

In order to characterize purely periodic points of complex Pisot expansions, we introduce the sofic cover of a complex Pisot numeration system and its natural extension $\hat{X}_\lambda$ in Section 4.

By using a construction theorem of Markov partitions for toral automorphisms developed by [1] and [5, 6], we obtain the other result of this paper:

THEOREM 5.2. *The following conditions are mutually equivalent.*

(1) $z \in \mathrm{Pur}(\lambda)$;

(2) $z \in \mathbb{Q}(\lambda) \cap \phi_e(\hat{X}_\lambda \setminus N')$ and $\rho(z) \in \hat{X}_\lambda$, *where $\hat{X}_\lambda$ is a natural extension of the Pisot numeration system.*

2. Basic concepts

We define a complex counterpart of the so-called β-expansion.

Let $P(X) = X^d - p_1X^{d-1} - \cdots - p_{d-1}X - p_d$, be an irreducible polynomial, where $p_i \in \mathbb{Z}$.

DEFINITION 2.1. A complex number λ is called a *complex Pisot number* if λ is an algebraic number which is a root of the equation $P(X) = 0$ and, in addition, roots $\lambda = \lambda_1, \bar{\lambda} = \lambda_2, \lambda_3, \dots, \lambda_d$ of $P(X) = 0$ satisfy the inequalities $|\lambda| = |\bar{\lambda}| > 1 > |\lambda_i|, 3 \leq i \leq d$. A complex number λ is called a *complex Pisot unit* if λ is an complex Pisot number and an algebraic unit.

For the complex Pisot number λ with the minimal polynomial $P(X)$, set

$$A_\lambda = \begin{pmatrix} 0 & 0 & \dots & 0 & p_d \\ 1 & 0 & \dots & 0 & p_{d-1} \\ 0 & 1 & \dots & 0 & p_{d-2} \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & p_1 \end{pmatrix}.$$

We call the matrix A_λ the *companion matrix* of the minimal polynomial $P(X)$.

We regard A_λ as a linear transformation on the $\mathbb{R}^d$, which maps each $\boldsymbol{x} \in \mathbb{R}^d$ to $A_\lambda \boldsymbol{x}$. There exist an A_λ-invariant expanding eigenspace $\mathcal{P}_e$ corresponding to λ, and contracting eigenspace $\mathcal{P}_c$ corresponding to other complex conjugate roots. Then $\mathbb{R}^d$ is decomposed as $\mathbb{R}^d = \mathcal{P}_e \oplus \mathcal{P}_c$. Since λ is a complex Pisot number, $\dim \mathcal{P}_e = 2$.
Let π_e(resp. π_c) be the projection from $\mathbb{R}^d$ to the eigenspace $\mathcal{P}_e$(resp. $\mathcal{P}_c$). Then it is easy to see that

$$A_\lambda \circ \pi_e = \pi_e \circ A_\lambda \quad \text{and} \quad A_\lambda \circ \pi_c = \pi_c \circ A_\lambda .$$

Let us consider the linear action A_λ on the linear subspace $\mathcal{P}_e$. Denote by μ the Lebesgue measure on $\mathbb{R}^d$, and by μ_e the Lebesgue measure on $\mathcal{P}_e$.

DEFINITION 2.2. We say that the companion matrix A_λ has a *complex Pisot numeration set* $X = \cup_{j \in J} X_j$ if there exist a finite index set J, a family $\mathcal{T} = \{X_j\}_{j \in J}$ of compact subsets of $\mathcal{P}_e$, a set $\mathcal{D} = \{d_1^{(j)}, d_2^{(j)}, \dots, d_{l_j}^{(j)}\}_{j \in J}$ of d-dimensional integral vectors, and a subset $W = \{W_1^{(j)}, W_2^{(j)}, \dots, W_{l_j}^{(j)}\}_{j \in J}$ of J, such that

1. $\overline{\operatorname{int} X_j} = X_j$, $\mu_e(X_j) > 0$ *and* $\mu_e(\partial X_j) = 0$ for all $j \in J$;
2. $A_\lambda X_j = \bigcup_{1 \leq k \leq l_j} (\pi_e d_k^{(j)} + X_{W_k^{(j)}})$ (disjoint) for all $j \in J$;
3. $X = \bigcup_{j \in J} X_j$ (disjoint),

where "$\bigcup_{j \in J} X_j$ (disjoint)" means that $\operatorname{int}(X_j) \cap \operatorname{int}(X_{j'}) = \emptyset$ if $j \neq j'$.

There are many matrices which have complex Pisot numeration sets. We present two simple examples here.

EXAMPLE 2.1 (Rauzy Fractal [2], [7]). Let λ be a complex Pisot number whose minimal polynomial is $P(X) = X^3 + X^2 + X - 1$. We have that $\lambda = -0.7718 + 1.1151i$, $\lambda_2 = -0.7718 - 1.1151i$, and $\lambda_3 = 0.5436$.

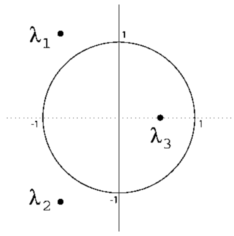

FIGURE 1

The companion matrix A_λ of λ is given by $A_\lambda = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ and λ_1, λ_2, and λ_3 satisfy $|\lambda_1| = |\lambda_2| > 1 > \lambda_3 > 0$. The configurations of the roots are illustrated by Figure 1.

In this case, it is known by [2] that there exist a finite index set $J = \{2 \wedge 3, 3 \wedge 1, 1 \wedge 2\}$, and a family $\mathcal{T} = \{X_{2\wedge 3}, X_{3\wedge 1}, X_{1\wedge 2}\}$ of compact subsets of $\mathcal{P}_e$, which is given by

$$X_{1\wedge 2} := \lim_{n\to\infty} A_\lambda^{-n} \pi_e E_2^n(\sigma)(\boldsymbol{e}_3, 1 \wedge 2);$$

$$X_{3\wedge 1} := \lim_{n\to\infty} A_\lambda^{-n} \pi_e E_2^n(\sigma)(\boldsymbol{e}_2, 3 \wedge 1);$$

$$X_{2\wedge 3} := \lim_{n\to\infty} A_\lambda^{-n} \pi_e E_2^n(\sigma)(\boldsymbol{e}_1, 2 \wedge 3),$$

(see [2] for the definitions of $E_2(\sigma)^n$ and $i \wedge j$).

We proved that the so-called Tile condition $\overline{\operatorname{int} X_j} = X_j$ is valid for all $j \in J$. In particular, $\mathcal{T}$ satisfies that

$$\begin{aligned} A_\lambda X_{1\wedge 2} &= \pi_e(-\boldsymbol{e}_2 - \boldsymbol{e}_3) + X_{2\wedge 3}; \\ A_\lambda X_{3\wedge 1} &= X_{1\wedge 2} \cup (\pi_e(-\boldsymbol{e}_2) + X_{2\wedge 3}); \\ A_\lambda X_{2\wedge 3} &= X_{2\wedge 3} \cup X_{3\wedge 1}, \end{aligned}$$

(see Figure 2).

In other words, a set $\mathcal{D} = \{d_1^{(j)}, d_2^{(j)}, \dots, d_{l_j}^{(j)}\}_{j\in J}$ of d-dimensional integral vectors and a subset $W = \{W_1^{(j)}, W_2^{(j)}, \dots, W_{l_j}^{(j)}\}_{j\in J}$ of J are given by

$$\begin{aligned} d_1^{(1\wedge 2)} &= -\boldsymbol{e}_2 - \boldsymbol{e}_3; & W_1^{(1\wedge 2)} &= 2 \wedge 3; \\ d_1^{(3\wedge 1)} &= \boldsymbol{0}; & W_1^{(3\wedge 1)} &= 1 \wedge 2; \\ d_2^{(3\wedge 1)} &= -\boldsymbol{e}_2; & W_2^{(3\wedge 1)} &= 2 \wedge 3; \\ d_1^{(2\wedge 3)} &= \boldsymbol{0}; & W_1^{(2\wedge 3)} &= 2 \wedge 3; \\ d_2^{(2\wedge 3)} &= \boldsymbol{0}; & W_2^{(2\wedge 3)} &= 3 \wedge 1. \end{aligned}$$

Therefore, $X = \cup_{i\in J} X_i$ is a complex Pisot numeration set.

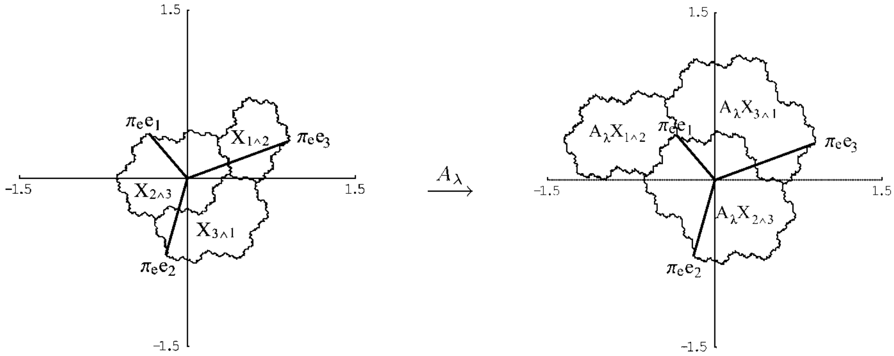

FIGURE 2

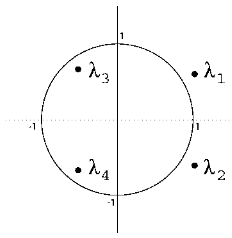

FIGURE 3

EXAMPLE 2.2 ([2]). Let λ be a complex Pisot number whose minimal polynomial is $P(X) = X^4 - X^3 + 1$. We have that $\lambda = 1.0189 + 0.6026i$, $\lambda_2 = 1.0189 - 0.6026i$, $\lambda_3 = -0.5189 + 0.6666i$, and $\lambda_4 = -0.5189 - 0.6666i$.

The companion matrix A_λ of λ is given by $A_\lambda = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $\lambda_1, \lambda_2, \lambda_3$, and λ_4 satisfy $|\lambda_1| = |\lambda_2| > 1 > |\lambda_3| = |\lambda_4| > 0$. The configurations of the roots are illustrated by Figure 3.

In this case, it is known by [2] that there exist a finite index set

$$J = \{1 \wedge 2, 1 \wedge 3, 1 \wedge 4, 2 \wedge 3, 2 \wedge 4, 3 \wedge 4\}$$

and a family $\mathcal{T} = \{X_{j\wedge k} \mid j \wedge k \in J\}$ which is given by

$$X_{1\wedge 2} := \lim_{n\to\infty} A_\lambda^{-n} \pi_e E_2(\sigma)^n (-\boldsymbol{e}_3 - \boldsymbol{e}_1 - \boldsymbol{e}_2, 1 \wedge 2);$$

$$X_{1\wedge 3} := \lim_{n\to\infty} A_\lambda^{-n} \pi_e E_2(\sigma)^n (\boldsymbol{e}_4 - \boldsymbol{e}_1 - \boldsymbol{e}_3, 1 \wedge 3);$$

$$X_{1\wedge 4} := \lim_{n\to\infty} A_\lambda^{-n} \pi_e E_2(\sigma)^n (-\boldsymbol{e}_3 - \boldsymbol{e}_1, 1 \wedge 4);$$

$$X_{2\wedge 3} := \lim_{n\to\infty} A_\lambda^{-n}\pi_e E_2(\sigma)^n\,(\boldsymbol{e}_4 - \boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3, 2\wedge 3)\,;$$
$$X_{2\wedge 4} := \lim_{n\to\infty} A_\lambda^{-n}\pi_e E_2(\sigma)^n\,(-\boldsymbol{e}_3 - \boldsymbol{e}_1 - \boldsymbol{e}_2, 2\wedge 4)\,;$$
$$X_{3\wedge 4} := \lim_{n\to\infty} A_\lambda^{-n}\pi_e E_2(\sigma)^n\,(-\boldsymbol{e}_3, 3\wedge 4)\,,$$

(see [2] for the definitions of $E_2(\sigma)^n$ and $i \wedge j$).

We proved that $\overline{\operatorname{int} X_j} = X_j$ for all $j \in J$. In particular, $\mathcal{T}$ satisfies that

$$\begin{aligned}
A_\lambda X_{1\wedge 2} &= X_{2\wedge 3} + \pi_e(-2\boldsymbol{e}_4 + \boldsymbol{e}_1)\\
A_\lambda X_{1\wedge 3} &= X_{2\wedge 4} + \pi_e \boldsymbol{e}_3\\
A_\lambda X_{1\wedge 4} &= (X_{2\wedge 4} + \pi_e(\boldsymbol{e}_3 + \boldsymbol{e}_1 - \boldsymbol{e}_4)) \cup (X_{1\wedge 2} + \pi_e \boldsymbol{e}_3)\\
A_\lambda X_{2\wedge 3} &= X_{3\wedge 4} + \pi_e(\boldsymbol{e}_1 + \boldsymbol{e}_2)\\
A_\lambda X_{2\wedge 4} &= (X_{3\wedge 4} + \pi_e(-\boldsymbol{e}_2 - \boldsymbol{e}_4)) \cup (X_{1\wedge 3} + \pi_e(-\boldsymbol{e}_2 - \boldsymbol{e}_4))\\
A_\lambda X_{3\wedge 4} &= X_{1\wedge 4} + \pi_e \boldsymbol{e}_3
\end{aligned}$$

(see Figure 4).

In other words, a set $\mathcal{D} = \{d_1^j, d_2^j, \ldots, d_{l_j}^j\}_{j\in J}$ of d-dimensional integral vectors and a subset $W = \{W_1^{(j)}, W_2^{(j)}, \ldots, W_{l_j}^{(j)}\}_{j\in J}$ of J are given by

$$\begin{aligned}
d_1^{(1\wedge 2)} &= -2\boldsymbol{e}_4 + \boldsymbol{e}_1\,; & W_1^{(1\wedge 2)} &= 2\wedge 3\,;\\
d_1^{(1\wedge 3)} &= \boldsymbol{e}_3\,; & W_1^{(1\wedge 3)} &= 2\wedge 4\,;\\
d_1^{(1\wedge 4)} &= \boldsymbol{e}_3 + \boldsymbol{e}_1 - \boldsymbol{e}_4\,; & W_1^{(1\wedge 4)} &= 2\wedge 4\,;\\
d_2^{(1\wedge 4)} &= \boldsymbol{e}_3\,; & W_2^{(1\wedge 4)} &= 1\wedge 2\,;\\
d_1^{(2\wedge 3)} &= \boldsymbol{e}_1 + \boldsymbol{e}_2\,; & W_1^{(2\wedge 3)} &= 3\wedge 4\,;\\
d_1^{(2\wedge 4)} &= -\boldsymbol{e}_2 - \boldsymbol{e}_4\,; & W_1^{(2\wedge 4)} &= 3\wedge 4\,;\\
d_2^{(2\wedge 4)} &= -\boldsymbol{e}_2 - \boldsymbol{e}_4\,; & W_2^{(2\wedge 4)} &= 1\wedge 3\,;\\
d_1^{(3\wedge 4)} &= \boldsymbol{e}_3\,; & W_1^{(3\wedge 4)} &= 1\wedge 4\,;
\end{aligned}$$

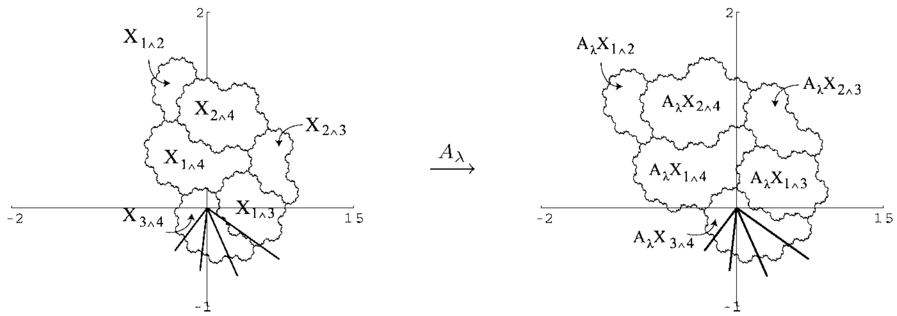

FIGURE 4

Therefore, $X = \cup_{i \in J} X_i$ is a complex Pisot numeration set. Especially, we see that $\mathbf{0} \in \operatorname{int} X$. Hence we have that $\mathbb{Q}(\lambda) \cap \phi_e(\hat{X}_\lambda \setminus N') \neq \emptyset$.

3. Complex Pisot expansion

Let Y denote the set $\cup_{j=1}^{N} \partial X_j$. We define an expanding transformation $\bar{A} : X \backslash Y \to X \backslash Y$ by

$$\bar{A}z = A_\lambda z - \pi_e d_k^{(j)} \quad \text{if } z \in \operatorname{int} X_j \text{ and } A_\lambda z \in (\pi_e d_k^{(j)} + \operatorname{int} X_{W_k^{(j)}})$$

and define iterations $\bar{A}^n$ of $\bar{A}$ by

$$\bar{A}^n(z) = A_\lambda \bar{A}^{n-1}(z) - \pi_e d_k^{(j_n)} \quad (n \geq 1)$$

if

$$\bar{A}^{n-1}(z) \in \operatorname{int} X_{j_{n-1}} \text{ and } A_\lambda \bar{A}^{n-1}(z) \in \left(\pi_e d_{k_{n-1}}^{(j_{n-1})} + \operatorname{int} X_{W_{k_{n-1}}^{(j_{n-1})}}\right).$$

Note that the iteration of the map $\bar{A}$ is well-defined on the set $X \setminus N$, where

$$N = \left\{ z \in X \mid \text{ there exists } n \in \mathbb{N} \text{ such that } \bar{A}^{n-1}(z) \in Y \right\}.$$

We consider the points in X whose orbits are disjoint from the boundary of any of the sets $\{X_j\}_{j=1}^N$.

Since $\mu_e(\partial X_j) = 0$ $(j \in J)$ and A is a linear transformation, we can define iterations of the expanding map $\bar{A}$ for μ_e-a.e. $z \in X$.

Thus, for μ_e-a.e. $z \in X$, we have the infinite sequence $\mathbf{d} = (d_{k_0}^{(j_0)}, d_{k_1}^{(j_1)}, \ldots, d_{k_{n-1}}^{(j_{n-1})}, \ldots) \in (\mathbb{Z}^d)^{\mathbb{N}}$.

DEFINITION 3.1. We call the sequence $\mathbf{d}$ the *digit sequence* of $z \in X$. We obtain the following formal series

$$z = \sum_{n=1}^{\infty} A_\lambda^{-n} \pi_e d_{k_{n-1}}^{(j_{n-1})}.$$

Therefore, we can define the formal series for μ_e-a.e. $z \in X$. Moreover, there is a unique formal series for any given $z \in X \setminus N$.

Let $\boldsymbol{u}_1$ (resp.$\boldsymbol{u}_2$) be an eigenvector corresponding to λ (resp. $\bar{\lambda}$), i.e.

$$A_\lambda \boldsymbol{u}_1 = \lambda \boldsymbol{u}_1 \text{ (resp. } A_\lambda \boldsymbol{u}_2 = \bar{\lambda} \boldsymbol{u}_2).$$

Now, we change a basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ of $\mathcal{P}_e$ to another basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ by the transition matrix

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

Actually, $\boldsymbol{v}_1 = (\boldsymbol{u}_1 + \boldsymbol{u}_2)/2$ and $\boldsymbol{v}_2 = (\boldsymbol{u}_2 - \boldsymbol{u}_1)/2i$. Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ satisfies that

$$A_\lambda \boldsymbol{v}_1 = a\boldsymbol{v}_1 + b\boldsymbol{v}_2\,;$$
$$A_\lambda \boldsymbol{v}_2 = b\boldsymbol{v}_1 - a\boldsymbol{v}_2\,,$$

where $\lambda = a + bi$. Therefore, there exists a linear map $\phi_e : \mathcal{P}_e \to \mathbb{C}$ satisfying

1. $\phi_e(A_\lambda \boldsymbol{x}) = \lambda\phi_e(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{P}_e = \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$;
2. $\phi_e(\boldsymbol{v}_1) = 1$;
3. $\phi_e(\boldsymbol{v}_2) = i$,

where $\mathcal{L}(\boldsymbol{v}, \boldsymbol{w})$ means a linear span of the vectors $\{\boldsymbol{v}, \boldsymbol{w}\}$.

Let λ be a complex Pisot number. Throughout the remainder of this paper, we assume that the companion matrix of the minimal polynomial for λ has a complex Pisot numeration set.

For any $z \in \phi_e(X \setminus N)$, we can define the digit sequence $\mathbf{d} = (d_{k_0}^{(j_0)}, d_{k_1}^{(j_1)}, \dots, d_{k_{n-1}}^{(j_{n-1})}, \dots)$. Let $a_n = \phi_e(\pi_e d_{k_{n-1}}^{(j_{n-1})})$. Since A_λ is the companion matrix, $\pi_e d_{k_{n-1}}^{(j_{n-1})}$ belongs to $\mathbb{Z}[\lambda]$. Therefore a_n also belongs to $\mathbb{Z}[\lambda]$. We define a map $T : \phi_e(X \setminus N) \to \phi_e(X \setminus N)$ by

$$T(z) = \lambda z - a_1\,,$$

and iterations T^n of the map T by setting for $z \in \phi_e(X \setminus N)$,

$$T^n(z) = \lambda T^{n-1}(z) - a_n \ (n \geq 1)\,.$$

Notice that the expanding map T and its iterations are well-defined for μ_e-a.e. $z \in \phi_e(X)$.

DEFINITION 3.2. We define the *complex Pisot expansion*:

$$z = \sum_{n=1}^{\infty} \frac{a_n}{\lambda^n}\,.$$

The complex Pisot expansion is well-defined for μ_e-a.e. $z \in \phi_e(X)$. The complex Pisot expansion of any $z \in \phi_e(X \setminus N)$ is unique.

DEFINITION 3.3. We say that a complex number $z \in \phi_e(X \setminus N)$ has a *periodic complex Pisot expansion* if there exist integers $p, N \geq 1$ such that $a_{n+p} = a_n$ for every integer $n \geq N$. Let us denote by $\mathrm{Per}(\lambda)$ the set of points in $\phi_e(X \setminus N)$ which have periodic complex Pisot expansions.

We say that a complex number $z \in \phi_e(X \setminus N)$ has a *purely periodic complex Pisot expansion* if there exists an integer $p \geq 1$ such that $a_{n+p} = a_n$ for every integer $n \geq 1$. Let us denote by $\mathrm{Pur}(\lambda)$ the set of points in $\phi_e(X \setminus N)$ which have purely periodic complex Pisot expansions.

One of our main results is stated as follows.

THEOREM 3.1. *Let λ be a complex Pisot unit. Assume that the companion matrix of the minimal polynomial for λ has a complex Pisot numeration set X. Then it holds that*

$$\mathrm{Per}(\lambda) = \mathbb{Q}(\lambda) \cap \phi_e(X \setminus N)\,.$$

This is the complex Pisot version of the K. Schmidt's theorem, and the proof of the theorem can be given by using essentially the same idea.

To prove Theorem 3.1, we need a few definitions and lemmas.

We designate the conjugate roots of λ as follows:

$$\lambda = \lambda_1, \lambda_2 = \bar{\lambda}, \lambda_3, \lambda_4 = \bar{\lambda}_3, \ldots, \lambda_{2s-1}, \lambda_{2s} = \bar{\lambda}_{2s-1}, \tilde{\lambda}_1 = \lambda_{2s+1}, \ldots, \tilde{\lambda}_t = \lambda_{2s+t} = \lambda_d\,,$$

where $\lambda_k \in \mathbb{C}$ $(k = 1, \ldots, s)$ and $\tilde{\lambda}_k \in \mathbb{R}$ $(k = 1, \ldots, t)$.

Also, let $\boldsymbol{u}_k$(resp. $\tilde{\boldsymbol{u}}_k$) be an eigenvector of the matrix A corresponding to the eigenvalue λ_k(resp. $\tilde{\lambda}_k$). Define real vectors $\{\boldsymbol{v}_k\}_{k=1}^{2s+t}$ by

1. $\boldsymbol{v}_{2k-1} = (\boldsymbol{u}_{2k-1} + \boldsymbol{u}_{2k})/2$ $(k = 1, 2, \ldots, s)$.
2. $\boldsymbol{v}_{2k} = (\boldsymbol{u}_{2k} - \boldsymbol{u}_{2k-1})/2i$ $(k = 1, 2, \ldots, s)$.
3. $\boldsymbol{v}_{2s+k} = \boldsymbol{u}_{2s+k}$ $(k = 1, 2, \ldots, t)$.

Note that

$$\mathbb{R}^d = \sum_{k=1}^{s} \oplus \mathcal{L}(\boldsymbol{v}_{2k-1}, \boldsymbol{v}_{2k}) \bigoplus \sum_{k=1}^{t} \oplus \mathcal{L}(\boldsymbol{v}_{2k+t}). \tag{1}$$

Let $\phi : \mathbb{R}^d \to \mathbb{C}^s \times \mathbb{R}^t$

1. $\phi(\mathcal{L}(\boldsymbol{v}_{2k-1}, \boldsymbol{v}_{2k})) \simeq \mathbb{C}$ $(k = 1, 2, \ldots, s)$, i.e., $\phi(\boldsymbol{v}_{2k-1}) = \boldsymbol{e}_{2k-1}$ and $\phi(\boldsymbol{v}_{2k}) = \boldsymbol{e}_{2k}$.
2. $\phi(\mathcal{L}(\boldsymbol{v}_{2s+k})) \simeq \mathbb{R}$ $(k = 1, 2, \ldots, t)$, i.e. $\phi(\boldsymbol{v}_{2s+k}) = \boldsymbol{e}_{2s+k}$.

Actually,

$$\phi(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{2s-1}, \boldsymbol{v}_{2s}, \boldsymbol{v}_{2s+1}, \ldots, \boldsymbol{v}_d) = \begin{pmatrix} 0 & & & & O & a_d \\ 1 & 0 & & & & a_{d-1} \\ 0 & 1 & 0 & & & a_{d-2} \\ & 0 & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots\ 0 & a_2 \\ & O & & 0 & 1 & a_1 \end{pmatrix}. \tag{2}$$

By using the direct sum decomposition (1), we obtain a unique representation of $z \in \mathbb{R}^d$:

$$z = \sum_{k=1}^{s} (z_{2k-1}\boldsymbol{v}_{2k-1} + z_{2k}\boldsymbol{v}_{2k}) + \sum_{k=1}^{t} z_{2s+k}\boldsymbol{v}_{2s+k}\,,$$

where $\{z_k\}_{k=1}^{2s+t} \subset \mathbb{C}$.

Let us define maps $\{\phi_{c_k}\}_{k=2}^{2s+t}$ by

$$\phi_{c_k}(z) = \begin{cases} z_{2k-1}\boldsymbol{v}_{2k-1} + z_{2k}\boldsymbol{v}_{2k} \in \mathcal{L}(\boldsymbol{v}_{2k-1}, \boldsymbol{v}_{2k}) & (k = 2, 3, \ldots, s)\,; \\ z_k\boldsymbol{v}_k \in \mathcal{L}(\boldsymbol{v}_k) & (k = 2s+1, \ldots, 2s+t)\,. \end{cases}$$

Notice that $\phi_e = \phi|_{\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)}$, and

$$\phi_{c_k}(z) = \begin{cases} \phi|_{\mathcal{L}(\boldsymbol{v}_{2k-1}, \boldsymbol{v}_{2k})} & (k = 2, 3, \ldots, s)\,; \\ \phi|_{\mathcal{L}(\boldsymbol{v}_k)} & (k = 2s+1, \ldots, 2s+t)\,. \end{cases}$$

LEMMA 3.1. *Let ϕ_{c_k} be as above. Then*

$$\phi_{c_k}(A(z)) = \lambda_k \phi_{c_k}(z)$$

for $k = 2, 3, \ldots, s, 2s+1, \ldots, 2s+t$.

PROOF. This lemma is trivial in the case $k \in \{2s+1, \ldots, 2s+t\}$. We consider the case $k \in \{2, 3, \ldots, s\}$. Let $z = z_1\boldsymbol{v}_{2k-1} + z_2\boldsymbol{v}_{2k} \in \mathcal{L}(\boldsymbol{v}_{2k-1}, \boldsymbol{v}_{2k})$ and $\lambda_k = a_k + b_k i$ $(a_k, b_k \in \mathbb{R})$. It follows that

$$\begin{aligned} A(z) &= A(z_1\boldsymbol{v}_{2k-1} + z_2\boldsymbol{v}_{2k}) \\ &= A(\boldsymbol{v}_{2k-1},\ \boldsymbol{v}_{2k}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= (a\boldsymbol{v}_{2k-1} - b\boldsymbol{v}_{2k},\ b\boldsymbol{v}_{2k-1} - a\boldsymbol{v}_{2k}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= (\boldsymbol{v}_{2k-1},\ \boldsymbol{v}_{2k}) \begin{pmatrix} a_k & b_k \\ b_k & -a_k \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= \lambda_k(\boldsymbol{v}_{2k-1},\ \boldsymbol{v}_{2k}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= \lambda_k z\,. \end{aligned}$$

Since $A(z) \in \mathcal{L}(\boldsymbol{v}_{2k-1}, \boldsymbol{v}_{2k})$, it follows that $\phi_{c_k}(Az) = \lambda_k z$ for each $k \in \{2, 3, \ldots, s\}$. $\square$

Since $\{1, \lambda, \lambda^2, \ldots, \lambda^{d-1}\}$ is a basis for the field $\mathbb{Q}(\lambda)$, we can define a map $\rho : \mathbb{Q}(\lambda) \to \mathbb{Q}^d$ by

$$\rho(z_0 + z_1\lambda + \cdots z_{d-1}\lambda^{d-1}) = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{d-1} \end{pmatrix}.$$

We call the map ρ the canonical map. It is obvious that for any $z \in \mathbb{Q}(\lambda)$,

$$\rho(\lambda z) = A\rho(z) \ (\operatorname{mod} \mathbb{Z}^d)\,.$$

Now we need a lemma.

LEMMA 3.2. *Let $z \in \mathbb{Q}(\lambda)$ be represented in the form*

$$z = \frac{1}{q}\sum_{i=0}^{d-1} p_i\lambda^i\,, \quad p_i, q \in \mathbb{Z},\ q > 0\,,$$

where q is the least integer such that $qz \in \mathbb{Z}[\lambda]$. Let $z_n \in \mathbb{Q}(\lambda)$ be given by

$$z_n = \lambda^n\left(z - \sum_{k=1}^{n}\frac{a_k}{\lambda^k}\right), \tag{3}$$

where $\{a_k\}_{k=1}^{\infty}$ is the complex Pisot expansion of z. Then for every integer $n \geq 0$, there exists a unique element $(r_1^{(n)}, \dots, r_d^{(n)}) = \boldsymbol{r}^{(n)} \in \mathbb{Z}^d$ with

$$z_n = \frac{1}{q}\sum_{k=1}^{d} r_k^{(n)}\lambda^{-k}\,.$$

PROOF. We prove this lemma by the induction argument.

If $n = 0$, the lemma is true because $\{1, \lambda^{-1}, \dots, \lambda^{-d+1}\}$ is a basis of the field $\mathbb{Q}(\lambda)$.

Assume that the lemma is true when $n = j$. There exists a unique element $\boldsymbol{r}^{(j)} = (r_1^{(j)}, \dots, r_d^{(j)}) \in \mathbb{Z}^d$ such that

$$z_j = \frac{1}{q}\sum_{k=1}^{d} r_k^{(j)}\lambda^{-k}\,.$$

It follows from the definition that

$$z_{j+1} = \lambda^{j+1}\left(z - \sum_{k=1}^{j+1}\frac{a_k}{\lambda^k}\right) = \lambda z_j - a_{j+1}\,.$$

Since a_k belongs to the ring $\mathbb{Z}[\lambda]$, z_{j+1} belongs to the field $\mathbb{Q}(\lambda)$. Therefore, the lemma follows from the fact that $\{1, \lambda^{-1}, \dots, \lambda^{-d+1}\}$ is a basis for the field $\mathbb{Q}(\lambda)$. □

Let $z \in \mathbb{Q}(\lambda)$. Since $\{1, \lambda, \dots, \lambda^{d-1}\}$ is a basis for $\mathbb{Q}(\lambda)$, there exists a polynomial $P_z(X) = z_1 + z_2X + \cdots + z_{d-1}X^{d-1}$ ($z_i \in \mathbb{Q}$) satisfying $z = P_z(\lambda)$. We define maps $\zeta_j : \mathbb{Q}(\lambda) \to \mathbb{Q}(\lambda_j)$ ($j = 1, \dots, 2s, 2s+1, \dots, 2s+t$) by

$$\zeta_j(z) = P_z(\lambda_j)\,,$$

and define a map

$$\zeta : \mathbb{Q}(\lambda) \to \mathbb{Q}(\lambda_3) \times \mathbb{Q}(\lambda_5) \times \cdots \times \mathbb{Q}(\lambda_{2s-1}) \times \mathbb{Q}(\lambda_{2s+1}) \times \mathbb{Q}(\lambda_{2s+2}) \times \cdots \times \mathbb{Q}(\lambda_{2s+t})$$

by

$$\zeta(z) = (\zeta_3(z), \dots, \zeta_{2s-1}(z), \zeta_{2s+1}(z), \zeta_{2s+2}(z), \dots, \zeta_{2s+t}(z))\,.$$

PROOF OF THEOREM 3.1. Let $z \in \mathbb{Q}(\lambda) \cap \phi_e(X \setminus N)$ be arbitrary. Let z_n be as in Lemma 3.2 and write

$$z = \frac{1}{q}\sum_{i=0}^{d-1} p_i \lambda^i\,, \quad p_i, q \in \mathbb{Z},\ q > 0\,;$$

$$\zeta_i(z_n) = \frac{1}{q}\sum_{k=1}^{d} r_k^{(n)} \lambda_i^{-k}\,, \quad n \geq 1 \quad (i = 1, \ldots, d)\,.$$

Since $z \in \mathbb{Q}(\lambda) \cap \phi_e(X \setminus N)$, we have

$$|z_n| = |\zeta_1(z_n)| = |\zeta_2(z_n)| < ||\phi_e(X)|| < +\infty$$

for all $n \geq 1$.

Define K and η by

$$K = \max\{|\zeta_e(\pi_e(d_k^{(j)}))|, |\zeta_{c_j}(\pi_{c_j}(d_k^{(j)}))| \mid d_k^{(j)} \in \mathcal{D}, 1 \leq j \leq d\}\,;$$
$$\eta = \max_{i=3,\ldots,d} |\lambda_i| < 1\,.$$

Then the equation (2) induces the inequalities

$$|\zeta_i(z_n)| = \left|\lambda_i^n\left(\frac{1}{q}\sum_{k=0}^{d-1} p_i\lambda_i^k - \sum_{l=0}^{n}\zeta_{c_i}(d_{k_l}^{(j_l)})\lambda_i^{-l}\right)\right| \leq \frac{1}{q}\sum_{k=0}^{d-1}|p_i|\eta^{n+k} + K\sum_{k=0}^{n}\eta^k < +\infty$$

for every $i = 3, \ldots, d$.

Since

$$\mathbb{V}_n = q\begin{pmatrix}\zeta_1(z_n)\\ \vdots \\ \zeta_d(z_n)\end{pmatrix} = \begin{pmatrix}\lambda_1^{-1} & \lambda_1^{-2} & \ldots & \lambda_1^{-d}\\ \vdots & \vdots & \vdots & \vdots\\ \lambda_d^{-1} & \lambda_d^{-2} & \ldots & \lambda_d^{-d}\end{pmatrix}\begin{pmatrix}r_n^{(1)}\\ \vdots \\ r_n^{(d)}\end{pmatrix} \tag{4}$$

and $|\zeta_i(z^n)| < ||\phi_e(X)|| < +\infty$ $(1 \leq i \leq d)$, the vectors $\{\mathbb{V}_n \mid n \geq 0\}$ have bounded norms. Since the matrix in (3) is nonsingular, the vectors $\{\boldsymbol{r}_n \mid n \geq 0\}$ have bounded norms. Therefore we have that $\boldsymbol{r}_{m+n} = \boldsymbol{r}_n$ for some $m, n \geq 1$, and hence $z_{m+n} = z_n$. □

4. Symbolic dynamics of complex Pisot expansions

Let us define a directed graph $G = (V, E, i, t)$ from a complex Pisot numeration system by

1. $V = J$;
2. $E = \left\{\binom{j}{k} \mid j \in V, \quad 1 \leq k \leq l_j\right\}$;
3. $i : E \to V, \binom{j}{k} \mapsto j$;
4. $t : E \to V, \binom{j}{k} \mapsto W_k^{(j)}$.

Note that the graph G is uniquely determined.

EXAMPLE 4.1 (Rauzy Fractal [7]). The graph of Example 2.1 is given by Figure 5.

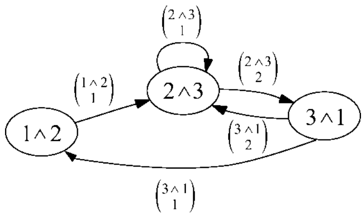

FIGURE 5

EXAMPLE 4.2 ([2]). The graph of Example 2.2 is given by Figure 6.

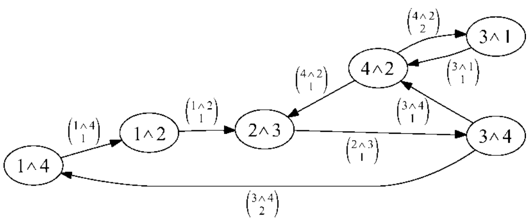

FIGURE 6

Let us call a symbolic space right-sided (left-sided) if it is one-sided extending to the right (left, respectively).

DEFINITION 4.1. Let Ω_λ^{j+} denote the right-sided symbolic space

$$\left\{ (d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \dots) \;\middle|\; j_1 = j \in V,\ t\begin{pmatrix} j_{p-1} \\ k_{p-1} \end{pmatrix} = j_p \ (p \in \mathbb{N}) \right\} .$$

Then we denote by Ω_λ^{+} the right-sided symbolic space $\bigcup_{j \in J} \Omega_\lambda^{j+}$.

Let us define

$$\hat{X}_\lambda^{j+} = \left\{ \sum_{n=1}^{\infty} \pi_e A^{-n} d_{k_{n-1}}^{(j_{n-1})} \;\middle|\; (d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \dots) \in \Omega_\lambda^{j+} \right\} ,$$

and

$$\hat{X}_\lambda^{+} = \left\{ \sum_{n=1}^{\infty} \pi_e A^{-n} d_{k_{n-1}}^{(j_{n-1})} \;\middle|\; (d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \dots) \in \Omega_\lambda^{+} \right\} .$$

It is obvious that $\hat{X}_\lambda^{+} = \bigcup_{j \in J} \hat{X}_\lambda^{j+}$.

DEFINITION 4.2. Define $\varphi : \hat{\Omega}_\lambda^+ \to \hat{X}_\lambda^+$ by

$$\varphi\big((d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \dots)\big) = \sum_{n=1}^{\infty} A^{-n} \pi_e d_{k_{n-1}}^{(j_{n-1})} .$$

Note that $(\hat{\Omega}_\lambda^+, \sigma)$ is a sofic cover of $(X, \hat{A})$ and φ is a factor map from $\hat{\Omega}_\lambda^+$ to X.

PROPOSITION 4.1. *$\varphi(\hat{\Omega}_\lambda^{j+}) = X_j$ holds for all $j \in J$.*

PROOF. It is easy to see that the set $\{\varphi(\hat{\Omega}_\lambda^{j+})\}_{j \in I}$ is the family of the compact sets and satisfies the set equation stated in Definition 2.2. On the other hand, we see that $X_j \backslash N \subset \varphi(\hat{\Omega}_\lambda^{j+})$, $X_j \subset \mathrm{cl}(\gamma_j \backslash N)$ and so $X_j \subset \varphi(\hat{\Omega}_\lambda^{j+})$. Therefore, from the uniqueness of attractors by the graph-directed iterated function system theorem [4], we have $\varphi(\Omega_\lambda^{(j)}) = X_j$. $\square$

Let $\hat{\Omega}_\lambda^{j-}$ denote the left sided symbolic space

$$\left\{ \left(\dots d_{k_{-3}}^{(j_{-3})} d_{k_{-2}}^{(j_{-2})} d_{k_{-1}}^{(j_{-1})} \right) \,\middle|\, t\binom{j_0}{k_0} = j,\ t\binom{j_{-p}}{k_{-p}} = j_{-p+1}\ (p \in \mathbb{N}) \right\} .$$

Let us define a two sided symbolic space $\hat{\Omega}_\lambda$ by

$$\hat{\Omega}_\lambda = \left\{ \left(\dots d_{k_{-2}}^{(j_{-2})} d_{k_{-1}}^{(j_{-1})} .\, d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \dots \right) \,\middle|\, j_1 = j \in V, \quad t\binom{j_p}{k_p} = j_{p+1}\ (p \in \mathbb{Z}) \right\} .$$

Let us call $\hat{\Omega}_\lambda = \bigcup_{j \in J} \hat{\Omega}_\lambda^j$ the natural extension of Ω_λ^+.

DEFINITION 4.3. Define a set $\hat{X}_\lambda^{j-} \subset \mathcal{P}_c$ by

$$\hat{X}_\lambda^{j-} = \left\{ \sum_{n=0}^{\infty} \pi_c A^n d_{k_n}^{(j_n)} \,\middle|\, \left(\dots d_{k_{-2}}^{(j_{-2})} d_{k_{-1}}^{(j_{-1})} .\, d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \dots \right) \in \Omega_\lambda^{j-} \right\} .$$

Let $\hat{X}_\lambda^j = \hat{X}_\lambda^{j+} - \hat{X}_\lambda^{j-} \subset \mathbb{R}^d$.

Since $\{\hat{X}_\lambda^{j+}\}_{j=1}^N$ is a complex Pisot numeration set, we have that $\mathrm{int}\, \hat{X}^i \cap \mathrm{int}\, \hat{X}^j = \emptyset$ $(i \neq j)$. Set $\hat{X}_\lambda = \bigcup_{j \in J} \hat{X}_\lambda^j$.

DEFINITION 4.4. Define $\hat{\varphi} : \hat{\Omega}_\lambda \to \hat{X}_\lambda$ by

$$\hat{\varphi}\big((\dots d_{k_{-2}}^{(j_{-2})} d_{k_{-1}}^{(j_{-1})} .\, d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \dots)\big) = -\sum_{n=0}^{\infty} A^n \pi_c d_{k_{-n-1}}^{(j_{-n-1})} + \sum_{n=1}^{\infty} A^{-n} \pi_e d_{k_{n-1}}^{(j_{n-1})} .$$

PROPOSITION 4.2. *Define a map $\hat{A}$ on the space $\hat{X}$ by*

$$\hat{A}(x) = \hat{A}(x_1 - x_2) = Ax - d_{k_0}^{(j_0)} ,$$

where $x = x_1 - x_2 \in \hat{X}_\lambda^{j_1}$ $(x_1 \in \hat{X}_\lambda^{j_1+}, x_2 \in \hat{X}_\lambda^{j_1-})$.

Then it follows that

1. $\hat{\varphi}(\hat{\Omega}_\lambda) = \hat{X}_\lambda$;
2. $\hat{A} \circ \hat{\varphi} = \hat{\varphi} \circ \sigma$,

where σ *is the shift transformation on* $\hat{\Omega}_\lambda$.

PROOF. It follows from the definition of the map $\hat{\varphi}$ that $\hat{\varphi}(\hat{\Omega}_\lambda) = \hat{X}_\lambda$. Let $\mathbf{d} = (\dots d_{k_{-1}}^{(j_{-1})} . d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \dots) \in \hat{\Omega}_\lambda$. Then we have

$$\hat{\varphi}(\mathbf{d}) = -\sum_{n=0}^{\infty} A^n \pi_c d_{k_{-n-1}}^{(j_{-n-1})} + \sum_{n=1}^{\infty} A^{-n} \pi_e d_{k_{n-1}}^{(j_{n-1})} .$$

On the other hand, we have

$$\begin{aligned}
\hat{A}\big(\hat{\varphi}(\mathbf{d})\big) &= -\sum_{n=0}^{\infty} A^{n+1} \pi_c d_{k_{-n-1}}^{(j_{-n-1})} - \varphi(d_{k_0}^{(j_0)}) + \pi_e(d_{k_0}^{(j_0)}) + \sum_{n=2}^{\infty} A^{-n-1} \pi_e d_{k_{n-1}}^{(j_{n-1})} \\
&= -\sum_{n=0}^{\infty} A^{n+1} \pi_c d_{k_{-n-1}}^{(j_{-n-1})} - \pi_c(d_{k_0}^{(j_0)}) + \sum_{n=2}^{\infty} A^{-n+1} \pi_e d_{k_{n-1}}^{(j_{n-1})} \\
&= -\sum_{n=-1}^{\infty} A^{n+1} \pi_c d_{k_{-n-1}}^{(j_{-n-1})} + \sum_{n=2}^{\infty} A^{-n+1} \pi_e d_{k_{n-1}}^{(j_{n-1})} \\
&= -\sum_{n=0}^{\infty} A^{n} \pi_c d_{k_{-n}}^{(j_{-n})} + \sum_{n=1}^{\infty} A^{-n} \pi_e d_{k_{n}}^{(j_{n})} \\
&= \hat{\varphi}(\dots d_{k_{-1}}^{(j_{-1})} d_{k_0}^{(j_0)} . d_{k_1}^{(j_1)} d_{k_2}^{(j_2)} \dots) \\
&= \hat{\varphi}(\sigma(\mathbf{d})) .
\end{aligned}$$

□

Note that $\pi_e(\hat{X}_\lambda) = X$. We set

$$N' = \left\{ z \in \hat{X}_\lambda \;\middle|\; \text{there exists } n \in \mathbb{N} \text{ such that } \bar{A}^{n-1}(\pi_e(z)) \in \bigcup_{j=1}^{N} \partial X_j \right\} .$$

5. Main results

We give below a characterization of purely periodic points of complex Pisot expansions.

We define a map $\hat{T} : \phi(\hat{X}_\lambda) \to \phi(\hat{X}_\lambda)$ in the following way. Let $x \in \hat{X}_\lambda$, $z = \phi(x) = (z_1, z_2, \ldots, z_s, z_{s+1}, \ldots, z_{s+t})^t \in \phi(\hat{X}_\lambda) \subset \mathbb{C}^s \times \mathbb{R}^t$, and

$$\hat{T}(z) = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \lambda_3 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_{2s-1} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \tilde{\lambda}_1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & \tilde{\lambda}_t \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_s \\ z_{s+1} \\ \vdots \\ z_{s+t} \end{pmatrix} - \begin{pmatrix} \phi_e(d) \\ \phi_{c_1}(d) \\ \vdots \\ \phi_{c_s}(d) \\ \phi_{c_{s+1}}(d) \\ \vdots \\ \phi_{c_{s+t}}(d) \end{pmatrix},$$

where $d = d_{k_0}^{(j_0)}$.

Then we can see that $\phi \circ \hat{A} = \hat{T} \circ \phi$, that is, the diagram

$$\begin{array}{ccc} \hat{X}_\lambda & \xrightarrow{\hat{A}} & \hat{X}_\lambda \\ {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\phi}\big\downarrow \\ \phi(\hat{X}_\lambda) & \xrightarrow{\hat{T}} & \phi(\hat{X}_\lambda) \end{array}$$

commutes. Let us define a map $\tilde{\varphi} : \hat{\Omega}_\lambda \to \mathbb{C}^s \times \mathbb{R}^t$ by

$$\begin{aligned} &\tilde{\varphi}(\ldots d_{k_{-2}}^{(j_{-2})} d_{k_{-1}}^{(j_{-1})} . d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} d_{k_2}^{(j_2)} \ldots) \\ &= \left(\sum_{n=1}^{\infty} \frac{\phi_e(\pi_e(d_{k_{n-1}}^{(j_{n-1})}))}{\lambda^n}, -\sum_{n=0}^{\infty} \phi_{c_3}(\pi_{c_3}(d_{-k_{n-1}}^{(j_{n-1})}))\lambda_3^n, \ldots, -\sum_{n=0}^{\infty} \phi_{c_d}(\pi_{c_d}(d_{-k_{n-1}}^{(j_{n-1})}))\lambda_d^n \right). \end{aligned}$$

Set $\hat{K}_\lambda = \tilde{\varphi}(\hat{\Omega}_\lambda)$. We then obtain the following theorem.

THEOREM 5.1. *Let λ be a complex Pisot unit. Assume that there is a complex Pisot numeration set $X = \bigcup_{j \in J} X_j$. Then the following conditions are mutually equivalent.*

(1) $z \in \mathrm{Pur}(\lambda)$;

(2) $z \in \mathbb{Q}(\lambda) \cap \phi_e(\hat{X}_\lambda \setminus N')$ *and* $(z, \zeta(z)) \in \hat{K}_\lambda$.

PROOF. (i) Suppose $z = 0.\overline{a_1 \ldots a_p} \in \mathrm{Pur}(\lambda)$, where $a_i = \phi_e(\pi_e(d_{k_{i-1}}^{(j_{i-1})}))$. Since $a_i \in \mathbb{Z}[\lambda]$, $z \in \mathbb{Q}(\lambda) \cap \phi_e(\hat{X}_\lambda \setminus N')$. Therefore, we obtain

$$z = \frac{a_1\lambda^{p-1} + \cdots + a_{p-1}\lambda + a_p}{\lambda^p - 1}.$$

For $k \geq 3$, the k-th coordinate of $\zeta(z)$ is given by

$$-(\phi_{c_k}(\pi_{c_k}(d_{k_{p-1}}^{(j_{p-1})})) + \phi_{c_k}(\pi_{c_k}(d_{k_{p-2}}^{(j_{p-2})}))\lambda_k + \cdots + \phi_{c_k}(\pi_{c_k}(d_{k_0}^{(j_0)}))\lambda_k^p)(1 + \lambda_k^p + \lambda_k^{2p} + \cdots)$$

$$= \frac{\phi_{c_k}(\pi_{c_k}(d_{k_0}^{(j_0)}))\lambda_k^{p-1} + \cdots + \phi_{c_k}(\pi_{c_k}(d_{k_{p-2}}^{(j_{p-2})}))\lambda_k + \phi_{c_k}(\pi_{c_k}(d_{k_{p-1}}^{(j_{p-1})}))}{\lambda_k^p - 1}$$
$$= \zeta_k(z)\,.$$

Thus we have $(z, \zeta(z)) \in \hat{K}_\lambda$.

(ii) Suppose that $z \in \mathbb{Q}(\lambda) \cap \phi_e(\hat{X}_\lambda \setminus N')$ and $(z, \zeta(z)) \in \hat{K}_\lambda$. Let b be the least integer such that $bz \in \mathbb{Z}[\lambda]$. Set

$$\mathcal{R}_b = \{(z, \zeta(z)) \,:\, z \in b^{-1}\mathbb{Z}[\lambda] \cap \phi_e(\hat{X}_\lambda \setminus N')\}\,.$$

Then $\mathcal{R}_b$ is a finite set because $\phi_e(\hat{X}_\lambda)$ is bounded. Notice that $\hat{T}(z, \zeta(z)) = (\hat{T}z, \zeta(\hat{T}(z)))$. Since there is a digit $d \in \mathcal{D}$ such that $\hat{T}z = \phi_e\big(Az - d_{k_0}^{(j_0)}\big) \in b^{-1}\mathbb{Z}[\lambda]$ and $\hat{T}(\phi_e(\hat{X}_\lambda)) = \phi_e(\hat{X}_\lambda)$, we obtain that $\hat{T}(\mathcal{R}_b) \subseteq \mathcal{R}_b$. For $(z, \zeta(z)) \in \mathcal{R}_b$, there exists a sequence $(w, u) \in \hat{\Omega}_\lambda$ such that $\hat{\zeta}(w, u) = (z, \zeta(z))$.

Let $y = \lambda^{-1}(z + z_0)$. Then $\hat{\varphi} \circ \sigma^{-1}(w, u) = (y, \varphi(y))$. Therefore $(y, \zeta(y)) \in \phi_e(X)$ and $\hat{T}(y, \zeta(y)) = (z, \zeta(z))$. Consequently we can state that $\hat{T}$ is surjective on $\mathcal{R}_b$. Since λ is an algebraic unit, we know that $y \in b^{-1}\mathbb{Z}[\lambda]$, and hence $(y, \zeta(y)) \in \mathcal{R}_b$. Since $\hat{T}|_{\mathcal{R}_b}$ is a one-to-one map, there exists an integer n such that

$$(z, \zeta(z)) = \hat{T}^n(z, \zeta(z)) = (\hat{T}^n(z), \zeta(\hat{T}^n(z)))\,.$$

Thus we obtain that $z = \hat{T}^n z$. □

Now we can state the main theorem.

THEOREM 5.2. *Let λ be a complex Pisot number. Then the following conditions are mutually equivalent.*

(1) $z \in \mathrm{Pur}(\lambda)$;

(2) $z \in \mathbb{Q}(\lambda) \cap \phi_e(\hat{X}_\lambda \setminus N')$ *and* $\rho(z) \in \hat{X}_\lambda$.

Before proving the main theorem, we shall show the following proposition.

PROPOSITION 5.1. $\phi^{-1}(\hat{K}_\lambda) = \hat{X}_\lambda$ *holds.*

PROOF. It is clear from (2) that

$$\phi^{-1} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d)\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & i & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & i & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= (\boldsymbol{v}_1 - i\boldsymbol{v}_2, \boldsymbol{v}_3 - i\boldsymbol{v}_4, \ldots, \boldsymbol{v}_{2s-1} - i\boldsymbol{v}_{2s}, \boldsymbol{v}_{2s+1}, \ldots, \boldsymbol{v}_d)\,.$$

Therefore, we obtain

$$\begin{aligned}
\phi^{-1}(\hat{K}_\lambda) &= \Bigg\{ \phi^{-1}\left(\sum_{n=1}^{\infty} \pi_e(d_{k_{n-1}}^{(j_{n-1})})\lambda^{-n}, -\sum_{n=0}^{\infty} \pi_{c_3}(d_{k_{-n+1}}^{(j_{-n+1})})\lambda_3^n, \ldots, -\sum_{n=0}^{\infty} \pi_{d_3}(d_{k_{-n+1}}^{(j_{-n+1})})\lambda_d^n \right)^t \\
&\qquad \Bigg| \, \boldsymbol{d} = (\ldots d_{k_{-1}}^{(j_1)}.\, d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \ldots) \in \hat{\Omega}_\lambda \Bigg\} \\
&= \Bigg\{ \sum_{n=1}^{\infty} \pi_e(d_{k_{n-1}}^{(j_{n-1})})\lambda^{-n}(\boldsymbol{v}_1 - i\boldsymbol{v}_2) - \sum_{n=0}^{\infty} \pi_{c_3}(d_{k_{-n+1}}^{(j_{-n+1})})\lambda_3^n(\boldsymbol{v}_3 - i\boldsymbol{v}_4) \cdots \\
&\qquad - \sum_{n=0}^{\infty} \pi_{d_3}(d_{k_{-n+1}}^{(j_{-n+1})})\lambda_d^n(\boldsymbol{v}_d) \Bigg| \, \boldsymbol{d} = (\ldots d_{k_{-1}}^{(j_{-1})}.\, d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \ldots) \in \hat{\Omega}_\lambda \Bigg\} \\
&= \Bigg\{ \sum_{n=1}^{\infty} \pi_e(d_{k_{n-1}}^{(j_{n-1})})A^{-n}(\boldsymbol{v}_1 - i\boldsymbol{v}_2) - \sum_{n=0}^{\infty} \pi_{c_3}(d_{k_{-n+1}}^{(j_{-n+1})})A^n(\boldsymbol{v}_3 - i\boldsymbol{v}_4) \cdots \\
&\qquad - \sum_{n=0}^{\infty} \pi_{d_3}(d_{k_{-n+1}}^{(j_{-n+1})})A^n(\boldsymbol{v}_d) \Bigg| \, \boldsymbol{d} = (\ldots d_{k_{-1}}^{(j_{-1})}.\, d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \ldots) \in \hat{\Omega}_\lambda \Bigg\} \\
&= \Bigg\{ \sum_{n=1}^{\infty} A^{-n}\pi_e(d_{k_{n-1}}^{(j_{n-1})}) - \sum_{n=0}^{\infty} A^n\pi_{c_3}(d_{k_{-n+1}}^{(j_{-n+1})}) \cdots - \sum_{n=0}^{\infty} A^n\pi_{d_3}(d_{k_{-n+1}}^{(j_{-n+1})}) \, \Bigg| \\
&\qquad \boldsymbol{d} = (\ldots d_{k_{-1}}^{(j_{-1})}.\, d_{k_0}^{(j_0)} d_{k_1}^{(j_1)} \ldots) \in \hat{\Omega}_\lambda \Bigg\} \\
&= \hat{X}_\lambda\,.
\end{aligned}$$

□

LEMMA 5.1. *Let $z \in \mathbb{Q}(\lambda)$. Then*

$$\phi^{-1}\begin{pmatrix} z \\ \zeta(z) \end{pmatrix} = \rho(z)\,. \tag{5}$$

PROOF. We prove the lemma by steps. If $z = 1$, then it is trivial.
Next we show that if (4) is true for z, then it is also true for λz. We have

$$\begin{aligned}
\phi^{-1}\begin{pmatrix} \lambda z \\ \zeta(\lambda z) \end{pmatrix} &= \phi^{-1} I_\lambda \begin{pmatrix} z \\ \zeta(z) \end{pmatrix} \\
&= \big(\lambda(\boldsymbol{v}_1 - i\boldsymbol{v}_2), \lambda_3(\boldsymbol{v}_3 - i\boldsymbol{v}_4), \ldots, \lambda_{2s-1}(\boldsymbol{v}_{2s-1} - i\boldsymbol{v}_{2s}), \\
&\qquad \lambda_{2s+1}\boldsymbol{v}_{2s+1}, \ldots, \lambda_{2s+t}\boldsymbol{v}_{2s+t}\big) \begin{pmatrix} z \\ \zeta(z) \end{pmatrix}
\end{aligned}$$

$$= A\,((\boldsymbol{v}_1 - i\boldsymbol{v}_2), (\boldsymbol{v}_3 - i\boldsymbol{v}_4), \dots, (\boldsymbol{v}_{2s-1} - i\boldsymbol{v}_{2s}), \boldsymbol{v}_{2s+1}, \dots, \boldsymbol{v}_{2s+t}) \begin{pmatrix} z \\ \zeta(z) \end{pmatrix}$$
$$= A\phi^{-1} \begin{pmatrix} z \\ \zeta(z) \end{pmatrix}$$
$$= A\rho(z)$$
$$= \rho(\lambda z)\,.$$

Hence (4) holds for $z = \lambda^k, 0 \le k \le d-1$. Note that both ϕ^{-1} and ζ are linear maps. Therefore (4) holds for all $z \in \mathbb{Q}(\lambda)$. □

PROOF OF THEOREM 5.2. From Proposition 5.1 and Lemma 5.1, we can show that

$$(z, \zeta(z)) \in \hat{K}_\lambda \;\Leftrightarrow \phi^{-1} \begin{pmatrix} z \\ \zeta(z) \end{pmatrix} \in \phi^{-1}(\hat{K}_\lambda) \;\Leftrightarrow \rho(z) \in \hat{X}_\lambda.$$

Hence the main theorem follows from Theorem 5.1. □

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