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Predual Spaces of Morrey Spaces with Non-doubling Measures

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Abstract. In the present paper, we investigate the predual of the Morrey spaces with non-doubling measures. We also study the modified maximal function, singular integrals and commutators on the predual spaces.

1. Introduction

Recently analysis of the non-doubling measures has been developed very rapidly, stemming from the pioneering works [6, 7, 11, 12], for example. Many related function spaces, for example, the Hardy spaces, the BMO spaces, the Triebel-Lizorkin spaces, the Besov spaces and the Morrey spaces, are considered [2, 4, 9, 11, 12].

In many literatures dealing with the theory of non-homogeneous spaces we postulate some growth condition on the measure μ;

$$\mu \text{ is a Radon measure on } \mathbf{R}^d \text{ satisfying } \mu(B(x,r)) \le c_0\, r^n\,, \quad 0 < n \le d\,, \tag{1}$$

where $B(x,r)$ is an open ball centered at x of radius r. However, the theory can be developed, even if we do not assume the growth condition (1). Nazarov, Treil and Volberg proved the boundedness of the modified maximal operator assuming that the measure is just a Radon measure on metric measure spaces [7]. In [9] the authors defined the Morrey spaces for Radon measures on $\mathbf{R}^d$. Here and below by a "cube" we mean a compact cube whose edges are parallel to the coordinate axes. For $x = (x_1, \ldots, x_d)$, $l > 0$, $Q(x,l)$ denotes the set $\{(y_1, \ldots, y_d) : |y_j - x_j| \le l,\ j = 1, 2, \ldots, d\}$. Conversely if $Q = Q(x,l)$, we denote $z_Q = x$ and $\ell(Q) = l$. $\mathcal{Q}(\mu)$ denotes the set of all cubes with positive μ-measure. Let $1 \le q \le p < \infty$. We define $\mathcal{M}^p_q(\mu)$ to be the set of all μ-measurable functions f for which the norm

$$\|f : \mathcal{M}^p_q(\mu)\| = \sup_{Q \in \mathcal{Q}(\mu)} \mu(2Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q \, d\mu \right)^{\frac{1}{q}} < \infty\,.$$

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Here and below, given $\kappa > 0$ and a cube Q, by $\kappa\, Q$ we denote the cube R such that $z_R = z_Q$, $\ell(R) = \kappa\, \ell(Q)$.

One of the key properties the norm $\| \cdot : \mathcal{M}^p_q(\mu)\|$ enjoys is that the function space $\mathcal{M}^p_q(\mu)$ remains unchanged, if we replace the norm $\|f : \mathcal{M}^p_q(\mu)\|$ by

$$\|f : \mathcal{M}^p_q(\kappa, \mu)\| := \sup_{Q \in \mathcal{Q}(\mu)} \mu(\kappa\, Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q \, d\mu \right)^{\frac{1}{q}}, \ \kappa > 1 .$$

Speaking precisely, there exists $C = C_{\kappa,d} > 1$ such that

$$C^{-1} \, \|f : \mathcal{M}^p_q(\mu)\| \le \|f : \mathcal{M}^p_q(\kappa, \mu)\| \le C \, \|f : \mathcal{M}^p_q(\mu)\| \tag{2}$$

for all μ-measurable functions f. Define $\mathcal{M}^p_q(\kappa, \mu)$ as a set of all measurable functions $f \in L^q_{loc}(\mu)$-functions for which the norm $\|f : \mathcal{M}^p_q(\mu)\|$ is finite. In [9], for example, we considered the modified uncentered maximal operator given by

$$M_\kappa f(x) = \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu(\kappa\, Q)} \int_Q |f| d\mu . \tag{3}$$

If $\kappa > 1$, $1 < q \le p < \infty$, then we have shown $\| M_\kappa f : \mathcal{M}^p_q(\mu)\| \le c_\kappa \, \|f : \mathcal{M}^p_q(\mu)\|$. For details we refer to [9, Theorems 2.3, 2.4].

The aim of this paper is to specify the predual of the Morrey spaces with the underlying measure μ non-doubling. Given a Banach space X, we say that Y is a predual of X, if the topological dual of Y is isomorphic to X. In [13] the predual spaces of the Morrey spaces on $\mathbf{R}^d$ were considered for the Lebesgue measure. In [1] the predual spaces for the Lebesgue measure were considered in order to investigate the capacity of subsets in $\mathbf{R}^d$. However, our approach is different from that in [1]. In [5] Y. Komori and T. Mizuhara used the block functions to define the predual of the Morrey spaces. In this paper we consider their non-homogeneous version following [5].

Finally let us describe the organization of this paper. Section 2 will be devoted to some examples that will facilitate us to grasp what the Morrey spaces are. In Section 3 we define the "modified" predual of the Morrey spaces by means of the block functions. Section 4 is devoted to the study of the behavior of the modified maximal function M_κ on the predual spaces. In the last section, Section 5, we investigate various integral operators, singular integrals and commutators. Most of our results can be extended to the vector-valued inequality.

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2. Some Examples

In order to be accustomed with the definition, let us exhibit some examples. We shall give an example of the element in $\mathcal{M}^p_q(\mu)$.

PROPOSITION 2.1. *Consider the case $d = 1$. Let $\mathcal{H}^1$ be a 1-dimensional Lebesgue measure. Set $\mu(x) = e^{2x}\mathcal{H}^1(x)$ and $f(x) = e^{-x}$, $x \in \mathbf{R}$. Then we have*

$$f \in \mathcal{M}_1^2(\mu) \setminus \mathcal{M}_2^2(\mu) = \mathcal{M}_1^2(\mu) \setminus L^2(\mu)\,.$$

In particular $\mathcal{M}_1^2(\mu)$ is not isomorphic to $\mathcal{M}_2^2(\mu)$.

PROOF. By definition, we have $\|f : \mathcal{M}_2^2(\mu)\| = \int_{\mathbf{R}} 1dx = \infty$, disproving $f \in \mathcal{M}_2^2(\mu)$.

Next we show that $f \in \mathcal{M}_1^2(\mu)$. Again we write out $\|f : \mathcal{M}_1^2(\mu)\|$ in full.

$$\begin{aligned}\|f : \mathcal{M}_1^2(\mu)\| &= \sup_{x\in\mathbf{R}, l>0} \frac{\int_{x-l}^{x+l} e^y dy}{\sqrt{\mu((x-2l, x+2l))}} = \sup_{x\in\mathbf{R}, l>0} \frac{\int_{x-l}^{x+l} e^y dy}{\sqrt{\int_{x-2l}^{x+2l} e^{2y} dy}} \\ &= \sup_{x\in\mathbf{R}, l>0} \frac{e^{x+l} - e^{x-l}}{\sqrt{\frac{1}{2}(e^{2x+4l} - e^{2x-4l})}} = \sup_{l>0} \frac{e^l - e^{-l}}{\sqrt{\frac{1}{2}(e^{4l} - e^{-4l})}} < \infty\,,\end{aligned}$$

proving $f \in \mathcal{M}_1^2(\mu)$.

As a corollary we see that C_c^∞ is not dense in $\mathcal{M}_q^p(\mu)$.

PROPOSITION 2.2. *Let $1 \le q < p < \infty$. Then C_c^∞ is not dense in $\mathcal{M}_q^p(\mu)$.*

PROOF. We assume $p = 2$, the general case being similar. A similar calculation shows that $f \in \mathcal{M}_q^2(\mu)$ with $1 \le q < 2$ and the distance from C_c^∞ is not 0 due to self-similarity of f and μ, disproving that C_c^∞ is dense in $\mathcal{M}_q^p(\mu)$.

As we have seen, what is surprising about $\mathcal{M}_q^p(\kappa, \mu)$ is the independence of the parameter κ. We can define $\mathcal{M}_q^p(1, \mu)$ by inserting $\kappa = 1$ in the definition of $\mathcal{M}_q^p(\kappa, \mu)$. However, it can happen $\mathcal{M}_q^p(1, \mu)$ and $\mathcal{M}_q^p(2, \mu)$ are not isomorphic. We exhibit a counterexample showing that $\mathcal{M}_1^2(1, \mu)$ is not isomorphic to $\mathcal{M}_1^2(2, \mu)$.

EXAMPLE 2.3. Let $d = 2$ and dx_1dx_2 be the 2-dimensional Lebesgue measure. We set $\mu(x) = e^{2x_2}dx_1dx_2$ and $f(x) = e^{-x_2}$. Then $f \in \mathcal{M}_1^2(2, \mu) \setminus \mathcal{M}_1^2(1, \mu)$.

PROOF. Let $Q = [y-l, y+l] \times [z-l, z+l]$ be a cube. Then we have

$$\frac{\int_Q |f| d\mu}{\sqrt{\mu(Q)}} = \frac{\sqrt{l}\int_{z-l}^{z+l} e^{x_2} dx_2}{\sqrt{\int_{z-l}^{z+l} e^{2x_2} dx_2}} = \frac{\sqrt{l}(e^{z+l} - e^{z-l})}{\sqrt{\frac{1}{2}(e^{2z+2l} - e^{2z-2l})}} = \sqrt{\frac{2l(e^l - e^{-l})}{e^l + e^{-l}}}\,.$$

Thus taking the supremum over Q, we see that this quantity becomes ∞. As a result we conclude $f \notin \mathcal{M}_1^2(1, \mu)$.

Now we will prove $f \in \mathcal{M}_1^2(2,\mu)$. In fact a similar calculation gives

$$\mu(2Q)^{-1/2}\int_Q |f|d\mu = \sqrt{\frac{2l(e^l - e^{-l})}{e^{3l}+e^l+e^{-l}+e^{-3l}}} \leq c < \infty\,.$$

As a consequence the proof of this proposition is finished.

3. Predual of the Morrey spaces

In this section we give a definition of the predual spaces.

DEFINITION 3.1. Let $1 \leq p \leq q < \infty$. A μ-measurable function A is said to be a (p,q)-block if there is a cube $Q \in \mathcal{Q}(\mu)$ supporting A and $\|A\|_q \leq \mu(2Q)^{1/q-1/p}$. If one is to stress the cube Q supporting A, then one says A is a (Q,p,q)-block. As is easily verified by Hölder's inequality, any (p,q)-block has the $L^p(\mu)$ norm less than 1.

DEFINITION 3.2. Let $1 \leq p \leq q < \infty$. Then define a function space $\mathcal{H}_q^p(\mu)$ by

$$\mathcal{H}_q^p(\mu) := \left\{ f \in L^p(\mu) : f = \sum_{j \in \mathbf{N}} \lambda_j A_j,\ \{\lambda_j\} \in l^1,\ \text{each } A_j \text{ is a } (p,q)\text{-block} \right\}.$$

Define $\|f : \mathcal{H}_q^p(\mu)\|$ for $f \in \mathcal{H}_q^p(\mu)$ by $\|f : \mathcal{H}_q^p(\mu)\| := \inf_\lambda \left(\sum_{j=1}^\infty |\lambda_j|\right)$, where λ runs over the admissible expressions $f = \sum_{j\in\mathbf{N}} \lambda_j A_j,\ \{\lambda_j\} \in l^1,\ A_j$ is a (p,q)-block for all $j \in \mathbf{N}$.

It is easy to see from the definition that $\mathcal{H}_q^p(\mu)$, $1 \leq p \leq q < \infty$ is a Banach space and that $\mathcal{H}_q^p(\mu)$ is embedded into $L^p(\mu)$ continuously. The following proposition, asserting that $\mathcal{H}_q^p(\mu)$ carries a structure of lattice, also follows immediately from the definition.

PROPOSITION 3.3. *Let $1 \leq p \leq q < \infty$. If f is a μ-measurable function which is majorized by $g \in \mathcal{H}_q^p(\mu)$, that is, $|f(x)| \leq |g(x)|$ for μ-a.e. $x \in \mathbf{R}^d$, then $f \in \mathcal{H}_q^p(\mu)$ with a norm estimate $\|f : \mathcal{H}_q^p(\mu)\| \leq \|g : \mathcal{H}_q^p(\mu)\|$.*

$\mathcal{H}_q^p(\mu)^*$, the dual of $\mathcal{H}_q^p(\mu)$, is $\mathcal{M}_{q'}^{p'}(\mu)$ in the following sense, where u' denotes the harmonic conjugate of $u \in [1,\infty]$:

$$u' = \begin{cases} \infty & (u=1) \\ \dfrac{u}{u-1} & (1<u<\infty) \\ 1 & (u=\infty) \end{cases}$$

THEOREM 3.4. *Suppose that $1 < p \leq q < \infty$.*

1. *If $f \in \mathcal{M}_{q'}^{p'}(\mu)$, then $L_f : g \mapsto \int_{\mathbf{R}^d} f \cdot g\, d\mu$ is a continuous functional on $\mathcal{H}_q^p(\mu)$.*
2. *Conversely any continuous functional L on $\mathcal{H}_q^p(\mu)$ is realized with $f \in \mathcal{M}_{q'}^{p'}(\mu)$.*

3. *The correspondence* $f \in \mathcal{M}^{p'}_{q'}(\mu) \mapsto L_f \in \mathcal{H}^p_q(\mu)^*$ *is an isomorphism. Furthermore*

$$\|f : \mathcal{M}^{p'}_{q'}(\mu)\| = \sup_{h \in \mathcal{H}^p_q(\mu)\setminus\{0\}} \frac{\left|\int_{\mathbf{R}^d} f \cdot h d\mu\right|}{\|h : \mathcal{H}^p_q(\mu)\|} \tag{4}$$

$$\|g : \mathcal{H}^p_q(\mu)\| = \sup_{h \in \mathcal{M}^{p'}_{q'}(\mu)\setminus\{0\}} \frac{\left|\int_{\mathbf{R}^d} g \cdot h d\mu\right|}{\|h : \mathcal{M}^{p'}_{q'}(\mu)\|}.$$

for all $f \in \mathcal{M}^{p'}_{q'}(\mu)$ *and* $g \in \mathcal{H}^p_q(\mu)$.

PROOF. The proof of 1 is straightforward so we omit it. In fact it is easy to establish that $|L_f|_* \le \| f : \mathcal{M}^{p'}_{q'}(\mu)\|$, where $|L_f|_* := \|L_f\|_{\mathcal{H}^p_q(\mu)\to\mathbf{C}}$ denotes the operator norm on $\mathcal{H}^p_q(\mu)^*$.

Let us prove 2. Take a cube $Q_0 \in \mathcal{Q}(\mu)$ and let $Q_j = 2^j Q_0$ for $j \in \mathbf{N}$. For the sake of simplicity we denote $L^q(Q_j, \mu)$ by the set of $L^q(\mu)$-functions supported on Q_j. Since we can regard every element in $L^q(Q_j, \mu)$ as a (p, q)-block modulo multiplicative constants, the functional $g \mapsto L(g)$ is well-defined and bounded on $L^q(Q_j, \mu)$. Thus the duality $L^q(\mu)$-$L^{q'}(\mu)$ gives us $f_j \in L^{q'}(Q_j, \mu)$ satisfying $L(g) = \int_{Q_j} f_j \cdot g d\mu$ for all $g \in L^q(Q_j, \mu)$. By the uniqueness of this theorem we can find an $L^q_{loc}(\mu)$-function f such that $f|_{Q_j} = f_j|_{Q_j}$ μ-a.e..

We shall establish that $f \in \mathcal{M}^{p'}_{q'}(\mu)$, which amounts to estimating

$$\mathrm{I} := \mu(2Q)^{\frac{1}{p'}-\frac{1}{q'}} \left(\int_Q |f|^{q'} d\mu\right)^{\frac{1}{q'}}$$

for a fixed $Q \in \mathcal{Q}(\mu)$.

For fixed $Q \in \mathcal{Q}(\mu)$ and a function f we set $g(x) := \chi_Q(x)\overline{sgn(f(x))}|f(x)|^{q'-1}$. Then we can write

$$\mathrm{I} = \mu(2Q)^{1/p'-1/q'} \left(\int_Q f \cdot g d\mu\right)^{1/q'} = \mu(2Q)^{1/p'-1/q'} (L(g))^{1/q'}.$$

Since the function $\frac{\mu(2Q)^{1/p'-1/q'}}{\|g\|_q} g = \frac{\mu(2Q)^{1/q-1/p}}{\|g\|_q} g$ is a (p, q)-block, we have

$$|L(g)| \le |L|_* \, \mu(2Q)^{-1/p'+1/q'} \|g\|_q .$$

As a result we conclude $\mathrm{I} \le |L|_*$. This is the desired result. The proof of 3 is already included in those of 1 and 2.

We supple ment Theorem 3.4 by stating the limiting case when $p = 1$, whose proof is obtained in the same way as Theorem 3.4.

THEOREM 3.5. *Suppose that* $1 \le q < \infty$.

(a) *Let* $f \in L^\infty(\mu)$. *Then* $L_f : g \mapsto \int_{\mathbf{R}^d} f \cdot g d\mu$ *is a continuous functional on* $\mathcal{H}^1_q(\mu)$.

(b) *Conversely every continuous functional* L *on* $\mathcal{H}^1_q(\mu)$ *can be realized with* $f \in L^\infty(\mu)$.

(c) *The mapping* $f \in L^\infty(\mu) \mapsto L_f \in \mathcal{H}^1_q(\mu)^*$ *is an isomorphism. Furthermore*

$$\|f\|_\infty = \sup_{g \in \mathcal{H}^1_q(\mu) \setminus \{0\}} \frac{\left|\int_{\mathbf{R}^d} f \cdot g d\mu\right|}{\|g : \mathcal{H}^1_q(\mu)\|}$$

$$\|g : \mathcal{H}^1_q(\mu)\| = \sup_{f \in L^\infty(\mu) \setminus \{0\}} \frac{\left|\int_{\mathbf{R}^d} f \cdot g d\mu\right|}{\|f\|_\infty}.$$

COROLLARY 3.6. *Let* $1 < p \le q < \infty$ *and assume* $f \in \mathcal{H}^p_q(\mu)$. *Then*

$$\|f : \mathcal{H}^p_q(\mu)\| = \sup_{\substack{g \in \mathcal{M}^{p'}_{q'}(\mu) \\ \|g : \mathcal{M}^{p'}_{q'}(\mu)\| = 1}} \left| \int_{\mathbf{R}^d} f \cdot g d\mu \right|.$$

Before we finish Section 3, a helpful remark may be in order.

The number 2 appearing in Definition 3.1 does not count so much. Any number will do in defining $\mathcal{H}^p_q(\mu)$. To formulate this fact more precisely, we make the following definition.

DEFINITION 3.7. Let $1 \le p \le q < \infty$, $K > 1$. One says that a μ-measurable function A is a (p, q, K)-block if A is supported on some $Q \in \mathcal{Q}(\mu)$ and $\|A\|_q \le \mu(K\,Q)^{\frac{1}{q} - \frac{1}{p}}$. If there is need to specify Q, said to be A is a (Q, p, q, K)-block.

With this definition in mind, let us formulate an important observation in $\mathcal{H}^p_q(\mu)$.

PROPOSITION 3.8. *Set*

$$\mathcal{H}^p_q(K, \mu) := \left\{ f \in L^p(\mu) : f = \sum_{j \in \mathbf{N}} \lambda_j A_j,\ \{\lambda_j\} \in l^1,\ \text{each } A_j \text{ is a } (p, q, K)\text{-block} \right\}$$

and $\|f : \mathcal{H}^p_q(K, \mu)\| := \inf_\lambda \left(\sum_{j=1}^\infty |\lambda_j| \right)$, *where* λ *runs over the admissible expressions*

$$f = \sum_{j \in \mathbf{N}} \lambda_j A_j,\ \{\lambda_j\} \in l^1,\ A_j \text{ is a } (p, q, K)\text{-block for all } j \in \mathbf{N}.$$

Then $\mathcal{H}^p_q(K, \mu)$ *coincides with* $\mathcal{H}^p_q(\mu)$ *as a set and their norms are mutually equivalent.*

PROOF. We can assume that $K_2 > K_1 > 1$ by symmetry with respect to K_1, K_2. By the monotonicity of the norm $\| \cdot : \mathcal{H}^p_q(K, \mu)\|$ with respect to K, it can be assumed even that

$K_2 = 2K_1 - 1 > 1$. Finally we note that it suffices to prove Proposition 3.8 at a level of blocks ; it is enough to prove $\| A : \mathcal{H}^p_q(K_2, \mu)\| \le c$ for each (p, q, K_1)-block A.

Suppose A is a (Q, p, q, K_1)-block. Then bisect Q and label $Q_1, \dots, Q_N$, $N \le 2^d$ to those which μ charges. Then a geometric observation shows that the distance from z_{Q_j} to the boundary of $K_1\, Q$ is $K_2 \ell(Q_j)$. From this we obtain $K_2\, Q_j \subset K_1\, Q$. As a consequence $\chi_{Q_j} \cdot A$ is a (Q_j, p, q, K_2)-block, which shows that A is decomposed into a sum of at most 2^d (p, q, K_2)-blocks. Proposition 3.8 is therefore proved.

4. Maximal operators

Having cleared up the definition and elementary properties of the predual spaces $\mathcal{H}^p_q$, we now turn to the modified maximal operator M_κ, $\kappa > 1$, given by (3).

$\mathcal{H}^p_q(\mu)$ **maximal inequality.** First we prove M_κ maps $\mathcal{H}^p_q(\mu)$ to itself, if $1 < p \le q < \infty$ and $\kappa > 1$.

Let us set

$$K := \frac{7\kappa}{\kappa - 1}, \quad K_1 := \frac{2\kappa}{\kappa + 1}, \quad K_2 := \frac{4\kappa}{3\kappa + 1}. \tag{5}$$

THEOREM 4.1. *Let $1 < p \le q < \infty$ and $\kappa > 1$. Then M_κ is bounded on $\mathcal{H}^p_q(\mu)$.*
To prove Theorem 4.1 we have only to prove the following.

LEMMA 4.2. *Let $\kappa > 1$. Suppose that A is a $(Q, p, q, 2K)$-block, where K is given by (5). Then there exist a countable sequence $R_{l,m} \in \mathcal{Q}(\mu) \cup \{\mathbf{R}^d\}$, $l \in \mathbf{N}$, $m = 1, 2, \dots, N$ and a sequence $\{f_{l,m}\}$ of μ-measurable functions with the following properties. Here N depends only on κ.*

(a) *The pointwise estimate $M_\kappa A(x) \le \sum_{l \in \mathbf{N}} \sum_{m=1}^N f_{l,m}(x)$, $x \in \mathbf{R}^d \setminus K\, Q$ holds.*

(b) *$\mu\left(K_1 R_{l,m}\right) \sim 2^l \mu\left(\frac{3}{2} Q\right)$, where the implicit constants appearing in $\sim$ do not depend on Q, κ and l.*

(c) *$0 \le f_{l,m}(x) \le \frac{\|A\|_1}{2^{l-1}\mu\left(\frac{3}{2}Q\right)} \chi_{K_2\, R_{l,m}}(x)$ for $x \in \mathbf{R}^d$.*

In particular there exists $c = c_{\kappa,p,q}$ so that, for every $(Q, p, q, 2K)$-block A,

$$\| M_\kappa A : \mathcal{H}^p_q(\mu)\| \le c. \tag{6}$$

Indeed, once we accept Lemma 4.2, we can prove Theorem 4.1 in the following way.

First, suppose that we are given $f \in \mathcal{H}^p_q(\mu)$. Then we can find a sequence $\{A_j\}_{j \in \mathbf{N}}$ of $(Q, p, q, 2K)$-block and $\lambda = \{\lambda_j\}_{j \in \mathbf{N}}$ such that

$$f = \sum_{j=1}^\infty \lambda_j\, A_j, \ \sum_{j=1}^\infty |\lambda_j| \le 2\|f : \mathcal{H}^p_q(\mu)\|.$$

By Lemma 4.2 the function $g := \sum_{j=1}^{\infty} |\lambda_j| M_\kappa A_j$ satisfies

$$|f(x)| \le g(x)\ \mu\text{-a.e. and } \|g : \mathcal{H}^p_q(\mu)\| \le c \sum_{j=1}^{\infty} |\lambda_j| \le c\, \|f : \mathcal{H}^p_q(\mu)\| .$$

In view of Proposition 3.3, we obtain

$$\|M_\kappa f : \mathcal{H}^p_q(\mu)\| \le \|g : \mathcal{H}^p_q(\mu)\| \le c\, \|f : \mathcal{H}^p_q(\mu)\| .$$

PROOF OF Lemma 4.2. First of all, we construct the desired cubes and the desired functions. Let $x \in \mathbf{R}^d \setminus K\,Q$. A cube $R \in \mathcal{Q}(\mu)$ satisfies $\frac{3}{2}Q \subset \frac{\kappa+1}{2}R$ whenever R intersects Q. Consequently it follows that

$$M_\kappa A(x) \le \sup_{x \in R} \frac{\|A\|_1}{\mu(\kappa\, R)} \le \sup_{\frac{3}{2}Q \cup \{x\} \subset R} \frac{\|A\|_1}{\mu\,(K_1 R)} ,$$

for $x \in \mathbf{R}^d \setminus K\,Q$. Set

$$Z_l := \left\{ R \in \mathcal{Q}(\mu) \cup \{\mathbf{R}^d\} : \frac{3}{2}Q \subset R,\ 2^{l-1}\mu\left(\frac{3}{2}Q\right) \le \mu\,(K_1 R) < 2^l \mu\left(\frac{3}{2}Q\right) \right\}$$

for $l \in \mathbf{N}$. Although the union $\bigcup_{R \in Z_l} R$ can be unbounded, we are still in the position of applying Besicovitch's covering lemma to obtain $R_{l,1}, \dots, R_{l,N} \in Z_l$, with N independent of l, such that $\bigcup_{R \in Z_l} R \subset \bigcup_{m=1}^{N} K_2\, R_{l,m}$ holds. If $\bigcup_{R \in Z_l} R$ is unbounded, it suffices to set $R_{l,1} = \cdots = R_{l,N} = \mathbf{R}^d$. We define $f_{l,m} := \chi_{K_2\, R_{l,m} \setminus K\,Q} \cdot M_\kappa A$. Then we have $0 \le f_{l,m}(x) \le \frac{2^{-l+1}}{\mu\left(\frac{3}{2}Q\right)} \|A\|_1$ for $x \in \mathbf{R}^d$. Thus, we have found the decomposition described in Lemma 4.2.

Let us prove (6). We decompose $M_\kappa A$ according to $K\,Q$, that is, we split $M_\kappa A$ by $M_\kappa A = A_1 + A_2$ with $A_1 = \chi_{K\,Q} \cdot M_\kappa A$ and $A_2 = \chi_{\mathbf{R}^d \setminus K\,Q} \cdot M_\kappa A$. Accordingly the estimate was split into those of $\|\, A_1 : \mathcal{H}^p_q(\mu)\|$ and $\|\, A_2 : \mathcal{H}^p_q(\mu)\|$.

The estimate of $\|\, A_1 : \mathcal{H}^p_q(\mu)\|$ is simple. It is known that M_κ is $L^q(\mu)$-bounded. We refer to [8, Theorem 1.6] or [11, p. 127] for the proof. If we set the operator norm $C_0 := \|M_\kappa\|_{L^q(\mu) \to L^q(\mu)}$, then we see that $\frac{A_1}{C_0}$ is a $(p, q, 2)$-block. As a result the estimate of $\|\, A_1 : \mathcal{H}^p_q(\mu)\|$ is valid. To obtain the norm estimate of A_2, we observe

$$\|f_{l,m}\|_q \le \frac{c\,\mu\left(K_2\, R_{l,m}\right)^{\frac{1}{q}}}{2^l} \left(\frac{\int_Q |A|^q d\mu}{\mu\left(\frac{3}{2}Q\right)} \right)^{\frac{1}{q}}$$

$$\leq \frac{c\,\mu\left(K_2\,R_{l,m}\right)^{\frac{1}{q}}}{2^l \mu\left(\frac{3}{2}Q\right)^{\frac{1}{p}}}$$

$$\leq \frac{c\,\mu\left(K_1 R_{l,m}\right)^{\frac{1}{q}-\frac{1}{p}}}{2^{l\left(1-\frac{1}{p}\right)}}\,.$$

From this we deduce $2^{l\left(1-\frac{1}{p}\right)} f_{l,m}$ is a $\left(K_2\,R_{l,m},\,p,\,q,\,\frac{K_1}{K_2}\right)$ -block. Since a pointwise estimate $|A_2(x)| \leq \sum_{l\in\mathbf{N}}\sum_{m=1}^{N} f_{l,m}(x)$ holds, A_2 belongs to $\mathcal{H}^p_q(\mu)$ and its norm is bounded by some absolute constant.

Vector-valued extension. Here we consider a vector-version as the variant of the previous section. We intend to prove

THEOREM 4.3. *Suppose that $1 < p \leq q < \infty$ and $1 < r \leq \infty$. Then there exists a constant $c > 0$ such that*

$$\left\| \left(\sum_{j=1}^{\infty}(M_\kappa f_j)^r\right)^{\frac{1}{r}} : \mathcal{H}^p_q(\mu)\right\| \leq c \left\| \left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}} : \mathcal{H}^p_q(\mu)\right\|$$

for any sequence of measurable functions $\{f_j\}_{j=1}^{\infty}$.

PROOF. To prove Theorem 4.3 we may assume that the right-hand-side is finite. Otherwise there is nothing to prove. We may also assume that $\sum_{j=1}^{\infty}|f_j|^r > 0$ for μ-a.e., because we have only to incorporate $f_0 := \tilde{\varepsilon}\phi$ with $\tilde{\varepsilon} > 0$ small, where $\phi \in \mathcal{H}^p_q(\mu)$ is a function that does not vanish for μ-a.e. $x \in \mathbf{R}^d$. Furthermore, it can be assumed even that the f_j are zero μ-a.e. for all j larger than some j_0. Let $\varepsilon > 0$ be given. By Proposition 3.8 we can find a sequence of blocks $\{A_k\}_{j\in\mathbf{N}}$ satisfying

$$\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}} = \sum_{k\in\mathbf{N}}\lambda_k A_k \quad \text{and} \quad \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}} : \mathcal{H}^p_q(\mu)\right\| \leq \left(\sum_{k=1}^{\infty}|\lambda_k|\right) + \varepsilon\,.$$

More specifically we let A_k be a $(Q_k, p, q, 2K)$-block.

We define a partition of the unit by setting

$$W_k := |\lambda_k A_k| \left(\sum_{i=1}^{\infty}|\lambda_i A_i|\right)^{-1}, \quad k \in \mathbf{N}\,.$$

Note that $\sum_{k=1}^{\infty} W_k(x) = 1$ for μ-a.e. $x \in \mathbf{R}^d$. The triangle inequality gives us

$$\left\| \left(\sum_{j=1}^{\infty} (M_\kappa f_j)^r \right)^{\frac{1}{r}} : \mathcal{H}^p_q(\mu) \right\| \le \sum_{k=1}^{\infty} \left\| \left(\sum_{j=1}^{\infty} (M_\kappa [W_k \cdot f_j])^r \right)^{\frac{1}{r}} : \mathcal{H}^p_q(\mu) \right\| .$$

As before, we decompose the estimate of the summand with respect to $K\,Q_k$ for each k. Let us set

$$\mathrm{I}_k := \left\| \left(\sum_{j=1}^{\infty} \chi_{K\,Q_k} \cdot (M_\kappa [W_k \cdot f_j])^r \right)^{\frac{1}{r}} : \mathcal{H}^p_q(\mu) \right\|$$

$$\mathrm{II}_k := \left\| \left(\sum_{j=1}^{\infty} \chi_{\mathbf{R}^d \setminus K\,Q_k} \cdot (M_\kappa [W_k \cdot f_j])^r \right)^{\frac{1}{r}} : \mathcal{H}^p_q(\mu) \right\| .$$

The estimate of I_k is simple again. Invoking Fefferman-Stein inequality, established in [8, Theorem 1.7], we see, for each k,

$$\begin{aligned}
\mathrm{I}_k &\le c \left\| \left(\sum_{j=1}^{\infty} \chi_{K\,Q_k} \cdot (M_\kappa [W_k \cdot f_j])^r \right)^{\frac{1}{r}} \right\|_q \cdot \mu(2K\,Q_k)^{\frac{1}{p}-\frac{1}{q}} \\
&\le c \left\| \left(\sum_{j=1}^{\infty} |W_k|^r \cdot |f_j|^r \right)^{\frac{1}{r}} \right\|_q \cdot \mu(2K\,Q_k)^{\frac{1}{p}-\frac{1}{q}} \\
&= c \left\| \frac{\lambda_k A_k \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}}}{\sum_{k \in \mathbf{N}} |\lambda_k A_k|} \right\|_q \cdot \mu(2K\,Q_k)^{\frac{1}{p}-\frac{1}{q}} \\
&\le c\,|\lambda_k| \cdot \|A_k\|_q \cdot \mu(2K\,Q_k)^{\frac{1}{p}-\frac{1}{q}} \le c\,|\lambda_k| .
\end{aligned}$$

Consequently adding this inequality over $k \in \mathbf{N}$, we obtain

$$\sum_{k=1}^{\infty} \mathrm{I}_k \le c \sum_{j=1}^{\infty} |\lambda_j| \tag{7}$$

and the estimate of this term is valid.

Now let us turn our attention to II_k. We shall prove $\sum_{k=1}^{\infty} \mathrm{II}_k \le c\,\|\lambda : l^1\|$ with the help of Lemma 4.2.

By the aforementioned lemma there exist countable sets of cubes $\{R_{k,l,m}\}$ and functions $\{f_{j,k,l,m}\}$ with the following properties. Here K_1, K_2 are given by (5).

(a) $M_\kappa[W_k \cdot f_j](x) \le \sum_{l=1}^\infty \sum_{m=1}^N f_{j,k,l,m}(x)$ for all $j, k \in \mathbf{N}$ and $x \in \mathbf{R}^d \setminus K\,Q_k$.

(b) $\mu\left(K_1 R_{k,l,m}\right) \sim 2^l \mu\left(\frac{3}{2}Q_k\right)$ for all $k, l \in \mathbf{N}$ and $m = 1, 2, \ldots, N$.

(c) $0 \le f_{j,k,l,m}(x) \le \frac{\int_{Q_k} W_k \cdot |f_j| d\mu}{2^{l-1}\mu\left(\frac{3}{2}Q_k\right)} \chi_{K_2\, R_{k,l,m}}$ for all $j, k, l \in \mathbf{N}$ and $m = 1, 2, \ldots, N$.

By virtue of (a) we have

$$\sum_{j=1}^\infty M_\kappa[W_k \cdot f_j](x)^r \le \sum_{j=1}^\infty \left(\sum_{l=1}^\infty \sum_{m=1}^N f_{j,k,l,m}(x)\right)^r \le c \sum_{j=1}^\infty \sum_{m=1}^N \left(\sum_{l=1}^\infty f_{j,k,l,m}(x)\right)^r$$

for each $x \in \mathbf{R}^d \setminus K\,Q_k$. We now invoke (c) to obtain

$$\sum_{l=1}^\infty f_{j,k,l,m}(x) \le c \sum_{l=1}^\infty \frac{\int_{Q_k} W_k \cdot |f_j| d\mu}{2^l \mu\left(\frac{3}{2}Q_k\right)} \chi_{K_2\, R_{k,l,m}}(x)\,.$$

Let $0 < \varepsilon < 1 - \frac{1}{p}$. Then by the Hölder's inequality, we obtain a pointwise estimate

$$\sum_{l=1}^\infty \frac{\int_{Q_k} W_k \cdot |f_j| d\mu}{2^l \mu\left(\frac{3}{2}Q_k\right)} \chi_{K_2\, R_{k,l,m}} \le c \left\{\sum_{l=1}^\infty \left(\frac{\int_{Q_k} W_k \cdot |f_j| d\mu}{2^{l(1-\varepsilon)} \mu\left(\frac{3}{2}Q_k\right)}\right)^r \chi_{K_2\, R_{k,l,m}}\right\}^{\frac{1}{r}}.$$

If we insert this estimate to the functions in question, we obtain

$$\begin{aligned}\left(\sum_{j=1}^\infty M_\kappa[W_k \cdot f_j](x)^r\right)^{\frac{1}{r}} &\le c \left\{\sum_{j,l=1}^\infty \sum_{m=1}^N \left(\frac{\int_{Q_k} W_k \cdot |f_j| d\mu}{2^{l(1-\varepsilon)} \mu\left(\frac{3}{2}Q_k\right)}\right)^r \chi_{K_2\, R_{k,l,m}}(x)\right\}^{\frac{1}{r}} \\ &\le c \sum_{m=1}^N \left\{\sum_{j,l=1}^\infty \left(\frac{\int_{Q_k} W_k \cdot |f_j| d\mu}{2^{l(1-\varepsilon)} \mu\left(\frac{3}{2}Q_k\right)} \chi_{K_2\, R_{k,l,m}}(x)\right)^r\right\}^{\frac{1}{r}}\end{aligned}$$

for all $x \in \mathbf{R}^d \setminus K\,Q_k$. By applying the Minkowski inequality the above quantity is estimated by

$$\begin{aligned}&c \sum_{m=1}^N \int_{Q_k} \left\{\sum_{j,l=1}^\infty \left(\frac{2^{-l(1-\varepsilon)}}{\mu\left(\frac{3}{2}Q_k\right)} W_k \cdot |f_j| \cdot \chi_{K_2\, R_{k,l,m}}\right)^r\right\}^{\frac{1}{r}} d\mu \\ &= c \sum_{m=1}^N \frac{1}{\mu\left(\frac{3}{2}Q_k\right)} \int_{Q_k} \left\{\sum_{j,l=1}^\infty 2^{-lr(1-\varepsilon)} \chi_{K_2\, R_{k,l,m}} {W_k}^r \cdot |f_j|^r\right\}^{\frac{1}{r}} d\mu\,. \qquad (8)\end{aligned}$$

Taking into account the definition of W_k, we obtain

$$W_k(z)^r \left(\sum_{j=1}^{\infty} |f_j(z)|^r \right) \leq |\lambda_k|^r \cdot |A_k(z)|^r . \tag{9}$$

Furthermore, since $r > 1$, we are in the position of using $(a+b)^{\frac{1}{r}} \leq a^{\frac{1}{r}} + b^{\frac{1}{r}}$ for $a, b > 0$. Hence, we have

$$\left(\sum_{l=1}^{\infty} 2^{-lr(1-\varepsilon)} \chi_{K_2\, R_{k,l,m}}(x) \right)^{\frac{1}{r}} \leq \sum_{l=1}^{\infty} 2^{-l(1-\varepsilon)} \chi_{K_2\, R_{k,l,m}}(x) . \tag{10}$$

Inequalities (8)–(10) give us, for $x \in \mathbf{R}^d \setminus K\, Q$,

$$\left(\sum_{j=1}^{\infty} M_\kappa [W_k \cdot f_j](x)^r \right)^{\frac{1}{r}} \leq \frac{c\, |\lambda_k| \cdot \|A_k\|_1}{\mu\left(\frac{3}{2} Q_k\right)} \sum_{m=1}^{N} \sum_{l=1}^{\infty} 2^{-l(1-\varepsilon)} \chi_{K_2\, R_{k,l,m}}(x) .$$

Finally observe $\frac{2^{-\frac{l}{p}} \|A_k\|_1}{\mu\left(\frac{3}{2} Q_k\right)} \chi_{K_2\, R_{k,l,m}}$ is a $\left(K_2\, R_{k,l,m}, p, q, \frac{K_1}{K_2}\right)$ -block modulo multiplicative constants independent of l. Indeed, with the help of (b) and the Hölder inequality

$$\begin{aligned}
\left\| \frac{2^{-\frac{l}{p}} \|A_k\|_1}{\mu\left(\frac{3}{2} Q_k\right)} \chi_{K_2\, R_{k,l,m}} \right\|_q &\leq \left\| 2^{-\frac{l}{p}} \left(\frac{\int_{Q_k} |A_k|^q d\mu}{\mu\left(\frac{3}{2} Q_k\right)} \right)^{\frac{1}{q}} \chi_{K_2\, R_{k,l,m}} \right\|_q \\
&\leq 2^{-\frac{l}{p}} \mu \left(\frac{3}{2} Q_k \right)^{-\frac{1}{p}} \mu\left(K_2\, R_{k,l,m}\right)^{\frac{1}{q}} \\
&\leq 2^{-\frac{l}{p}} \mu \left(\frac{3}{2} Q_k \right)^{-\frac{1}{p}} \mu\left(K_1\, R_{k,l,m}\right)^{\frac{1}{q}} \\
&\leq c\, \mu\left(K_1\, R_{k,l,m}\right)^{\frac{1}{q}-\frac{1}{p}} .
\end{aligned}$$

It follows from these observations that

$$\left(\sum_{j=1}^{\infty} M_\kappa [W_k \cdot f_j](x)^r \right)^{\frac{1}{r}} \leq c \sum_{m=1}^{N} \sum_{l=1}^{\infty} \frac{|\lambda_k| \cdot \|A_k\|_1 2^{-\frac{l}{p}}}{2^{l(1-\varepsilon-\frac{1}{p})} \mu\left(\frac{3}{2} Q_k\right)} \chi_{K_2\, R_{k,l,m}}(x)$$

for $x \in \mathbf{R}^d \setminus K\, Q_k$. This estimate is summable over $k \in \mathbf{N}$ to a quantity less than $c \sum_{j=1}^{\infty} |\lambda_j|$ after taking the $\mathcal{H}^p_q(\mu)$-norm :

$$\sum_{k=1}^{\infty} \mathrm{II}_k \leq c \sum_{k,l=1}^{\infty} \sum_{m=1}^{N} \frac{|\lambda_k|}{2^{l(1-\varepsilon-\frac{1}{p})}} \leq c \sum_{j=1}^{\infty} |\lambda_j| . \tag{11}$$

(7) and (11) prove the theorem.

5. Boundedness of the other operators

In this section we prove the boundedness of the linear operators.

Singular integral operators. Here we assume the growth condition (1).

Recall that $T : L^2(\mu) \to L^2(\mu)$ is a singular integral operator, if there exists a function K that satisfies three properties listed below.

(a) There exists $c > 0$ such that $|K(x, y)| \le \frac{c}{|x-y|^n}$ for all $x \ne y$.

(b) There exist $\varepsilon > 0$ and $c > 0$ so that

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \le c \frac{|x - z|^\varepsilon}{|x - y|^{n+\varepsilon}}$$

for every $x, y, z \in \mathbf{R}^d$ with $|x - y| \ge 2|x - z| > 0$.

(c) If f is a bounded μ-measurable function with a compact support, then we have $Tf(x) = \int_{\mathbf{R}^d} K(x, y) f(y) d\mu(y)$ for μ-a.e. $x \notin \operatorname{supp}(f)$.

In [7] it was shown that T can be extended to an $L^p(\mu)$-bounded linear operator for $1 < p < \infty$. Furthermore we have shown that T is $\mathcal{M}^p_q(\mu)$-bounded for $1 < q \le p < \infty$ [9, Theorem 6.6]. We are to deduce [9, Theorem 6.6] reversely from the $\mathcal{H}^p_q(\mu)$-boundedness of T. The definition of the singular integral operators on $\mathcal{M}^p_q(\mu)$ in [9, Definition 6.4] was very awkward : In [9, Definition 6.4] we have defined Tf by

$$Tf(x) = \lim_{m\to\infty} \left(T[\chi_{\{|y|\le 2m\}} f](x) + \int_{\{|y|>2m\}} K(x, y) f(y) d\mu(y) \right)$$

for $f \in \mathcal{M}^p_q(\mu)$ with $1 < q \le p < \infty$.

Once we accept the $\mathcal{H}^p_q(\mu)$-boundedness of T, we can give a natural definition of Tf for $f \in \mathcal{M}^p_q(\mu)$ with $1 < q \le p < \infty$; we can redefine Tf as the unique element $h \in \mathcal{M}^p_q(\mu)$ satisfying $\int_{\mathbf{R}^d} h \cdot g d\mu = \int_{\mathbf{R}^d} f \cdot T^* g d\mu$ for all $g \in \mathcal{H}^{p'}_{q'}(\mu)$. We note that the $\mathcal{M}^p_q(\mu)$-boundedness of T can be obtained conversely from this formula and (4). This is why we intend to prove the $\mathcal{H}^p_q(\mu)$-boundedness of T without using the boundedness of T on $\mathcal{M}^p_q(\mu)$ established in [9, Theorem 6.6]. The same can be said for commutators whose boundedness we proved in [10, Theorems 4.5, 4.6].

THEOREM 5.1. *Suppose that $1 < p \le q < \infty$. Then T is $\mathcal{H}^p_q(\mu)$-bounded.*

PROOF. The proof is similar to Theorem 5.4 and we omit it.

Commutators. Here we postulate on μ the growth condition (1). Before we formulate our theorems, let us recall the definition of the RBMO spaces due to Tolsa [11]. Given two cubes $Q \subset R$ with $Q \in \mathcal{Q}(\mu)$, we write

$$\delta(Q, R) := \int_{\ell(Q)}^{\ell(R)} \frac{\mu(B(z_Q, l))}{l^n} \frac{dl}{l} \text{ and } K_{Q,R} = 1 + \delta(Q, R)\,,$$

where z_Q denotes the center of the cube Q. A cube $Q \in \mathcal{Q}(\mu)$ is said to be doubling, if $\mu(2Q) \le 2^{d+1}\mu(Q)$. The set of all doubling cubes will be denoted by $\mathcal{Q}(\mu, 2)$. Given $Q \in \mathcal{Q}(\mu)$, we set Q^* as the smallest doubling cube R of the form $R = 2^j Q$ with $j = 0, 1, \ldots$.[1]

Let us recall that Tolsa defined $\|f\|_*$ as follows:

$$\|f\|_* := \sup_{Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f - m_{Q^*}(f)| d\mu + \sup_{\substack{Q \subset R \\ Q, R \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}}\,,$$

where $m_Q(f) := \frac{1}{\mu(Q)} \int_Q f d\mu$. Tolsa defined a new BMO for the growth measures, which is suitable for the Calderón-Zygmund theory. We say that $f \in L^1_{loc}(\mu)$ is a member of RBMO if it satisfies $\|f\|_* < \infty$. Further details may be found in [11, Section 2].

LEMMA 5.2. [11, Corollary 3.5] *Let $f \in RBMO$.*

(a) *There exist $c, c' > 0$ independent of f so that*

$$\mu(Q \cap \{|f - m_{Q^*}(f)| > \lambda\}) \le c\, \mu\left(\frac{3}{2}Q\right) \exp\left(-\frac{c'\lambda}{\|f\|_*}\right)$$

for every $\lambda > 0$ and every cube $Q \in \mathcal{Q}(\mu)$.

(b) *Let $1 \le q < \infty$. Then there exists a constant c independent of f, so that, for every cube $Q \in \mathcal{Q}(\mu)$,*

$$\left(\frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f - m_{Q^*}(f)|^q d\mu\right)^{\frac{1}{q}} \le c\, \|f\|_*\,.$$

As for the Morrey boundedness of commutator generated by RBMO and the singular integral operators, we have the following result.

THEOREM 5.3 ([10, Theorem 4.5]). *Suppose that $a \in RBMO$. Let $1 < q \le p < \infty$ and T be a singular integral linear operator with associated kernel K. Then the commutator $[a, T]f(x) := \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} (a(x) - a(y))K(x, y) f(y) d\mu(y)$ extends to a bounded operator on $\mathcal{M}^p_q(\mu)$.*

Here we prove the following boundedness of $\mathcal{H}^p_q(\mu)$.

THEOREM 5.4. *Let $1 < p \le q < \infty$. Let $a \in RBMO$ and T be a singular integral operator. Then $[a, T]$ is $\mathcal{H}^p_q(\mu)$-bounded.*

[1] By the growth condition (1) there are a lot of big doubling cubes. Precisely speaking, given a cube $Q \in \mathcal{Q}(\mu)$, we can find $j \in \mathbf{N}$ with $2^j Q \in \mathcal{Q}(\mu, 2)$ (see [11]).

PROOF. As before it suffices to show that there exists a constant $c > 0$ such that

$$\| [a, T]A : \mathcal{H}^p_q(\mu)\| \le c$$

for every $(Q, p, q, 4)$-block. We decompose $[a, T]A$ with respect to $2Q$. We set $B_1 = \chi_{2Q} \cdot [a, T]A$ and $B_2 = [a, T]A - B_1$. Then by [10, Theorem 4.5] we see that B_1 is a $(2Q, p, q, 4)$-block modulo multiplicative constants. As for B_2 we decompose further.

$$\begin{aligned} B_2(x) = &\int_Q (a(x) - m_{Q^*}(a))K(x, y)A(y)d\mu(y) \\ &+ \int_Q (m_{Q^*}(a) - a(y))K(x, y)A(y)d\mu(y) . \end{aligned}$$

Set

$$\begin{aligned} C_{1,j}(x) &= \chi_{2^{j+1}Q\setminus 2^jQ}(x) \int_Q (a(x) - m_{Q^*}(a))K(x, y)A(y)d\mu(y) \\ C_{2,j}(x) &= \chi_{2^{j+1}Q\setminus 2^jQ}(x) \int_Q (m_{Q^*}(a) - a(y))K(x, y)A(y)d\mu(y) . \end{aligned}$$

Then by the kernel condition we obtain

$$\begin{aligned} |C_{1,j}(x)| &\le \frac{c\, \chi_{2^{j+1}Q}(x)}{\ell(2^jQ)^n} |a(x) - m_{Q^*}(a)| \cdot \|A\|_1 \\ |C_{2,j}(x)| &\le \frac{c\, \chi_{2^{j+1}Q}(x)}{\ell(2^jQ)^n} \int_Q |m_{Q^*}(a) - a| \cdot |A| d\mu . \end{aligned}$$

As a result we obtain

$$\begin{aligned} \|C_{1,j}\|_q &\le \frac{c\, \chi_{2^{j+1}Q}(x)}{\ell(2^jQ)^n} \left(\int_{2^{j+1}Q} |a - m_{Q^*}(a)|^q d\mu\right)^{\frac{1}{q}} \|A\|_1 \\ \|C_{2,j}\|_q &\le \frac{c\, \chi_{2^{j+1}Q}(x)}{\ell(2^jQ)^n} \left(\int_Q |m_{Q^*}(a) - a|^{q'} d\mu\right)^{\frac{1}{q'}} \|A\|_q \mu(2^{j+1}Q)^{\frac{1}{q}} . \end{aligned}$$

Since A is a block, we have $\|A\|_q \le \mu(4Q)^{\frac{1}{q}-\frac{1}{p}}$ and $\|A\|_1 \le \mu(4Q)^{1-\frac{1}{p}}$. As for the term containing a, Lemma 5.2 yields $\left(\int_Q |m_{Q^*}(a) - a|^{q'} d\mu\right)^{\frac{1}{q'}} \le \mu(4Q)^{\frac{1}{q'}} \|a\|_*$, from which we deduce that $\left(\int_{2^{j+1}Q} |a - m_{Q^*}(a)|^q d\mu\right)^{\frac{1}{q}} \le c\, j \mu(2^{j+2}Q)^{\frac{1}{q}}$. As a consequence we obtain

$$\begin{aligned} \|C_{1,j}\|_q + \|C_{2,j}\|_q &\le \frac{c\, j \chi_{2^{j+1}Q}(x)}{\ell(2^jQ)^n} \mu(Q)^{1-\frac{1}{p}} \mu(2^{j+2}Q)^{\frac{1}{q}} \|a\|_* \\ &\le c\, j \cdot 2^{-j\left(1-\frac{1}{p}\right)} \chi_{2^{j+1}Q}(x) \mu(2^{j+2}Q)^{\frac{1}{q}-\frac{1}{p}} , \end{aligned}$$

which implies that $\|C_{1,j} : \mathcal{H}_q^p\| + \|C_{2,j} : \mathcal{H}_q^p\| \leq c\, j \cdot 2^{-j\left(1-\frac{1}{p}\right)}$. Now that $[a,T]A = B_1 + \sum_{j=1}^{\infty}(C_{1,j} + C_{2,j})$, we see that $\|\,[a,T]A : \mathcal{H}_q^p\| \leq c$.

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