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# The Fundamental Groups of Certain One-Dimensional Spaces

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**Abstract.** An infinitary version of edge path groups is introduced for applications to non-locally simply connected spaces (see Figure 1 in the text).

(1) Edge path groups in this paper are subgroups of the free $\sigma$-product of copies of the integer group $\mathbf{Z}$, which is isomorphic to the fundamental groups of the Hawaiian earring of $I$-many circles for some index set $I$.

(2) Let $Y$ be a subspace of the real line in the Euclidean plane $\mathbf{R}^2$ and $\mathcal{C}$ the set of all connected components of $Y$. Then, the fundamental group of $\mathbf{R}^2 \setminus Y$ is isomorphic to a free product of infinitely many non-trivial groups, if and only if there exists an accumulation point of $\mathcal{C}$ in $Y \cup \{\infty\} \cup -\infty$.

## 1. Introduction and summary.

As we have shown in [8], the fundamental group of a 1-dimensional compact space is isomorphic to a subgroup of the inverse limit of finitely generated free groups. However, the result gives us little information about its group theoretic properties so far, even if $X$ is a Peano continuum, i.e. a locally connected, connected, compact metric space. On the other hand, we have investigated the fundamental groups of certain non-locally simply connected spaces using the notion of free $\sigma$-products [4, 3, 6, 7] and have gotten some group theoretic results. In the present paper, we are interested in an infinitary version of edge path groups. The edge path group is defined for an infinite simplicial complex. Our infinitary version is not aimed at an investigation of infinite simplicial complexes, but at that of spaces which are not locally simply connected, particularly 1-dimensional spaces like (1), (2) and (3) in Figure 1, where there are infinitely many small circles or triangles. The space (1) is called the Hawaiian earring and is the plane continuum $\mathbf{H} = \bigcup_{n=1}^{\infty} C_n$, where $C_n = \{(x, y) : (x - 1/n)^2 + y^2 = 1/n^2\}$. The spaces (2) and (3) are similar to the Sierpinski gasket and its three-dimensional analogue. To state our main results, we recall a free $\sigma$-product of groups $G_i (i \in I)$ [4]. The notation "$X \Subset Y$" means that $X$ is a finite subset of $Y$. We assume $G_i \cap G_j = \{e\}$ for distinct $i, j \in I$. A $\sigma$-word $W : \overline{W} \to \bigcup\{G_i : i \in I\}$ is a function such that $\overline{W}$ is a countable linearly ordered set and $\{\alpha \in \overline{W} : W(\alpha) \in G_i\}$ is finite for each $i \in I$. The set of $\sigma$-words is denoted by $\mathcal{W}^{\sigma}(G_i : i \in I)$. For $F \Subset I$, $W_F$ is the word of finite length obtained by picking all elements in $\bigcup_{i \in F} G_i$ from $W$, i.e. $\overline{W_F} = \{\alpha \in \overline{W} : W(\alpha) \in \bigcup_{i \in F} G_i\}$ and $W_F = W \upharpoonright \overline{W_F}$. Two $\sigma$-words $V, W$ are equivalent, if $V_F = W_F$ holds in the free product $*_{i \in F} G_i$ for every

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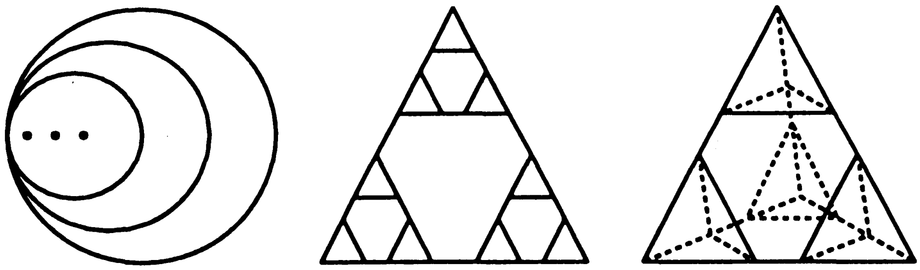

(1) (2) (3)

FIGURE 1

$F \Subset I$. The equivalence class containing $W$ is denoted by $[W]$. The group operation on the equivalence classes is defined by the concatenation. Then, $\{[W] : W \in \mathcal{W}^\sigma(G_i : i \in I)\}$ forms a group which we call the free $\sigma$-product $\mathsf{X}^\sigma_{i\in I} G_i$. The fundamental group of the Hawaiian earring of $I$-many circles is isomorphic to $\mathsf{X}^\sigma_{i\in I} \mathbf{Z}_i$, where $\mathbf{Z}_i$ is a copy of the integer group $\mathbf{Z}$. We refer the reader for this fact as well as basic properties of free $\sigma$-products to [4]. Throughout the present paper, we use the word 'a word' for 'a $\sigma$-word.' We also write 'a word $W$' instead of 'an element $[W]$' as the usual case of a word of finite length, when no confusion will occur. The notation $V \cong W$ means that words $V$ and $W$ are the same, i.e. there is an order isomorphism $i : \overline{V} \to \overline{W}$ such that $V(\alpha) = W(i(\alpha))$ for $\alpha \in \overline{V}$. The first theorem can be regarded as an infinitary version of a theorem for edge path groups.

THEOREM 1.1. *Let $X$ be a space with the following properties: There exist arcs $A_i$ ($i \in I$) with the end points $\dot{A}_i = \{u_i, v_i\}$ and a closed set $D$ such that*
(1) *$X = D \cup \bigcup_{i\in I} A_i$ and $D \cap A_i = \dot{A}_i$ for each $i$,*
(2) *$A_i \setminus \dot{A}_i$ is open and $u_i \neq v_i$ for each $i$,*
(3) *$D$ contains no arc.*
*Then, the fundamental group of $X$ is isomorphic to a subgroup of $\mathsf{X}^\sigma_{i\in I} \mathbf{Z}_i$.*

The next theorem generalizes [4, Corollary 2.5] which implies the same result for the fundamental group of the Hawaiian earring.

THEOREM 1.2. *Let $X$ be a Peano continuum satisfying the hypothesis of Theorem 1.1. Suppose that there exists $K \subset I$ such that $D \cup \bigcup_{i\in K} A_i$ is contractible and locally connected. Then, for any homomorphism $h : \pi_1(X, x) \to *_{n<\omega} H_n$ there exists $m < \omega$ such that $\mathrm{Im}(h) \leq *_{n<m} H_n$. Consequenthy, the fundamental group of $X$ is not isomorphic to a free product of infinitely many non-trivial groups.*

Consequently,

COROLLARY 1.3. *None of the fundamental groups of the spaces* (1), (2) *and* (3) *in Figure* 1 *is isomorphic to a free product of infinitely many non-trivial groups.*

The situation of Theorems 1.1 and 1.2 would be better understood by applying these theorems to the spaces (1), (2) and (3) in Figure 1. See Figure 2 at the proof of Corollary 1.3 in section 2 for it. A set $D$ is the set of dotted points and the edges are the arcs connecting the dotted points. In Section 3 we continue an investigation of the fundamental groups of certain subspaces of the plane [6] and prove

THEOREM 1.4. *Let* $X = \mathbf{R}^2 \setminus Y \times \{0\}$ *for* $Y \subset \mathbf{R}$, $x \in X$ *and* $C$ *be the set of all connected groups of* $Y$.

*Then,* $\pi_1(X, x)$ *is isomorphic to a free product of infinitely many non-trivial groups, if and only if there exists an accumulation point of* $C$ *in* $Y \cup \{\infty\} \cup \{-\infty\}$.

In the last section, we investigate homomorphisms from $\pi_1(\mathbf{H}, o)$ to $\pi_1(X, x)$ for spaces $X$ satisfying the conditions in Theorem 1.1. As a corollary, we prove: None of the fundamental groups of the spaces (2) and (3) in Figure 1 is isomorphic to $\pi_1(\mathbf{H}, o)$ (The point $o$ denotes the origin $(0, 0)$).

The following question is still open. The same question was also asked by J. Cannon and G. Conner [1].

QUESTION 1.5([6]). (See the addendum.) Is the fundamental group of the Sierpinski gasket (or carpet) isomorphic to a subgroup of that of the Hawaiian earring?

REMARK 1.6. Let $\mathbf{H}_I$ be the Hawaiian earring of $I$-many copies $C_i (i \in I)$ of the circle $\{(x, y) : (x-1)^2 + y^2 = 1\}$. More precisely, we identify all points corresponding to $(0, 0)$ to a single point $o$ and the topology is described by specifying neighborhood bases. The neighborhood base of each point $x \neq o$ is the same as the standard base of the circle and a neighborhood of $o$ is of the form: $\bigcup_{i \in F} U_i \cup \bigcup_{j \in I \setminus F} C_i$, where $F$ is a finite subset of $I$ and $U_i$ is a neighborhood of $(0.0)$ in $C_i$ with respect to the standard topology of the circle. Under the condition of Theorem 1.1, let $\sigma : X \to \mathbf{H}_I$ be the continuous map so that $\sigma(D) = \{o\}$ and $\sigma \restriction A_i$ is a relative homeomorphism of $(A_i, \dot{A}_i)$ to $(C_i, \{o\})$. The proof of Theorem 1.1 implies that $\sigma_* : \pi_1(X, x) \to \pi_1(\mathbf{H}_I, o)$ is injective for $x \in D$.

REMARK 1.7. Theorem 1.2 generalizes [4, Corollary 2.5], which treats only the case of the Hawaiian earring. In a recent preprint, J. Cannon and G. Conner [1, Theorem 5.1] proved a closely related statement to Theorem 1.2 for Peano continua by a rather different method. The infinitary version of edge path groups of the present paper was used in [5] for the free topological linear space over a pseudo arc.

## 2. Proof of Theorems 1.1 and 1.2.

For a path $f : [s, t] \to X$, define $\tilde{f} : [0, 1] \to X$ by: $\tilde{f}(u) = f((1-u)s + ut)$. We simply say two paths $f : [s, t] \to X$ and $g : [s', t'] \to X$ are *homotopic*, if $f(s) = g(s')$ and $f(t) = g(t')$ and the two paths $\tilde{f}$ and $\tilde{g}$ are homotopic relative to $\{0, 1\}$. For a space $X$ satisfying the hypothesis of Theorem 1.1, we introduce some auxiliary notions. We fix a base point $x_0 \in D$. For a loop $f$ with base point $x_0$, $[f] \in \pi_1(X, x_0)$ denotes its homotopy class relative to $\{0, 1\}$. For each $A_i$, let $\varphi_i : [0, 1] \to A_i$ be a homeomorphism. A path $f : [s, t] \to X$ with $f(s), f(t) \in D$ is *proper*, if the following hold: For any $s < a < t$ with

$f(a) \in A_i \setminus \dot{A}_i = \mathrm{int}(A_i)$, there exist $s \le u < v \le t$ such that $u < a < v$, $f(u), f(v) \in \dot{A}_i$, $f(u) \neq f(v)$ and $\mathrm{Im}(f \restriction (u, v)) \subset \mathrm{int}(A_i)$.

LEMMA 2.1. *Let X be a space satisfying the hypothesis of Theorem* 1.1. *Then, for any path* $f : [0, 1] \to X$ *with* $f(0), f(1) \in D$, *there exists a proper path g which is homotopic to f.*

PROOF. Let $O = \bigcup_{i \in I} \mathrm{int}(A_i)$ and $f^{-1}(O) = \bigcup_{m<\mu}(a_m, b_m)$, where $(a_m, b_m) \cap (a_n, b_n) = \emptyset$ for $m \neq n$ and $\mu \le \omega$. Define a homotopy $H : [0, 1] \times [0, 1] \to X$ by:

$$H(s,t) = \begin{cases} f(s) & \text{if } s \in (a_n, b_n) \text{ and } f(a_n) \neq f(b_n) \\ \varphi_i((1-t)\varphi_i^{-1} f(s)) & \text{if } s \in (a_n, b_n) \text{ and } f(a_n) = f(b_n) = u_i \\ \varphi_i(t + (1-t)\varphi_i^{-1} f(s)) & \text{if } s \in (a_n, b_n) \text{ and } f(a_n) = f(b_n) = v_i \\ f(s) & \text{otherwise.} \end{cases}$$

Since the diameters of $\mathrm{Im}(f) \cap A_i$ converge to 0, $H$ is continuous and $g(s) = H(s, 1)$ is a proper path. □

For a proper path $f : [s, t] \to X$ with $f(s), f(t) \in D$, we define a word $W^f$ as follows: Let $O = \bigcup_{i \in I} \mathrm{int}(A_i)$ and $f^{-1}(O) = \bigcup_{m<\mu}(a_m, b_m)$, where $(a_m, b_m) \cap (a_n, b_n) = \emptyset$ for distinct $m, n$ and $\mu \le \omega$. Let $\overline{W^f} = \mu$ and $m \prec n$ if $a_m < a_n$ for $m, n \in \mu$. Let

$$W^f(m) = \begin{cases} i & \text{if } \mathrm{Im} f \restriction [a_m, b_m] = A_i \text{ and } f(a_m) = u_i\,, \\ -i & \text{if } \mathrm{Im} f \restriction [a_m, b_m] = A_i \text{ and } f(a_m) = v_i\,, \end{cases}$$

where $i$ is the generator of $\mathbf{Z}_i$.

Though the next lemma is almost the same as [5, Lemma 4.5], we present its proof for the reader's convenience.

LEMMA 2.2. *Let f be a proper path with* $f(0), f(1) \in D$. *If* $W^f = e$, *then f is a loop, i.e.* $f(0) = f(1)$.

PROOF. Suppose that $i$ or $-i$ appears in $W^f$. Since $W^f = e$, we have $0 \le c_0 < c_1 < \cdots c_{2n} < c_{2n+1} \le 1$ so that $f(c_{2j}) = f(c_{2j+1}) \in \{u_i, v_i\}$, $W^{f \restriction [c_{2j}, c_{2j+1}]} = e$ and $\mathrm{Im}(f \restriction [0, c_0]) \cap \mathrm{int}(A_i) = \mathrm{Im}(f \restriction [c_{2j-1}, c_{2j}]) \cap \mathrm{int}(A_i) = \mathrm{Im}(f \restriction [c_{2n+1}, 1]) \cap \mathrm{int}(A_i) = \emptyset$. Let $g(x) = f(c_{2j}) = f(c_{2j+1})$ for $c_{2j} \le x \le c_{2j+1}$ and $g(x) = f(x)$ otherwise. Then, $g$ is a proper path and $W^g = e$ and $\mathrm{Im}(g) \cap \mathrm{int}(A_i) = \emptyset$. We enumerate $i$'s in $I$ for which $\pm i$'s appear in $W^f$ without repetition and let $\{i_n : n < \nu\}$ to be its enumeration, where $\nu \le \omega$. Let $f_0 = f$. Inductively, we apply the above reformation to $f_n$ using $A_{i_n}$ in the $n$-th step and obtain $f_{n+1}$. There would be a case where $\pm i_n$ does not appear in $W^{f_n}$. We let $f_{n+1} = f_n$ on that case. By the construction, it is easy to see that $f_n$ $(n < \omega)$ converge to a continuous function $f_\infty$. Then, $\mathrm{Im}(f_\infty) \subseteq D$, $f_\infty(0) = f(0)$ and $f_\infty(1) = f(1)$. Since $D$ contains no arc, $f(0) = f(1)$. □

The following lemma is basically the same as in the one for edge path groups and its proof is similar to the proof in [8, Appendix B].

LEMMA 2.3. *Let $f : [0,1] \to X$ be a proper loop which is homotopic to the constant map. Then, $W^f = e$ holds in $\mathsf{X}^{\sigma}_{i\in I}\mathbf{Z}_i$.*

PROOF. By Lemma 2.1, we may assume that $f$ is a proper loop. Let $F \Subset I$, $X_F$ be the quotient space of $X$ obtained by identifying $D \cup \bigcup_{i \notin F} A_i$ with one point and $\sigma_F : X \to X_F$ be the quotient map. Then, $\sigma_F f$ is a null homotopic loop in a finite bouquet, whose fundamental group is canonically isomorphic to $*_{i\in F}\mathbf{Z}_i$. Therefore, $(W^f)_F = e$ for each $F \Subset I$, which implies $W^f = e$. □

Let $P_{x_0}$ be the set of all proper loops with base point $x_0$. By this lemma, a homomorphism $\xi : \pi_1(X, x_0) \to \mathsf{X}^{\sigma}_{i\in I}\mathbf{Z}_i$ is defined by the formula:

$\xi([f]) = W^f$ for $f \in P_{x_0}$, where $[f]$ denotes the homotopy class containing $f$.

LEMMA 2.4. *Let $f : [0,1] \to X$ be a proper path. Then, there exists a proper path $g : [0,1] \to X$ such that*

1. *$g(0) = f(0)$ and $g(1) = f(1)$;*
2. *$W^g$ is the reduced word of $W^f$; and*
3. *$g(x) = f(x)$ or $g(x) = f(s_0) = f(t_0)$, where $s_0 = \min\{s : g(y) = g(x)\ f\ or\ s \leq y \leq x\}$ and $t_0 = \max\{t : g(y) = g(x)\ f\ or\ x \leq y \leq t\}$.*

PROOF. We recall a reduction procedure of a non-reduced word [4, p. 245]. When $W^f$ is reduced, there is nothing to prove. Otherwise, there exists a non-empty subword of $W^f$ which is equivalent to the empty word, i.e. there are $0 \leq s_0 < t_0 \leq 1$ such that $f(s_0) = f(t_0) \in D$ and $W^{f\restriction[s_0,t_0]} = e$ by Lemma 2.2. Let $g(x) = f(x)$ for $0 \leq x < s_0$ or $t_0 < x \leq 1$ and $g(x) = f(s_0)$ for $s_0 \leq x \leq t_0$. Repeat transfinitely the process obtaining $g$ from $f$. It will stop in countable steps. Then, we get the desired path, the continuity of which follows from that of the original $f$. □

LEMMA 2.5. *Let $f \in P_{x_0}$ with $W^f = e$ and $\rho$ be a compatible metric on* $\mathrm{Im}(f)$. *Suppose that $\rho(x_0, f(c_0)) = \max\{\rho(x_0, f(u)) : u \in [0,1]\} = d > 0$ and $\rho(x_0, f(u)) < d$ for $0 \leq u < c_0$. Then, the one of the following holds:*

(1) *There exist $s_0, t_0$ such that $s_0 < c_0 < t_0$, $f(s_0) = f(t_0) \in D$, $\rho(x_0, f(s_0)) = d/2$, and $W^{f\restriction[s_0,t_0]} = e$.*

(2) *There exist $s_0, s_1, t_0, t_1$ and $A_j$ satisfying:*

- *$s_0 < s_1 \leq t_1 < t_0$, $f(s_0) = f(t_0) \in \{u_j, v_j\}$, $f(s_1) = f(t_1) \in \{u_j, v_j\}$, and* $\mathrm{Im}(f \restriction [s_0, s_1]) = \mathrm{Im}(f \restriction [t_1, t_0]) = A_j$;
- *$s_0 < c_0$;*
- *there is $s_0 < s < c_0$ such that $f(s) \in A_j$ and $\rho(x_0, f(s)) = d/2$;*
- *$W^{f\restriction[s_1,t_1]} = e$.*

PROOF. Case 1: $f(c_0) \in D$. Let $g : [0, c_0] \to X$ and $h : [c_0, 1] \to X$ be the paths given by Lemma 2.4, which are corresponding to $f \restriction [0, c_0]$ and $f \restriction [c_0, 1]$ respectively. We remark that $W^{f\restriction[0,c_0]} \neq e$ and $W^{f\restriction[c_0,1]} \neq e$ by Lemma 2.2.

1(i). If there exists $0 < s < c_0$ such that $g(s) \in D$ and $\rho(x_0, g(s)) = d/2$, $g(s_0) = g(s) = f(s_0)$ and $W^{f\restriction[s_0,c_0]} = W^{g\restriction[s_0,c_0]}$ hold for $s_0 = \min\{x : g(y) = g(s) \text{ for } x \le y \le s\}$. Since $W^g$ and $W^h$ are reduced and $W^g W^h = W^f = e$, $W^g$ is the inverse word $(W^h)^{-1}$ [4, Corollary 1.5]. Therefore, there exists $c_0 < t_0 < 1$ such that $h(t_0) = f(t_0) = g(s_0)$ and $W^{h\restriction[c_0,t_0]}$ is the inverse word of $W^{g\restriction[s_0,c_0]}$. Then, $W^{f\restriction[s_0,t_0]} = W^{g\restriction[s_0,c_0]} W^{h\restriction[c_0,t_0]} = e$ holds, that is, the case (1) holds.

1(ii). Otherwise, there exists $0 < s < c_0$ and $A_j$ such that $g(s) \in \mathrm{int}(A_j)$ and $\rho(x_0, g(s)) = d/2$. Since $g$ is proper, we get the unique $s_0 < s_1$ such that $g(s_0), g(s_1) \in \dot{A}_j$, $s_0 < s < s_1 \le c_0$, $\mathrm{Im}(f \restriction [s_0, s_1]) = A_j$. Since $g(s) \in \mathrm{int}(A_j)$, $f(x) = g(x)$ for $s_0 \le x \le s_1$. As in Case 1(i), we get $t_0, t_1$ so that $c_0 \le t_1 < t_0$, $h(t_0) = g(s_0)$, $h(t_1) = g(s_1)$, $\mathrm{Im}(f \restriction (t_1, t_0)) = \mathrm{int}(A_j)$ and $W^{h\restriction[c_0,t_1]}$ is the inverse word of $W^{g\restriction[s_1,c_0]}$.

Case 2: $f(c_0) \in A_k$ for some $k$. Let $c_1 = \min\{u : c_0 < u,\ f(u) \in D\}$. Let $g : [0, c_1] \to X$ and $h : [c_1, 1] \to X$ be the paths obtained by applying Lemma 2.4 for $f \restriction [0, c_1]$ and $f \restriction [c_1, 1]$ respectively. Then, $W^{g\restriction[0,c_1]} \ne e$ and $W^{h\restriction[c_1,1]} \ne e$ by the assumption on $c_0$ of this case. Then, we deal with two cases similarly to the sub-cases in Case 1. We omit the detail, but just remark that $c_0 < s_1$ may hold. □

The remaining part of the proof of Theorem 1.1 is analogous to those in [4, 3, 6, 8].

PROOF OF THEOREM 1.1. It suffices to show that a homomorphism $\xi : \pi_1(X, x_0) \to \mathbf{X}^{\sigma}_{i\in I}\mathbf{Z}_i$ is injective. Let $f \in P_{x_0}$ with $W^f = e$. We shall show that $f$ is null homotopic. Since $\mathrm{Im}(f)$ is metrizable, we take a compatible metric $\rho$ on $\mathrm{Im}(f)$. We define parts of a homotopy $H : [0, 1] \times [0, 1] \to X$ between $f$ and the constant map at $x_0$ and also auxiliary notions by induction. To describe our inductive definition, we introduce some notion.

Let $Seq$ be the set of finite sequences of nonnegative integers. For $\sigma \in Seq$, the length of $\sigma$ is denoted by $|\sigma|$ and $\sigma = \langle \sigma(0), \cdots, \sigma(n-1)\rangle$, where $n = |\sigma|$. The sequence obtained by adding $i$ to $\sigma$ is denoted by $\sigma * \langle i\rangle$ and the empty sequence is denoted by $\langle\ \rangle$. Let $R_{\langle\ \rangle} = [0, 1] \times [0, 1]$. Let $H(s, 1) = f(s)$ and $H(s, 0) = H(0, t) = H(1, t) = f(0)$ for $0 \le s, t \le 1$ and also $f_{\langle\ \rangle} = f$.

(Stage $\langle\ \rangle$) In this stage we define rectangles $R_\sigma$ for each sequence $\sigma$ of length 1 and define the map $H$ on the closure of the complement of $\bigcup\{R_{\langle n\rangle} : n < \omega\}$.

(Sub-stage 0) Let $d = \max\{\rho(f(0), f(s)) : s \in [0, 1]\}$. If $d = 0$, $f$ is constant and so we just define $H(s, t) = f(s)$. (Since we shall mimic this stage again, this obvious definition is necessary.) Otherwise, our construction is divided into cases according to the two cases in Lemma 2.5. In the case (1), we define

$$H(s,t) = \begin{cases} f(s), & \text{for } s \in [0, s_0] \cup [t_0, 1] \text{ and } 1/2 \le t \le 1, \\ f(s_0) = f(t_0), & \text{for } s_0 \le s \le t_0 \text{ and } t = 1/2, \end{cases}$$

and also $f_{\langle 0\rangle} = f \restriction [s_0, s_1]$ and $R_{\langle 0\rangle} = [s_0, s_1] \times [1/2, 1]$.

In the case (2), we define $H(s,t)$ as follows. Here we assume $\varphi_j(0) = f(s_0)$ and $\varphi_j(1) = f(s_1)$, but the other case can be treated similarly. Let

$$H(s,t) = \begin{cases} f(s), & \text{for } s \in [0, s_1] \cup [t_1, 1] \text{ and } 1/2 + 1/4 \le t \le 1, \\ f(s_1), & \text{for } s_1 \le s \le t_1 \text{ and } t = 1/2 + 1/4, \\ f(s), & \text{for } s \in [0, s_0] \cup [t_0, 1] \text{ and } 1/2 \le t \le 1/2 + 1/4, \\ \varphi_j(4(t - 1/2)\varphi_j^{-1} f(s)), & \text{for } s \in [0, s_1] \cup [t_1, 1] \text{ and } 1/2 \le t \le 1/2 + 1/4, \end{cases}$$

and also $f_{\langle 0 \rangle} = f \restriction [s_1, t_1]$ and $R_{\langle 0 \rangle} = [s_1, t_1] \times [1/2 + 1/4, 1]$. In case $s_1 = t_1$, we have defined $H$ on $[0, 1] \times [0, 1]$.

(Sub-stage $n + 1$) We have defined $H(s, 1/2^{n+1})$ for $0 \le s \le 1$ in Sub-stage $n$. We make the above procedure in Sub-stage 0 for $[0, 1] \times [0, 1/2^{n+1}]$ instead of $[0, 1] \times [0, 1]$ and the loop $s \mapsto H(s, 1/2^{n+1})$ instead of $f(s) = H(s, 1)$ to define a partial homotopy, $f_{\langle n+1 \rangle}$ and the rectangle $R_{\langle n+1 \rangle}$. Then, $H$ is defined on a subset of the rectangle $[0, 1] \times [1/2^{n+1}, 1/2^{n+2}]$ and in particular $H(s, 1/2^{n+2})$ is defined for $0 \le s \le 1$.

Here, we notice that the diameters of the loops $s \mapsto H(s, 1/2^n)$ $(n < \omega)$ converge to 0 and consequently the loops themselves converge to $f(0)$. To see this, assume the contrary. Since the above procedure does not increase the diameter of a loop, the assumption implies that the diameters of the loops $s \mapsto H(s, 1/2^n)$ converges to a positive number $d_0 > 0$. It means that the image $\mathrm{Im}\, f$ travels infinitely many times between points distant from $f(0)$ by at most $d_0/2$ and points distant from $f(0)$ by at least $d_0$. However this contradicts the continuity of $f$.

(Stage $\sigma$) We have defined a loop $f_\sigma$ with its domain $[a_\sigma, b_\sigma]$ and a rectangle $R_\sigma$ of form $[a_\sigma, b_\sigma] \times [y_\sigma, z_\sigma]$. Here, $H(s, z_\sigma) = f_\sigma(s)$ for $a_\sigma \le s \le b_\sigma$ and $H(s, y_\sigma)$ is constant for $a_\sigma \le s \le b_\sigma$. If $a_\sigma = b_\sigma$, i.e. $\mathrm{int}(R_\sigma) = \emptyset$, there is no sub-stages and we do not define $R_\tau$ and so on for any extensions $\tau$ of $\sigma$. Otherwise, we work as in Stage $\langle\ \rangle$ replacing $f_{\langle\ \rangle}$ by $f_\sigma$ and $R_{\langle\ \rangle}$ by $R_\sigma$. That is, we define the rectangles $R_{\sigma * \langle n \rangle}$, loops $f_{\sigma * \langle n \rangle}$ and a partial homotopy $H$ on $R_\sigma$ applying Lemma 2.5 repeatedly.

When the all stages are performed, we define the map $H$ on a dense subset of $[0, 1] \times [0, 1]$. However, there are still many points on which $H$ has not been defined. Let $x$ be such a point. For each $n$, there exists a unique $\sigma_n \in Seq$ such that $|\sigma_n| = n$ and $x$ belongs to the interior of $R_{\sigma_n}$. By the uniform continuity of $f$, the diameters of the loops $f_{\sigma_n}$ converge to 0. Therefore, there exists a unique convergent point in $\mathrm{Im}(f)$. Let $H(x)$ be the point.

Now, we verify the continuity of $H$. The continuity at the points defined right now is clear from the definition. Hence, it suffices to show the continuity at the boundaries for each rectangle $R_\sigma = [a_\sigma, b_\sigma] \times [y_\sigma, z_\sigma]$. Since $R_\tau \subseteq R_\sigma$ holds if $\tau$ is an extension of $\sigma$ and $\mathrm{int}(R_\sigma) \cap \mathrm{int}(R_\tau) \neq \emptyset$ holds if and only if one of $\sigma$ and $\tau$ is an extension of the other. It suffices to verify the continuity in each $R_\sigma = [a_\sigma, b_\sigma] \times [y_\sigma, z_\sigma]$. It is easy to see that $H$ is continuous on the side edges $\{a_\sigma, b_\sigma\} \times (y_\sigma, z_\sigma]$, because $H$ is defined on a neighborhood of each point of those edges at some stage. Observe that the loops $f_{\sigma * \langle n \rangle}$ $(n < \omega)$ converge to the base point of $f_\sigma$ for each $\sigma \in Seq$. Then, the continuity of $H$ at the lower edge $[a_\sigma, b_\sigma] \times \{y_\sigma\}$ follows from the argument of the continuity at the end of Sub-stage 0. What remains to be proved is the continuity at each point $x$ at the upper edge such that $H$ is not totally defined on a neighborhood of $x$ at any stage. In this case, there exists a unique sequence $\sigma_n \in Seq$ $(n < \omega)$ such that $|\sigma_n| = n$ and $x$ belongs to the upper edge of $R_{\sigma_n}$. Moreover, $x$ is not a corner point of $R_{\sigma_n}$. Therefore, the continuity at $x$ follows from the convergence of the values on edges of $R_{\sigma_n}$'s. □

To prove Theorem 1.2, we first show that $\pi_1(X, x_0)$ can be canonically embeddable into $\mathsf{X}^{\sigma}_{i\in I\setminus K}\mathbf{Z}_i$. As we have shown right now, a homomorphism $\xi : \pi_1(X, x_0) \to \mathsf{X}^{\sigma}_{i\in I}\mathbf{Z}_i$ is an injection. Let $J = I \setminus K$ and $p_J : \mathsf{X}^{\sigma}_{i\in I}\mathbf{Z}_i \to \mathsf{X}^{\sigma}_{i\in J}\mathbf{Z}_i$ be the projection. We show that $p_J\xi$ is injective.

LEMMA 2.6. *If $p_J(W^f) = e$ for $f \in P_{x_0}$, then $W^f = e$ holds.*

PROOF. Suppose that $W^f = e$ does not hold. By Lemma 2.4, we may assume $W^f$ is reduced. Then, there exists a finite subset $F$ of $I$ such that $(W^f)_F \neq e$. Since $D \cup \bigcup_{i\in K} A_i$ is contractible, $D \cup \bigcup_{i\in K} A_i \setminus \bigcup_{i\in F\cap K} \mathrm{int}(A_i)$ consists of finitely many contractible closed subspaces $T_k$ $(0 \leq k \leq n)$. By identifying each $T_k$ as a point $q_k$, we get the quotient space $X^*$ of $X$ and the quotient map $q : X \to X^*$. Now, $W^{qf} \neq e$. Since $W^{qf} \in *_{F\cap K}\mathbf{Z} * \mathsf{X}^{\sigma}_{i\in J}\mathbf{Z}_i$, the reduced word of $W^{qf}$ can be presented as $W_0 W_1 \cdots W_n$ satisfying the following properties:

1. Members of $*_{F\cap K}\mathbf{Z}$ and $\mathsf{X}^{\sigma}_{i\in J}\mathbf{Z}_i$ appear alternately.
2. There exist $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that $qf(t_m) \in \{q_k : 0 \leq k \leq n\}$ and $W^{qf\restriction[t_m,t_{m+1}]} = W_m$.

Since $D \cup \bigcup\{A_i : i \in K\}$ is contractible and $W_0 W_1 \cdots W_n$ is reduced, $qf(t_k) \neq qf(t_{k+1})$ for $k$ with $W_k \in *_{F\cap K}\mathbf{Z}$. On the other hand, since $p_J(W^{qf}) = p_J(W^f) = e$, there are $W_{m-1}, W_{m+1} \in \mathsf{X}^{\sigma}_{i\in J}\mathbf{Z}_i$ such that a word $W_{m-1}W_{m+1}$ is not reduced. Since $W_{m-1}$ and $W_{m+1}$ themselves are reduced, we conclude $qf(t_m) = qf(t_{m+1})$ by [4, Corollary 1.7], which is a contradiction. □

DEFINITION 2.7. A loop $f \in P_{x_0}$ has the *essential size* less than $\varepsilon$, if there exist $s < t$ and $y_0 \in D$ such that $f(s) = f(t) = y_0$, $f \restriction [0, s]$ and $f \restriction [t, 1]$ are paths in $D \cup \bigcup\{A_i : i \in K\}$ and the diameter of $\mathrm{Im}(f \restriction [s, t])$ is less than $\varepsilon$. An *essential part* of $f$ is a path $f \restriction [s, t]$.

LEMMA 2.8. *Let $h : \pi_1(X, x_0) \to *_{n<\omega}H_n$ be a homomorphism such that* $\mathrm{Im}(h)$ *is not contained in any $*_{n<m}H_n$. Then, for any $\varepsilon > 0$ and $m$ there exists $f \in P_{x_0}$ such that the essential size of $f$ is less than $\varepsilon$ but $h([f]) \notin *_{n<m}H_n$.*

PROOF. Assume the negation of the conclusion. Then, there exist $\varepsilon_0 > 0$ and $m$ such that $h([f]) \in *_{n<m}H_n$ holds if $f \in P_{x_0}$ has the essential size less than $\varepsilon_0$.

Since $D \cup \bigcup\{A_i : i \in K\}$ is locally path-connected, there exists $\varepsilon_1 < \varepsilon_0/2$ satisfying the following:

For any $u, v \in D$ with $\rho(u, v) < \varepsilon_1$, there exists a path $g$ from $u$ to $v$ in $D \cup \bigcup\{A_i : i \in K\}$ such that the diameter of $\mathrm{Im}(g)$ is less than $\varepsilon_0/2$.

Since $X$ is locally path-connected, the diameters of $A_i$'s converge to 0. Therefore, there exists $J' \Subset J$ such that the diameters of $A_i$'s are less than $\varepsilon_1$ for $i \in J \setminus J'$. For each $A_i$ $(i \in J)$, let $f_i \in P_{x_0}$ be so that $f_i \restriction [0, 1/3]$ is a path from $x_0$ to $u_i$ in $D \cup \bigcup\{A_i : i \in K\}$, $f_i \restriction [2/3, 1]$ is a path from $v_i$ to $x_0$ in $D \cup \bigcup\{A_i : i \in K\}$, and $f_i(s) = \varphi_i(3(s-1/3))$ for $1/3 \le s \le 2/3$. Since the essential size of $f_i$ is less than $\varepsilon_0$ for $i \in J \setminus J'$, we can choose $m_0 \ge m$ so that $h([f_i]) \in *_{n<m_0} H_n$ for all $i \in J$.

For any $f \in P_{x_0}$, there are $0 = t_0 < t_1 < \cdots < t_k = 1$ such that for $0 \le j < k$

$f \restriction [t_j, t_{j+1}]$ is homotopic to some $\varphi_i$ or $\varphi_i^{-1}$ $(i \in J')$,
or $\mathrm{Im}(f \restriction [t_j, t_{j+1}]) \cap \bigcup\{\mathrm{int}(A_i) : i \in J'\} = \emptyset$ and
$f \restriction [t_j, t_{j+1}]$ is a path with its size less than $\varepsilon_1$.

Then, we can adjust the above sequence $0 = t_0 < t_1 < \cdots < t_k = 1$ so that $f(t_j) \in D$, and each $f \restriction [t_j, t_{j+1}]$ is homotopic to some $\varphi_i$ $(i \in J)$ or is a path with its size less than $\varepsilon_0$. Since $D \cup \bigcup\{A_i : i \in K\}$ is contractible, we can decompose $f$ to proper loops $g$ which are homotopic to some $f_i$ or homotopic to a proper loop with the essential size less than $\varepsilon_0$, i.e. $h([g]) \in *_{n<m_0} H_n$ in either case. Therefore, $h([f]) \in *_{n<m} H_n$ holds for any proper loop $f$ with base point $x_0$, which is a contradiction. □

LEMMA 2.9. *Let $f_n \in P_{x_n} (n < \omega)$ and $f'_n \in P_{x_0} (n < \omega)$, where $x_n \in D$. Suppose that $(\mathrm{Im}(f_n) : n < \omega)$ converges a point $y_0 \in D$ and $f'_n \restriction [1/3, 2/3]$ and $f_n$ are homotopic for each $n$ and $\mathrm{Im}(f'_n \restriction [0, 1/3]) \cup \mathrm{Im}(f'_n \restriction [2/3, 1]) \subseteq D \cup \bigcup\{A_i : i \in K\}$. Then, there exists a homomorphism $h : \mathsf{X}_{n<\omega}\mathbf{Z}_n \to \pi_1(X, x_0)$ such that $h(\delta_n) = [f'_n]$ for each $n < \omega$, where $\delta_n$ is the generator of $\mathbf{Z}_n \le \mathsf{X}_{n<\omega}\mathbf{Z}_n$.*

PROOF. By the local connectivity of $X$, we can take $f''_n \in P_{y_0}$ so that $f''_n \restriction [1/3, 2/3]$ and $f_n$ are homotopic and $\mathrm{Im}(f''_n \restriction [0, 1/3]) \cup \mathrm{Im}(f''_n \restriction [2/3, 1]) \subseteq D \cup \bigcup\{A_i : i \in K\}$ and $\mathrm{Im}(f''_n)$ converge to $y_0$. Let $\psi : \mathbf{H} \to X$ be the continuous map defined by:

$$\psi\left(\frac{1}{n}\cos(\pi + 2\pi s) + \frac{1}{n}, \frac{1}{n}\sin(\pi + 2\pi s)\right) = f''_n(s),$$

that is, $C_n$ is mapped as in the same way as mapped by $f''_n$. Then, the induced homomorphism $\psi_* : \pi_1(\mathbf{H}, o) \to \pi_1(X, y_0) (\simeq \pi_1(X, x_0))$ is the desired homomorphism through the canonical isomorphism between $\pi_1(\mathbf{H}, o)$ and $\mathsf{X}_{n<\omega}\mathbf{Z}_n$ [4, Theorem A.1]. □

PROOF OF THEOREM 1.2. Suppose that $\mathrm{Im}(h)$ is not contained in any $*_{n<m} H_n$ for a homomorphism $h : \pi_1(X, x_0) \to *_{n<\omega} H_n$. According to Lemma 2.8, for each $m$ there exists $g_m \in P_{x_0}$ such that $g_m$ has the essential size less than $1/m$, but $h([g_m]) \notin *_{n<m} H_n$. Let $g'_m$ be an essential part of $f$. Since the diameters of $\mathrm{Im}(g'_m)$ converge to 0, there exist a point $y_0 \in D$ and a subsequence $m_n$ $(n < \omega)$ such that $\mathrm{Im}(g'_{m_n})$ converge to $y_0$. By Lemma 2.9, we get a homomorphism $\psi : \mathsf{X}_{n<\omega}\mathbf{Z}_n \to \pi_1(X, x_0)$ such that $\psi(\delta_n) = [g_{m_n}]$ for each $n < \omega$. But, this contradicts [4, Corollary 2.5], which says that $\mathrm{Im}(h \cdot \psi)$ is contained in some $*_{n<m} H_n$. □

PROOF OF COROLLARY 1.3. For the Hawaiian earring, let $D$ be the set consisting of all the points dotted in Figure 2(1) and $A_i (i \in K)$ be the edges in Figure 2(1). Then,

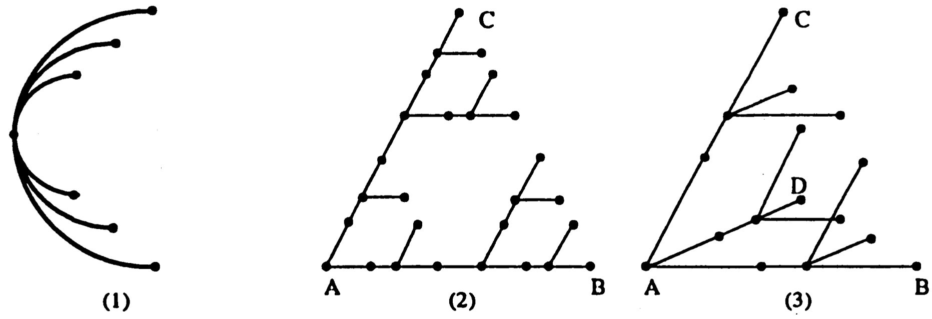


FIGURE 2

$D \cup \bigcup\{A_i : i \in K\}$ satisfies the conditions in Theorem 1.2 and we get the conclusion. For the spaces (2) and (3) in Figure 1, let $D$ be the points dotted in Figure 2(2) and 2(3) and take $K \subset I$ so that $A_i$ $(i \in I \setminus K)$ are the edges which are parallel to the line $BC$ and the plane $BCD$ respectively. Then, $D \cup \bigcup\{A_i : i \in K\}$ satisfies the conditions in Theorem 1.2 and we get the conclusion. $\square$

## 3. Proof of Theorem 1.4.

In this section, we continue to investigate the fundamental groups of the subspaces $\mathbf{R}^2 \setminus Y \times \{0\}$ of the plane [6]. In that paper, we characterized $Y \subset \mathbf{R}$ for which the fundamental group of $\mathbf{R}^2 \setminus Y \times \{0\}$ is isomorphic to that of the Hawaiian earring. Here, we shall characterize $Y \subset \mathbf{R}$ for which the fundamental group has the property in Theorem 1.4. First, we recall results from [6]. Let $\mathcal{C}$ be the set of all connected components of $Y$ and take a countable subset $D$ of $Y$ so that

1. $D \cap (u, v) \neq \emptyset$ if and only if $Y \cap (u, v) \neq \emptyset$ for $u, v \in \mathbf{R} \setminus Y$,
2. $D \cap (u, v)$ is a singleton if and only if $(u, v) \subset Y$ for $u, v \in \mathbf{R} \setminus Y$,
3. $D \cap C$ is empty for an unbounded $C \in \mathcal{C}$.

Then, $\pi_1(X, x)$ can be canonically represented as a subgroup $\{[W] : \mathcal{U}(D, Y)\}$ of a free $\sigma$-product $\times^\sigma_{d \in D} \mathbf{Z}_d$ [6, Theorem 3.2], where $[W]$ denotes the equivalence class containing $W$. In the following we use the subset $\mathcal{U}(D, Y)$ of $\mathcal{W}(\mathbf{Z}_d : d \in D)$ in [6], the definition and some properties of which are recalled below for the reader's convenience.

For a linearly ordered set $S$, let $S^{-1}$ be the linearly ordered set consisting of $-s$ for $s \in S$ such that $-s < -t$ if and only if $t < s$. Let $|s| = |-s| = s$ for $s \in S$. We remark that $-s$ is a letter consisting of '$-$' and '$+$,' but is not a real number even if $s$ is a real number. Let $\mathbf{Z}_d$ is a copy of $\mathbf{Z}$ whose generator is $d \in D$. Let $W \in \mathcal{W}(\mathbf{Z}_d :\in D)$ be a word such that each $W(\alpha)$ is $d$ or $-d$ for some $d \in D$. A word $V \in \mathcal{W}(\mathbf{Z}_d : d \in D)$ is a *component* of $W$, if $V$ is a maximal subword of $W$ which satisfies the following:

(+) There exist $u, v \in \mathbf{R} \setminus Y$ such that $u < v$ and $V : \overline{V} \to (u, v) \cap D$ is the order isomorphism,

or

$(-)$ There exist $u, v \in \mathbf{R} \setminus Y$ such that $u < v$ and $V : \overline{V} \to ((u, v) \cap D)^{-1}$ is the order isomorphism.

The set $\mathcal{U}(D, Y)$ consists of $W \in \mathcal{W}(\mathbf{Z}_d : d \in D)$ which satisfies the following:

(1) Each $W(\alpha)$ is $d$ or $-d$ for some $d \in D$,

(2) For any $\alpha \in \overline{W}$, there exists a component $V$ of $W$ such that $\alpha \in \overline{V}$,

(3) Let $\alpha_n \in \overline{W}$ $(n < \omega)$ be an increasing or decreasing sequence. If $\alpha_n$'s belong to different components $V_n$'s, i.e. $\alpha_n \in \overline{V_n}$ with $V_m \neq V_n$ $(m \neq n)$, there exists $x \in \mathbf{R} \setminus Y$ such that $\lim_{n\to\infty} |W(\alpha_n)| = x$.

Let $x \in X$ be a point in the upper half plane. (The choice of $x$ is made for the convenience of the definition of a word $W^f$.) The important facts are

1. $\mathcal{U}(D, Y)$ consists of all the $W^f$'s for loops $f$ in $X$ with base point $x$;
2. an isomorphism $\pi_1(X, x) \simeq \{[W] : \mathcal{U}(D, Y)\}$ is given by $f \mapsto W^f$.

Here, $W^f$ is given by the following.

Let $\bigcup_{\alpha\in L}(a_\alpha, b_\alpha) = f^{-1}(\mathbf{R} \times (-\infty, 0))$, where $(a_\alpha, b_\alpha) \cap (a_\beta, b_\beta) = \emptyset$ for $\alpha \neq \beta$. Let the domain $\overline{W^f}$ of the word $W^f$ is a subset of $L \times \mathbf{R}$ defined by

$$\begin{cases} \{\alpha\} \times (D \cap (f(a_\alpha), f(b_\alpha))) & \text{if } f(a_\alpha) < f(b_\alpha), \\ \{\alpha\} \times (D \cap (f(b_\alpha), f(a_\alpha)))^{-1} & \text{otherwise}, \end{cases}$$

and let $(\alpha, u) < (\beta, v)$, if $a_\alpha < a_\beta$, or $\alpha = \beta$ and $u < v$. Finally, let $W^f(\alpha, u) = u$.

From now on, we use the notation $\mathcal{U}(D, Y)$ not only as a set of words but also as a subgroup of $\mathsf{X}^\sigma_{d\in D}\mathbf{Z}_d$, since no confusion will occur. Though the next lemma can be proved in a purely algebraical manner, a precise description of $\mathcal{U}(D, Y)$ is needed for such a proof. Therefore, we prove the lemma with the help of a topological argument.

LEMMA 3.1. *Let $X = \mathbf{R}^2 \setminus Y \times \{0\}$ for $Y \subset \mathbf{R}$, $y \in Y$ and $x \in X$ be in the upper half plane. Suppose that there exist $u, v \in \mathbf{R} \setminus Y$ such that $u < y < v$. Then,*

$$\pi_1(X, x) \simeq \mathcal{U}(D \cap (-\infty, y), Y \cup [y, \infty)) * \mathbf{Z} * \mathcal{U}(D \cap (y, \infty), Y \cup (-\infty, y])$$

*holds.*

PROOF. Let $W^*$ be a word defined by: $\overline{W^*} = D \cap (u, v)$, $W^*(\alpha) = \alpha$ for $\alpha \in \overline{W^*}$. We show $\mathcal{U}(D, Y) \simeq \mathcal{U}(D \cap (-\infty, y), Y \cup [y, \infty)) * \langle W^* \rangle * \mathcal{U}(D \cap (y, \infty), Y \cup (-\infty, y])$ holds. For $W \in \mathcal{U}(D, Y)$, there is a loop $f$ in $X$ with its base point $x$ such that $W^f \cong W$. Therefore, $\max\{a : a \leq y, (a, 0) \in \mathrm{Im}(f)\} < y < \min\{a : a \geq y, (a, 0) \in \mathrm{Im}(f)\}$ holds and we can choose $d_0 \in D$ so that $\max\{a : a \leq y, (a, 0) \in \mathrm{Im}(f)\} < d_0 < \min\{a : a \geq y, (a, 0) \in \mathrm{Im}(f)\}$. Hence, there is a regular decomposition (see [6, Definition 3.8]) $W_0 \cdots W_n$ of $W$ such that each $W_i$ satisfies exactly one of the following:

1. $W_i \in \mathcal{U}(D \cap (-\infty, y), Y \cup [y, \infty))$;
2. $W_i \in \mathcal{U}(D \cap (y, \infty), Y \cup (-\infty, y])$;
3. $W_i$ is a component of $W$ containing $d_0$ or $-d_0$.

We remark that the number $n$ does not depend on the decompositions of this kind. In case $W_i$ is a component of $W$, there exist $U_i, V_i \in \mathcal{U}(D, Y)$ such that:

1. $W_i = U_i W^* V_i$ or $W_i = U_i (W^*)^{-1} V_i$;

2. If $U_i$ is non-empty, $U_i$ belongs to exactly one of $\mathcal{U}(D \cap (-\infty, y), Y \cup [y, \infty))$ and $\mathcal{U}(D \cap (y, \infty), Y \cup (-\infty, y])$. The same holds for $V_i$.

Therefore, we get a homomorphism from $\mathcal{U}(D, Y)$ to $\mathcal{U}(D \cap (-\infty, y), Y \cup [y, \infty)) * \langle W^* \rangle * \mathcal{U}(D \cap (y, \infty), Y \cup (-\infty, y])$ and now it is easy to see that this is an isomorphism. The lemma follows from [6, Theorem 3.2]. □

LEMMA 3.2. *Let $X = \mathbf{R}^2 \setminus Y \times \{0\}$ and $x \in X$ for $Y \subseteq \mathbf{R}$. Suppose that $Y$ contains an increasing sequence $y_n$ $(n < \omega)$ such that $\lim_{n\to\infty} y_n = \infty$ and $(y_n, y_{n+1}) \cap (\mathbf{R} \setminus Y) \neq \emptyset$ for each $n$. Then, $\pi_1(X, x)$ is isomorphic to an infinite free product of non-trivial components.*

PROOF. Choose $u_n \in \mathbf{R} \setminus Y$ so that $y_n < u_n < y_{n+1}$ and let $\overline{W_n} = (u_n, u_{n+1}) \cap D$ and $W_n : \overline{W_n} \to D$ be the identity. Then, each $W_n$ belongs to $\mathcal{U}(D, Y)$. It is easy to see that

$$\begin{aligned}\mathcal{U}(D, Y) \simeq *_{n<\omega} \langle W_n \rangle * \mathcal{U}(D \cap (-\infty, y_0), Y \cup [y_1, \infty)) * \\ *_{n<\omega} \mathcal{U}(D \cap (y_n, y_{n+1}), Y \cup (-\infty, y_n] \cup [y_{n+1}, \infty)),\end{aligned}$$

which implies the lemma. □

Next, we define a size of $W \in \mathcal{U}(D, Y)$ and prove a lemma corresponding to Lemma 2.8 in Section 2. Let the *size* of $W \in \mathcal{U}(D, Y)$ be the diameter of the subset $\bigcup\{C : |u| \in C \in \mathcal{C}$ for $u \in \mathrm{Im}(W)\}$ of the real line.

LEMMA 3.3. *Suppose that every accumulation point of $\mathcal{C}$ belongs to $\mathbf{R} \setminus Y$. Let $h : \mathcal{U}(D, Y) \to *_{n<\omega} H_n$ be a homomorphism such that $\mathrm{Im}(h)$ is not contained in any $*_{n<m} H_n$. Then, for any $\varepsilon > 0$ and $m$ there exists $W \in \mathcal{U}(D, Y)$ of the size less than $\varepsilon$ such that $h(W) \notin *_{n<m} H_n$.*

PROOF. Assume the negation of the conclusion. Then, there exist $\varepsilon_0 > 0$ and $m$ such that $h(W) \in *_{n<m} H_n$ holds if the size of $W \in \mathcal{U}(D, Y)$ is less than $\varepsilon_0$. By the hypothesis of $\mathcal{C}$, the lengths of $C$'s in $\mathcal{C}$ converge to 0. Therefore, there exists $\mathcal{C}' \Subset \mathcal{C}$ such that the diameter of $C \in \mathcal{C} \setminus \mathcal{C}'$ is less than $\varepsilon_0$. Let $\overline{V_C} = C \cap D$ and $V_C : \overline{V_C} \to D$ be the identity. We remark that a word $V_C$ may not belong to $\mathcal{U}(D, Y)$. We choose $u_C, v_C \in \mathbf{R} \setminus Y (C \in \mathcal{C}')$ so that $C \subseteq (u_C, v_C)$ and $(u_C, v_C) \cap (u_{C'}, v_{C'}) = \emptyset$ for distinct $C, C' \in \mathcal{C}$ and let $\overline{W_C} = (u_C, v_C) \cap D$ and $W_C : \overline{W_C} \to D$ be the identity. Now, $W_C$ belongs to $\mathcal{U}(D, Y)$. Then, we can choose $m_0 \geq m$ so that $h(W_C) \in *_{n<m_0} H_n$ for all $C \in \mathcal{C}'$.

For any $W \in \mathcal{U}(D, Y)$, we have a regular decomposition $U_0 \cdots U_k$ of $W$ so that the size of $W_i$ is less than $\varepsilon_0$ or $W_i$ contains only one word in $\{V_C, (V_C)^{-1} : C \in \mathcal{C}'\}$ as a subword. Then, we adjust $U_i$'s and get words $W_0 \cdots W_l$ such that:

1. $W = W_0 \cdots W_l$;

2. the size of $W_i$ is less than $\varepsilon_0$, $W_i \cong W_C$ or $W_i \cong (W_C)^{-1}$ for some $C \in \mathcal{C}'$.

Therefore, $h(W) \in *_{n<m_0} H_n$ holds for any $W \in \mathcal{U}(D, Y)$, which is a contradiction. □

PROOF OF THEOREM 1.4. In case $\infty$ or $-\infty$ is an accumulation point of $\mathcal{C}$, Lemma 3.2 implies the conclusion. Suppose that there exists an accumulation point $y_0 \in Y$ of $\mathcal{C}$.

We only deal with the case where there is an increasing sequence $\mathcal{C}$ which converges to $y_0$. If $[y_0, \infty) \subset Y$, the conclusion follows from Lemma 3.2. Otherwise, there are $u, v \in \mathbf{R}\backslash Y$ such that $u < y_0 < v$. By Lemma 3.1, $\pi_1(X, x) \simeq \mathcal{U}(D \cap (-\infty, y_0), Y \cup [y_0, \infty)) * \mathbf{Z} * \mathcal{U}(D \cap (y_0, \infty), Y \cup (-\infty, y_0])$ holds. Again by Lemma 3.2, $\mathcal{U}(D \cap (-\infty, y_0), Y \cup [y_0, \infty))$ is isomorphic to a free product of infinitely many non-trivial components.

On the other hand, suppose that there is no accumulation point of $\mathcal{C}$ in $Y \cup \{\infty, -\infty\}$ and $h : \mathcal{U}(D, Y) \to *_{n<\omega} H_n$ be an isomorphism for non-trivial $H_n$'s. By Lemma 3.3, for each $m$ there exists $W_m \in \mathcal{U}(D, Y)$ such that $W_m$ has the size less than $1/2^m$, but $h(W_m) \notin *_{n<m} H_n$. Then, there exist a point $y_0$ and a subsequence $m_n$ $(n < \omega)$ such that $(|u| : u \in \mathrm{Im}(W_{m_n}))$ converge to $y_0$. Since $y_0$ is an accumulation point of $\mathcal{C}$, $y_0$ does not belong to $Y$. Therefore, each $d$ or $-d$ appears only in finitely many $W_{m_n}$'s. By [4, Proposition 1.9], we get a homomorphism $\psi : \mathsf{X}_{n<\omega} \mathbf{Z}_n \to \mathsf{X}_{d \in D} \mathbf{Z}_d$ such that $\psi(\delta_n) = W_{m_n}$ for each $n < \omega$. Moreover, $\mathrm{Im}(\psi) \subset \mathcal{U}(D, Y)$ holds, since $\psi$ is standard in the sense of [6, Definition 2.2] (See the next section). But, this contradicts [4, Corollary 2.5], which implies that $\mathrm{Im}(h \cdot \psi)$ is contained in some $*_{n<m} H_n$. □

## 4. Spatial homomorphisms and standard homomorphisms.

First, we recall the notions in the title above from [6]. Let $(X, x)$ and $(Y, y)$ be pointed spaces. A homomorphism $h : \pi_1(X, x) \to \pi_1(Y, y)$ is *spatial*, if there exists a continuous map $f : X \to Y$ with $f(x) = y$ such that $f_* = h$, where $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ is a homomorphism naturally induced from $f$.

For a word $W \in \mathcal{W}^\sigma(G_i : i \in I)$, the $i$-length $l_i(W)$ is the number of elements of $G_i$ which appear in $W$. For an element $x$ in the free $\sigma$-product $\mathsf{X}^\sigma_{i \in I} G_i$, $l_i(x)$ is defined as $l_i(W)$ for the reduced word $W$ of $x$ [4, p. 247]. A sequence $(x_j : j \in J)$ of elements of $\mathsf{X}^\sigma_{i \in I} G_i$ is *proper*, if $\{j \in J : l_i(x_j) \neq 0\}$ is finite for each $i \in I$.

Let $G_i$ $(i \in I)$ and $H_j$ $(j \in J)$ be groups. A homomorphism $h : \mathsf{X}^\sigma_{i \in I} G_i \to \mathsf{X}^\sigma_{j \in J} H_j$ is *standard*, if $(h(g_i) : i \in I)$ is proper for any $g_i \in G_i$ $(i \in I)$ and $h(W) = V$ for a word $W \in \mathcal{W}^\sigma(G_i : i \in I)$, where $V$ is the word in $\mathcal{W}^\sigma(H_j : j \in J)$ defined as follows:

(1) $\overline{V} = \{(\alpha, \beta) : \alpha \in \overline{W}, \beta \in \overline{V_\alpha}\}$, where $V_\alpha$ is the reduced word of $h(W(\alpha))$,

(2) The order $(\alpha, \beta) < (\alpha', \beta')$ is lexicographical, i.e. $\alpha < \alpha'$, or $\alpha = \alpha'$ and $\beta < \beta'$,

(3) $V(\alpha, \beta) = V_\alpha(\beta)$ for $(\alpha, \beta) \in \overline{V}$.

In case $G_i$ and $H_j$ are copies of $\mathbf{Z}$ for each $i$ and $j$, there is an easy criterion for detecting standard homomorphisms.

LEMMA 4.1. *Let $h : \mathsf{X}_{n<\omega} \mathbf{Z}_n \to \mathsf{X}^\sigma_{i \in J} \mathbf{Z}_i$ be a homomorphism. Then, $h$ is standard if and only if $(h(\delta_n) : n < \omega)$ is a proper sequence.*

PROOF. The one direction is clear by the definition. If $(h(\delta_n) : n < \omega)$ is a proper sequence, this sequence can be extended to a standard homomorphism naturally [4, Proposition 1.9]. By [6, Lemma 2.5], $h$ should be the standard homomorphism. □

For $a \in \mathsf{X}^{\sigma}_{i\in I}\mathbf{Z}_i$, supp$(a)$ is the set of $i \in I$ for which an element of $\mathbf{Z}_i$ appear in the reduced word for $a$. Let $\varphi : \pi_1(\mathbf{H}, o) \to \mathsf{X}_{n<\omega}\mathbf{Z}_n$ be the canonical isomorphism, that is, $\delta_n$ corresponds to a winding of the $n$-th circle $C_n$ of $\mathbf{H}$ and let $\xi : \pi_1(X, x_0) \to \langle W^f : f \in P_{x_0}\rangle \hookrightarrow \mathsf{X}^{\sigma}_{i\in I}\mathbf{Z}_i$ the isomorphism given by Theorem 1.1 for $x_0 \in D$.

THEOREM 4.2. *Let $X$ be a space satisfying the hypothesis of Theorem* 1.1. *Let $g : \pi_1(\mathbf{H}, o) \to \pi_1(X, x_0)$ and $h : \mathsf{X}_{n<\omega}\mathbf{Z}_n \to \langle W^f : f \in P_{x_0}\rangle$ be homomorphisms such that $\xi g = h\varphi$.*

*Then, $g$ is spatial if and only if $h$ is standard.*

PROOF. Suppose that $g = f_*$ for a continuous map $f : \mathbf{H} \to X$ with $f(o) = x_0$. Since $(f(C_n) : n < \omega)$ converges to $x_0$, $h$ is standard. Conversely, suppose that $h$ is standard. It suffices to show that (supp$(h(\delta_n)) : n < \omega$) converges to $x_0$, when all $h(\delta_n)$'s are non-trivial, since the reduced word for $h(\delta_n)$ is $W^{f_n}$ for some $f_n \in P_{x_0}$ by Lemma 2.4. Suppose that supp$(h(\delta_n)) \not\subseteq O$ for an open neighborhood $O$ of $x_0$ and infinitely many $n$'s. We choose a smaller neighborhood $P$ of $x_0$ so that $\overline{P} \subseteq O$. By [6, Lemma 2.4], there exist $W_n$ and $V_n$ $(n < \omega)$ such that

1. $h(\delta_n) = W_n^{-1}V_nW_n$,
2. $W_n^{-1}V_nW_n$ is quasi-reduced,
3. $V_nW_n$ is reduced,
4. $V_nV_n$ are reduced or $V_n$ is a single word.

We can inductively choose an increasing sequence $n_k < n_{k+1}, m_k$ and $i_k, j_k \in I$ so that

1. $m_0 = 1, i_k \neq j_k, A_{i_k} \not\subseteq O, A_{j_k} \cap P \neq \emptyset$,
2. the right end $V_{n_k}W_{n_k}$ of $W_{n_k}^{-1}V_{n_k}^{m_k}W_{n_k}$ remains in the reduced word of
$$W_{n_0}^{-1}V_{n_0}^{m_0}W_{n_0}\cdots W_{n_k}^{-1}V_{n_k}^{m_k}W_{n_k},$$
3. $i_k, j_k \notin$ supp$(h(\delta_m))$ for any $m \geq n_k$.

Choose $f \in P_{x_0}$ so that $W^f$ is the reduced word for $h(\delta_0^{m_0}\cdots\delta_k^{m_k}\cdots)$. Since $W^f = h(\delta_0^{m_0}\cdots\delta_k^{m_k})\cdot c$, where neither $i_k$ nor $j_k$ appears in the reduced word for $c$, $i_k$ and $j_k$ appear in $W^f$. Therefore, $f$ travels the inside of $P$ and the outside of $O$ infinitely many times alternately, which is impossible. □

Next, we shall show that any homomorphism $h : \mathsf{X}_{n<\omega}\mathbf{Z}_n \to \langle W^f : f \in P_{x_0}\rangle$ is a conjugate to a standard homomorphism. To show this, we state an easy lemma, whose proof is omitted.

LEMMA 4.3. *Let $X$ be a space satisfying the hypothesis of Theorem* 1.1. *If $W^f \cong UV$ for a proper path $f : [0, 1] \to X$, where $U$ and $V$ are non-empty, then there exists $0 < a < 1$ such that $f(a) \in D$, $W^{f\restriction[0,a]} \cong U$, and $W^{f\restriction[a,1]} \cong V$.*

THEOREM 4.4. *Let $X$ be a space satisfying the hypothesis of Theorem* 1.1. *For any homomorphism $h : \mathsf{X}_{n<\omega}\mathbf{Z}_n \to \langle W^f : f \in P_{x_0}\rangle$, there exist $y_0 \in D$ and a standard homomorphism $\bar{h} : \mathsf{X}_{n<\omega}\mathbf{Z}_n \to \langle W^f : f \in P_{y_0}\rangle$ such that $h$ is conjugate to $\bar{h}$. More precisely, there exists a proper path $f$ from $x_0$ to $y_0$ such that $h(x) = (W^f)^{-1}\bar{h}(x)W^f$ for all $x \in \mathsf{X}_{n<\omega}\mathbf{Z}_n$.*

PROOF. Since $\langle W^f : f \in P_{x_0}\rangle \leq \mathsf{X}^{\sigma}_{i\in I}\mathbf{Z}_i$, $h$ is a conjugate to a standard homomorphism $\bar{h} : \mathsf{X}_{n<\omega}\mathbf{Z}_n \to \mathsf{X}^{\sigma}_{i\in I}\mathbf{Z}_i$, i.e. $h(x) = W^{-1}\bar{h}(x)W$ for $x \in \mathsf{X}_{n<\omega}\mathbf{Z}_n$, by [6, Theorem 2.3]. In case $h(\delta_n)$ is trivial for almost all $n$, $h$ itself is standard. Otherwise, the reduced word $W$ is unique and obtained by a certain limit. In the proof of [6, Theorem 4.1] (pp. 20–21), we actually proved:

For any homomorphism $h : \mathsf{X}_{n<\omega}\mathbf{Z}_n \to \mathsf{X}^{\sigma}_{i\in I}\mathbf{Z}_i$ satisfying $h(\delta_n) \neq e$ for infinitely many $n < \omega$ and the corresponding $W$, there exists $u \in \mathsf{X}_{n<\omega}\mathbf{Z}_n$ such that the reduced word for $h(u)$ is of the form $UW$ for some word $U$.

Now, by Lemmas 2.4 and 4.3, we get a proper path $f$ and $y_0 \in D$ such that $W^f \cong W$ and $f$ is a path from $y_0$ to $x_0$. Define $\bar{h}(x) = Wh(x)W^{-1}$. Then, $\bar{h}$ is a standard homomorphism to $\langle W^f : f \in P_{y_0}\rangle$, and we get the conclusion. □

COROLLARY 4.5. *None of the fundamental groups of the spaces* (2) *and* (3) *in Figure* 1 *is isomorphic to* $\pi_1(\mathbf{H}, o)$.

PROOF. Let $h : \pi_1(\mathbf{H}, o) \to \pi_1(X, x_0)$ be an isomorphism, where $X$ is One of the spaces (2) and (3) in Figure 1. Then, $h$ is a conjugate to a spatial homomorphism by Theorems 4.2 and 4.4. Hence, there is an isomorphism which is spatial. So, we may assume $h = f_*$ for a continuous map with $f(o) = x_0$. Choose $y_0 \in D$ with $y_0 \neq x_0$ so that $X$ is no semi-locally connected at $y_0$. There exist a retraction $r : X \to R = P \cup E$ and a neighborhood $O$ of $x_0$ such that

1. $P$ is a neighborhood of $y_0$,
2. $E$ is a 1-dimensional compact polyhedron,
3. $r(O) \subseteq E$.

Since $rf(C_n) \subseteq E$ for almost all $n$, $\mathrm{Im}((rf)_*)$ is finitely generated. However, since $R$ contains $P$, $\pi_1(R, y_0)$ is infinitely generated, which is a contradiction. □

REMARK 4.6. Theorems 4.2 and 4.4 generalize [6, Corollary 2.11] and [6, Theorem 4.1] is another generalization of [6, Corollary 2.11].

ADDENDUM. Recently, the author has answered Question 1.5 negatively [2]. More precisely, the following has been shown. Let $X$ be a one-dimensional space which contains a copy $C$ of a circle and $X$ be not locally semi-simply connected at any point on $C$. Then, the fundamental group $\pi_1(X, x_0)$ for $x_0 \in C$ cannot be embeddable into $\mathsf{X}^{\sigma}_{i\in I}G_i$ for $n$-slender groups $G_i$ $(i \in I)$. Consequently, $\pi_1(X, x_0)$ for $x_0 \in C$ cannot be embeddable into the fundamental group of the Hawaiian earring.

## References


[1] J. W. CANNON and G. R. CONNER, On the fundamental groups of one dimensional spaces, preprint.

[2] K. EDA, The fundamental groups of one-dimensional wild spaces and the Hawaiian earring, Proc. Amer. Math. Soc. (to appear).

[3] K. EDA, The first integral singular homology groups of one point unions, Quart. J. Math. **42** (1991), 443–456.

[4] K. EDA, Free $\sigma$-products and noncommutatively slender groups, J. Algebra **148** (1992), 243–263.

[ 5 ] K. EDA, The Dugundji extension theorem and extension degree, Colloq. Math. **118** (1995), 25–38.

[ 6 ] K. EDA, Free $\sigma$-products and fundamental groups of subspaces of the plane, Topology Appl. **84** (1998), 283–306.

[ 7 ] K. EDA, The non-commutative Specker phenomenon, J. Algebra **204** (1998), 95–107.

[ 8 ] K. EDA and K. KAWAMURA, The fundamental groups of 1-dimensional spaces, Topology Appl. **87** (1998), 163–172.

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