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On the Schur Indices of the Irreducible Characters of Finite Unitary Groups

Zyozyu OHMORI

Hokkaido University of Education

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Let q be a power of a fixed prime number p. Let G be the unitary group $U(n, q^2)$ of degree n with respect to a quadratic extension $\mathbf{F}_{q^2}/\mathbf{F}_q$ ($\mathbf{F}_{q^e}$ denotes a finite field with q^e elements). The character table of G can essentially be obtained from the character table of the general linear group $GL(n, q)$ by a simple formal change that q is everywhere replaced by $-q$ (Ennola conjecture [2]; V. Ennola [2], G. Lusztig and B. Srinivasan [10], R. Hotta and T. A. Springer [7], G. Lusztig, D. Kazhdan, N. Kawanaka [8]; for $n \leqq 5$, the character table of G had been calculated by Ennola ([2]: $n = 2, 3$) and S. Nozawa ([12, 13]: $n = 4, 5$)). The purpose of this paper is to give some results concerning the Schur indices of the irreducible characters of G.

In the following, if χ is a complex irreducible character of a finite group and F is a field of characteristic 0, then $F(\chi)$ will denote the field generated over F by the values of χ and $m_F(\chi)$ will denote the Schur index of χ with respect to F.

Let χ be any one of the irreducible characters of G. Then the following two results are known:

THEOREM A (R. Gow [5, Theorem A]). *We have* $m_{\mathbf{Q}}(\chi) \leqq 2$.

THEOREM B ([18, Theorem 3]). *For any prime number* $l \neq p$, *we have* $m_{\mathbf{Q}_l}(\chi) = 1$.

The local index $m_{\mathbf{R}}(\chi)$ can be calculated by the method of Frobenius and Schur (see, e.g., Feit [3, pp. 20–21]): put $\nu(\chi) = (1/|G|) \cdot \sum_{g \in G} \chi(g^2)$; then $\nu(\chi) = 1, 0$ or -1; if $\nu(\chi) = 1$ or 0, then $m_{\mathbf{R}}(\chi) = 1$ and if $\nu(\chi) = -1$, then $m_{\mathbf{R}}(\chi) = 2$. But I think that when n is large the actual practice of this method is difficult. Of course, if $\mathbf{R}(\chi) = \mathbf{C}$, then $m_{\mathbf{R}}(\chi) = 1$. In the remark at the end of §2 of this paper we shall give, in terms of Ennola's parametrization of the irreducible characters of G ([2]; see §1, 1.5), a necessary and sufficient condition subject for that $\mathbf{R}(\chi) = \mathbf{R}$ or $\mathbf{C}$.

As to the local index $m_{\mathbf{Q}_p}(\chi)$, the following fact is implicit in [16, §3] in the case where $p \neq 2$.

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THEOREM C. *If* $[\mathbf{Q}_p(\chi):\mathbf{Q}_p]$ *is even, then* $m_{\mathbf{Q}_p}(\chi)=1$.

This theorem will be proved in §4. Using Ennola's parametrization, one can see the action of the Galois group $G(F/\mathbf{Q}_p)$ of a certain finite Galois extension F of $\mathbf{Q}_p$ over $\mathbf{Q}_p$ on the χ explicitely (see Remark at the end of §2), so we can determine $[\mathbf{Q}_p(\chi):\mathbf{Q}_p]$.

In §2, we shall determine the local indices $m_{\mathbf{R}}(\chi)$ and $m_{\mathbf{Q}_p}(\chi)$ completely in the case where $p=2$ (Theorem 3). By N. Kawanaka's "multiplicity-one theorem" for the generalized Gelfand-Graev characters γ_A of G (see §1, 1.4 and Theorem 1), the problem will be reduced to the rationality-problem of the unipotent characters of G, and we use a result of [17]. In §3, using the characters γ_A, we shall obtain some partial results concerning the local Schur indices of some of the χ in the case where $p\neq 2$ (Theorem 4).

In §5, we shall give some sufficient conditions subject for that $m_{\mathbf{Q}}(\chi)=1$ (Theorems 7, 8). Logically, Theorems 7, 8 are contained in the results in §§2, 4. But Theorems 7, 8 seem to be useful. They depend on the following fact:

THEOREM D. (i) *Assume that* $p=2$. *If* $\mathbf{Q}(\chi)$ *contains a* $(q+1)$*-th root of unity* $\neq 1$, *then* $m_{\mathbf{Q}}(\chi)=1$.
(ii) *Assume that* $p\neq 2$. *If* $\mathbf{Q}(\chi)$ *contains a* $(q+1)$*-th root of unity* $\neq\pm 1$, *then* $m_{\mathbf{Q}}(\chi)=1$.

This theorem will be proved in §4. Let ε be an element of order $q+1$ in $\mathbf{F}_{q^2}$ and let ζ be a primitive $(q+1)$-th root of unity in $\mathbf{C}$. Let x be an element of G whose characteristic roots are of the form ε^i. When $n\leqq 5$ we can see the character table of G directly ([2, 12, 13]). Let z be a generator of the centre of G. Then, by Schur's lemma, we must have $\chi(z)=\zeta^i\chi(1)$ for some i. Assume, for instance, that $p\neq 2$. If $\zeta^i\neq\pm 1$, then, by Theorem D, we have $m_{\mathbf{Q}}(\chi)=1$. Even in the case where $\zeta^i=\pm 1$ we find that in some cases there is some x as above such that $\chi(x)=\pm\zeta^j$. So, if $\zeta^j\neq\pm 1$, then we can use Theorem D. Theorems 7, 8 are generalization of these practical facts to a general n.

Our results of this paper depend on Ennola's character theory of G ([2]); in the appendix we shall review his formulation.

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1. The irreducible characters of $U(n,q^2)$.

1.1. Partitions. Let n be a non-negative integer. If $n=n_1+n_2+\cdots+n_s$ where $n_1,n_2,\cdots,n_s$ are non-negative integers, then the symbol $[n_1,n_2,\cdots,n_s]$ will be called a partition of n; if the sequence $1',2',\cdots,s'$ is any permutation of the sequence $1,2,\cdots,s$, then we make a promise that $[n_{1'},n_{2'},\cdots,n_{s'}]=[n_1,n_2,\cdots,n_s]$; we also promise that $[n_1,n_2,\cdots,n_s,0,0,\cdots,0]=[n_1,n_2,\cdots,n_s]$. If a partition ρ of n has r_1 parts equal to 1, r_2 parts equal to 2, r_3 parts equal to 3, $\cdots$, then ρ will be often denoted as $[1^{r_1}2^{r_2}\cdots n^{r_n}]$ or $[1^{r_1}2^{r_2}3^{r_3}\cdots]$, and in this case we put

$$z_\rho=1^{r_1}(r_1)!\,2^{r_2}(r_2)!\,3^{r_3}(r_3)!\cdots n^{r_n}(r_n)!\,. \tag{1.1.1}$$

If ρ is a partition of n, then we write $|\rho| = n$. For $n = 0$, 0 will also denote the unique partition of the number zero. For any $n \geqq 0$, P_n will denote the set of all partitions of n. We set $P = \bigcup_{n \geqq 0} P_n$.

Let n be a non-negative integer. Then P_n has a lexicographical ordering: for $\mu, \nu \in P_n$, if $\mu = [m_1, \cdots, m_s]$ with $m_1 \geqq \cdots \geqq m_s \geqq 0$ and $\nu = [n_1, \cdots, n_t]$ with $n_1 \geqq \cdots \geqq n_t \geqq 0$, then we have $\mu > \nu$ if $m_1 > n_1$ or there is a number i such that $m_1 = n_1, \cdots, m_i = n_i$ and $m_{i+1} > n_{i+1}$. If μ is a partition of n, then $\tilde{\mu}$ will denote the conjugate partition of μ; if $\tilde{\mu} = [k_1, k_2, \cdots, k_s]$, then we put

$$n_\mu = \sum_{i=1}^{s} \frac{k_i(k_i - 1)}{2}. \tag{1.1.2}$$

Let m, n be non-negative integers. Let μ be a partition of m and let ν be a partition of n, and suppose that $\mu = [m_1, \cdots, m_s]$ and $\nu = [n_1, \cdots, n_t]$. Then we denote by $\mu + \nu$ the partition $[m_1, \cdots, m_s, n_1, \cdots, n_t]$ of $m + n$. If $s = t$, $m_1 \geqq \cdots \geqq m_s$ and $n_1 \geqq \cdots \geqq n_s$, then $\mu \cdot \nu$ will denote the partition $[m_1+n_1, \cdots, m_s+n_s]$ of $m+n$; for example, if $\mu = [1, 2, 3]$ and $\nu = [6, 4, 7, 5]$, then, noting that $[1, 2, 3] = [3, 2, 1, 0]$ and $[6, 4, 7, 5] = [7, 6, 5, 4]$, we have $\mu \cdot \nu = [3 + 7, 2 + 6, 1 + 5, 0 + 4] = [10, 8, 6, 4]$. If d is a positive integer and $\pi = [p_1, p_2, \cdots, p_s]$ is a partition of a non-negative integer v, then $d \cdot \pi$ will denote the partition $[dp_1, dp_2, \cdots, dp_s]$ of dv.

1.2. Irreducible characters of the symmetric groups. Let n be a positive integer. Then S_n denotes the symmetric group of order $n!$. The conjugacy classes of S_n and the irreducible characters of S_n can be naturally parametrized by the partitions of n (see, e.g., [11]); for $\lambda, \rho \in P_n$, χ^λ_ρ denotes the value of the irreducible character χ^λ of S_n corresponding to λ at the class of S_n corresponding to ρ. We have

$$\chi^{\tilde{\lambda}} = \mathrm{sgn} \cdot \chi^\lambda \quad (\lambda \in P_n); \tag{1.2.1}$$

$$\mathrm{sgn}(d \cdot \pi) = (-1)^{(d-1)v} \, \mathrm{sgn}(\pi) \quad (\pi \in P_v). \tag{1.2.2}$$

[(1.2.2) can be easily checked by the induction on v.]

Let $n, n_1, n_2, \cdots, n_s$ be positive integers such that $n = n_1 + n_2 + \cdots + n_s$. Then the product $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_s}$ can be naturally viewed as a subgroup of S_n. Set $P_{(n_1,n_2,\cdots,n_s)} = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_s}$. For $\lambda \in P_n$, set

$$\begin{aligned} \chi^\lambda \,\big|\, S_{n_1} \times S_{n_2} \times \cdots \times S_{n_s} &= \text{the restriction of } \chi^\lambda \text{ to } S_{n_1} \times S_{n_2} \times \cdots \times S_{n_s} \\ &= \sum_{(\lambda_1,\lambda_2,\cdots,\lambda_s) \in P_{(n_1,n_2,\cdots,n_s)}} c^\lambda_{\lambda_1 \lambda_2 \cdots \lambda_s} (\chi^{\lambda_1} \times \chi^{\lambda_2} \times \cdots \times \chi^{\lambda_s}). \end{aligned} \tag{1.2.3}$$

Here the $c^\lambda_{\lambda_1 \lambda_2 \cdots \lambda_s}$ are some non-negative integers. By Frobenius reciprocity Law, (1.2.3) is equivalent to the following:

$$(1.2.4)\qquad \mathrm{Ind}_{S_{n_1}\times S_{n_2}\times\cdots\times S_{n_s}}^{S_n}(\chi^{\lambda_1}\times\chi^{\lambda_2}\times\cdots\times\chi^{\lambda_s}) = \sum_{\lambda\in P_n} c^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}\chi^{\lambda}\quad ((\lambda_1,\lambda_2,\cdots,\lambda_s)\in P_{(n_1,n_2,\cdots,n_s)}).$$

For a partition μ of some non-negative integer, let s_μ denote the S-function corresponding to μ (see [11, p. 24]). Then the formula (1.2.4) is equivalent to:

$$(1.2.5)\qquad s_{\lambda_1}s_{\lambda_2}\cdots s_{\lambda_s} = \sum_{\lambda\in P_n} c^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s} s_\lambda$$

(see [11, I, (7.3), p. 61; I, 9, p. 68]). The integers $c^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}$ can be calculated by the Littlewood-Richardson rule ([11, (9.2), p. 68]). So, by [14, (2.4)], we get:

LEMMA 1. *If $\lambda > \lambda_1\cdot\lambda_2\cdot\cdots\cdot\lambda_s$ or $\lambda < \lambda_1+\lambda_2+\cdots+\lambda_s$, then $c^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}=0$. If $\lambda = \lambda_1\cdot\lambda_2\cdot\cdots\cdot\lambda_s$ or $\lambda=\lambda_1+\lambda_2+\cdots+\lambda_s$, then $c^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}=1$.*

LEMMA 2 ([14, (2.5)], A. V. Zelevinsky [21, 4.1]). *Let d, v be positive integers, and let $\nu\in P_v$, $\lambda\in P_{dv}$. Then*

$$\sum_{\pi\in P_v}\frac{1}{z_\pi}\chi^{\nu}_{\pi}\chi^{\lambda}_{d\cdot\pi} = \begin{cases} 1 & \text{if } \lambda = d\cdot\nu\,,\\ 0 & \text{if } \lambda > d\cdot\nu\,.\end{cases}$$

1.3. Hall polynomials and Green polynomials. Let t be a variable over $\mathbf{C}$. Let $n, n_1, n_2, \cdots, n_s$ be positive integers such that $n = n_1+n_2+\cdots+n_s$. For $\lambda\in P_n$ and $\lambda_i\in P_{n_i}$, $1\leqq i\leqq s$, Let $g^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}(t)$ be the Hall polynomial in t (see J. A. Green [6, pp. 411–2]; also see [11, II, 4]).

LEMMA 3 ([6, Theorem 4]; also see [11, II, (4.3), p. 93]). *Let the notation be as in (1.2.3). If $c^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}=0$, then the polynomial $g^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}(t)$ vanishes identically. If $c^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}\neq 0$, then the polynomial $g^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}(t)$ has the leading term $c^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}t^{n_\lambda-n_{\lambda_1}-n_{\lambda_2}-\cdots-n_{\lambda_s}}$.*

As $n_{\lambda_1\cdot\lambda_2\cdot\cdots\cdot\lambda_s} = n_{\lambda_1}+n_{\lambda_2}+\cdots+n_{\lambda_s}$, we have from Lemmas 1, 3 the following fact:

LEMMA 4. *If $\lambda = \lambda_1\cdot\lambda_2\cdot\cdots\cdot\lambda_s$, then $g^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}=1$. If $\lambda > \lambda_1\cdot\lambda_2\cdot\cdots\cdot\lambda_s$ or $\lambda < \lambda_1+\lambda_2+\cdots+\lambda_s$, then $g^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}(t)\equiv 0$.*

The following formula can be proved by a method similar to that in the proof of formula (16) of [6, p. 421]:

$$(1.3.1)\qquad g^{\lambda}_{\lambda_1\,\lambda_2\cdots\lambda_s}(t) = \sum_{\xi\in P_{n_1+\cdots+n_{s-1}}} g^{\xi}_{\lambda_1\cdots\lambda_{s-1}}(t)g^{\lambda}_{\xi\lambda_s}(t)\,.$$

Set $\phi_0(t)=1$ and, for a positive integer r, set

$$(1.3.2)\qquad \phi_r(t) = (1-t)(1-t^2)\cdots(1-t^r)\,.$$

For $\lambda\in P_n$, set

$$(1.3.3)\qquad a_\lambda(t) = t^{n-2n_\lambda}\prod_{i=1}^{m}\phi_{k_i-k_{i+1}}(t^{-1})$$

if $\tilde{\lambda} = [k_1, \cdots, k_m]$ with $k_1 \geqq \cdots \geqq k_m > 0$ $(k_{m+1} = 0)$.

For $\lambda, \rho \in P_n$, let $Q_\rho^\lambda(t)$ be the Green polynomial of $GL(n, q)$ in t ([6, Definition 4.2; also see [11, III, 7, p. 132]). For $\rho = [1^{r_1} 2^{r_2} \cdots n^{r_n}] \in P_n$, set

$$c_\rho(t) = (t-1)^{r_1} (t^2-1)^{r_2} \cdots (t^n-1)^{r_n} . \tag{1.3.4}$$

Then, by Theorem 10 of [6] (also see [11, III, (7.10), p. 132]), we have

$$\sum_{\lambda \in P_n} \frac{1}{a_\lambda(t)} Q_\rho^\lambda(t) Q_\sigma^\lambda(t) = \delta_{\rho\sigma} \cdot \frac{z_\rho}{c_\rho(t)} \quad (\rho, \sigma \in P_n) . \tag{1.3.5}$$

For $\lambda \in P_n$, let u_λ be one of the unipotent elements of $GL(n, q)$ whose Jordan canonical forms are of type λ; for $\rho \in P_n$, let Q_{ρ, GL_n} be the class function on $GL(n, q)$ defined by $Q_{\rho, GL_n}(u_\lambda) = Q_\rho^\lambda(q)$ for $\lambda \in P_n$, and $Q_{\rho, GL_n}(x) = 0$ if x is a non-unipotent element of $GL(n, q)$. Then the left hand side of (1.3.5) is equal to the inner product $(Q_{\rho, GL_n}, Q_{\sigma, GL_n})_{GL(n,q)}$. We note that, by the truth of Ennola conjecture, $Q_\rho^\lambda(-q)$ is a Green polynomial of $U(n, q^2)$.

By Lemma 4.4 of [6], we have

$$Q_{\rho+\sigma}^\nu(t) = \sum_{(\lambda, \mu) \in P_{(l,m)}} g_{\lambda\mu}^\nu(t) Q_\rho^\lambda(t) Q_\sigma^\mu(t) \quad (\rho \in P_l, \sigma \in P_m, \nu \in P_{l+m}) . \tag{1.3.6}$$

Then, by the induction on s, it follows from (1.3.1), (1.3.6) that

$$Q_{\rho_1+\cdots+\rho_s}^\lambda(t) = \sum_{(\lambda_1, \cdots, \lambda_s) \in P_{(n_1, \cdots, n_s)}} g_{\lambda_1 \cdots \lambda_s}^\lambda(t) Q_{\rho_1}^{\lambda_1}(t) \cdots Q_{\rho_s}^{\lambda_s}(t) \tag{1.3.7}$$

$$(\lambda \in P_n, \rho_i \in P_{n_i}, 1 \leqq i \leqq s) .$$

For $\rho, \lambda \in P_n$, set (see [11, III, (7.8), p. 132])

$$X_\rho^\lambda(t) = t^{n_\lambda} Q_\rho^\lambda(t^{-1}) . \tag{1.3.8}$$

LEMMA 5 ([14, (2.13)]; cf. N. Kawanaka [8, (3.2.19)]). *Let d, v be positive integers, and let $\nu \in P_v$, $\lambda \in P_{dv}$. Then*

$$\sum_{\pi \in P_v} \frac{1}{z_\pi} \chi_\pi^\nu X_{d \cdot \pi}^\lambda(t) = \begin{cases} 1 & \text{if } \lambda = d \cdot \nu \\ 0 & \text{if } \lambda > d \cdot \nu . \end{cases}$$

Using Lemmas 4, 5 and (1.3.7), by the consideration in [14, p. 705], we have

LEMMA 6. *Let $d_1, \cdots, d_s$, $v_1, \cdots, v_s$ be positive integers and let $\lambda \in P_{d_1 v_1 + \cdots + d_s v_s}$ and $\nu_i \in P_{v_i}$, $1 \leqq i \leqq s$. Then we have*

$$\sum_{(\pi_1, \cdots, \pi_s) \in P_{(v_1, \cdots, v_s)}} \frac{1}{z_{\pi_1} \cdots z_{\pi_s}} \chi_{\pi_1}^{\nu_1} \cdots \chi_{\pi_s}^{\nu_s} X_{d_1 \cdot \pi_1 + \cdots + d_s \cdot \pi_s}^\lambda(t)$$

$$= \begin{cases} 1 & \text{if } \lambda = (d_1 \cdot \nu_1) \cdot \cdots \cdot (d_s \cdot \nu_s) , \\ 0 & \text{if } \lambda > (d_1 \cdot \nu_1) \cdot \cdots \cdot (d_s \cdot \nu_s) . \end{cases}$$

1.4. Generalized Gelfand-Graev characters. In this subsection we shall review Kawanaka's construction of the generalized Gelfand-Graev characters of $U(n, q^2)$ ([8]).

Let $\mathbf{G} = GL_n(\bar{\mathbf{F}}_q)$, where $\bar{\mathbf{F}}_q$ is an algebraic closure of a finite field $\mathbf{F}_q$ with q elements of characteristic p, and let $\mathbf{L}$ be the Lie algebra of all $n \times n$ matrices over $\bar{\mathbf{F}}_q$. Let $F : \mathbf{G} \to \mathbf{G}$ be the Frobenius endomorphism of $\mathbf{G}$ with respect to a twisted $\mathbf{F}_q$-structure on $\mathbf{G}$, so that $\mathbf{G}^F = \{g \in \mathbf{G} \mid F(g) = g\}$ is isomorphic to $U(n, q^2)$. $\mathbf{L}$ can be regarded as the Lie algebra of the algebraic group $\mathbf{G}$ and F acts on $\mathbf{L}$.

Let $\mathbf{G}_0$ be the set of unipotent elements of $\mathbf{G}$ and let $\mathbf{L}_0$ be the set of nilpotent elements of $\mathbf{L}$. Let η be an element of $\mathbf{F}_{q^2}$ such that $\eta^q + \eta = 1$, and let $f : \mathbf{G}_0 \to \mathbf{L}_0$ be the bijection given by $f(x) = (x-1)(\eta^q x + \eta)^{-1}$. One can choose F so that $F \circ f = f \circ F$.

Let $A \in {\mathbf{L}_0}^F$. Then, by the theory of Jordan canonical forms, there are positive integers $d_1, d_2, \cdots, d_s$ such that $d_1 + d_2 + \cdots + d_s = n$ and column vectors $e_1, e_2, \cdots, e_s$ of size n over $\bar{\mathbf{F}}_q$, such that $A^{d_i} e_i = 0$ for $1 \leqq i \leqq s$ and that the $A^j e_i$, $1 \leqq i \leqq s$, $0 \leqq j \leqq d_i - 1$, form a basis of the (column) vector space ${\bar{\mathbf{F}}_q}^n$ over $\bar{\mathbf{F}}_q$. Let $\lambda : \bar{\mathbf{F}}_q^\times \to \mathbf{G}$ be the morphism of $\bar{\mathbf{F}}_q^\times$ given by

$$\lambda(x) A^j e_i = x^{1-d_i+2j} A e_i\,, \quad 1 \leqq i \leqq s\,,\ 0 \leqq j \leqq d_i - 1 \ (x \in \bar{\mathbf{F}}_q^\times)\,.$$

For an integer i, set

$$\mathbf{L}(i)_A = \{M \in \mathbf{L} \mid \lambda(x) M \lambda(x)^{-1} = x^i M, x \in \bar{\mathbf{F}}_q^\times\}\,.$$

Then we have $\mathbf{L} = \bigoplus_{i \in \mathbf{Z}} \mathbf{L}(i)_A$ (T. A. Springer and R. Steinberg [19, IV, E-83]). For $i \geqq 1$, set

$$\mathbf{u}_i = \bigoplus_{j \geqq i} \mathbf{L}(j)_A\,,$$

and let

$$\mathbf{U}_i = f^{-1}(\mathbf{u}_i)\,,$$

where we assume that f is defined over $\mathbf{F}_q$. Then, for each $i \geqq 1$, $\mathbf{U}_i$ is an F-stable unipotent subgroup of $\mathbf{G}$, and, for $j \geqq i$, $\mathbf{U}_j$ is a normal subgroup of $\mathbf{U}_i$ ([8, 1]). For $i \geqq 1$, set $U_i = {\mathbf{U}_i}^F$. Set $G = \mathbf{G}^F$.

Let $\kappa : \mathbf{L} \times \mathbf{L} \to \bar{\mathbf{F}}_q$ be the bilinear mapping on $\mathbf{L}$ given by $\kappa(M, N) = \mathrm{Tr}(MN)$ (in the following application, we have, for $g = [g_{ij}] \in \mathbf{G}$, $F(g) = w\,{}^t[{g_{ij}}^q]^{-1} w^{-1}$, where w is a permutation matrix, so that, for $x = [x_{ij}] \in \mathbf{L}$, $F(x) = -w\,{}^t[{x_{ij}}^q] w^{-1}$; hence we have $\kappa(F(M), F(N)) = \kappa(M, N)^q$, that is, κ is defined over $\mathbf{F}_q$). Let χ_0 be a fixed non-trivial additive character of $\mathbf{F}_q$ with values in $\mathbf{C}$. Let ξ_A be the linear character of U_2 defined by

$$\text{(1.4.1)} \qquad \xi_A(u) = \chi_0(\kappa(A^*, f(u)))\,, \quad u \in U_2\,,$$

where $A^* = -{}^t A$ (in the following application, we have $F(A^*) = A^*$). Let $m(A) = (1/2) \dim_{\bar{\mathbf{F}}} \mathbf{L}(1)_A$ (an integer). Then there is a linear subspace $\mathbf{s}$ of $\mathbf{L}(1)^F$, of dimension $m(A)$ over $\mathbf{F}_q$, such that

$$\kappa(A^*, [M, N]) = 0\,, \quad M, N \in \mathbf{s}$$

([8, p. 180]). Let

$$U_{1.5} = f^{-1}(\mathbf{s} + {\mathbf{u}_2}^F)\,.$$

Then $U_{1.5}$ is a subgroup of U_1 and the linear character ξ_A can be extended to a linear character $\xi_A^{\sim}$ of $U_{1.5}$ ([8, (1.3.2)(i), (iii)]). We put

$$\gamma_A = \mathrm{Ind}_{U_{1.5}}^{G}(\xi_A^{\sim}) . \tag{1.4.2}$$

We also write $\gamma_A = \gamma_\mu$, $\mu = [d_1, d_2, \cdots, d_s] \in P_n$.

Let

$$\theta_A = \theta_\mu = \mathrm{Ind}_{U_{1.5}}^{U_1}(\xi_A^{\sim}) . \tag{1.4.3}$$

Then, by [8, (1.3.7)], we have

$$\theta_A(u) = \begin{cases} q^{m(A)}\xi_A(u) & (u \in U_2) \\ 0 & (u \in U_1 - U_2) . \end{cases} \tag{1.4.4}$$

It follows that θ_A is an irreducible character of U_1 ([8, (1.3.8)]). We note that if $p = 2$ then θ_A is $\mathbf{Q}$-valued.

For $\mu, \lambda \in P_n$, we have ([8, (3.2.14), p. 194]):

$$\gamma_\mu(u_\lambda) = (-1)^{n_\mu + [n/2]} \sum_{\rho \in P_n} \frac{1}{z_\rho} \mathrm{sgn}(\rho) |c_\rho(-q)| X_\rho^\mu(-q) Q_\rho^\lambda(-q) . \tag{1.4.5}$$

Here u_λ is a unipotent element of $G = U(n, q^2)$ which belongs to the class $((t-1)^\lambda)$ (cf. Appendix).

THEOREM 1 ([8, (3.2.18)(iii), (3.2.24)(i)]). *For any irreducible character χ of G, there is a partition $\mu(\chi)$ of n such that $(\gamma_{\mu(\chi)}, \chi)_G = 1$.*

In (1.6) below, we shall give $\mu(\chi)$ explicitely (cf. [8, (3.2.18)(iii)]).

1.5. Ennola's parametrization. Let $G = U(n, q^2)$. In this subsection we review Ennola's parametrization of the irreducible characters of G ([2]). As to his character theory of G, we shall review it in the appendix.

Let s be a positive integer. Then a set $g = \{k, k(-q), k(-q)^2, \cdots, k(-q)^{s-1}\}$ of integers will be called an s-simplex with the roots $k(-q)^i$, $0 \leqq i \leqq s-1$, if the $k(-q)^i$ are all distinct modulo $q^s - (-1)^s$; we write $d(g) = s$. For two s-simplexes $g = \{k, k(-q), \cdots, k(-q)^{s-1}\}$, $g' = \{k', k'(-q), \cdots, k'(-q)^{s-1}\}$, we understand $g = g'$ if and only if $k' \equiv k(-q)^u$ (mod $q^s - (-1)^s$) for some non-negative integer u. Let S be the set of all s-simplexes for $s \geqq 1$. Let X be the set of all functions $\nu : S \to P$ such that

$$\sum_{g \in S} |\nu(g)| d(g) = n .$$

For $\nu \in X$, set (formally)

$$\chi_\nu = (\cdots g^{\nu(g)} \cdots) = (g_1{}^{\nu_1} \cdots g_N{}^{\nu_N}) , \tag{1.5.1}$$

where $g_1, \cdots, g_N$ are all the $g \in S$ such that $\nu(g) \neq 0$ and, for $1 \leqq i \leqq N$, $\nu_i = \nu(g_i)$. χ_ν is called a dual class of G. The χ_ν, $\nu \in X$, parametrize the irreducible characters of G. For $\nu \in X$, we identify χ_ν with the irreducible character of G corresponding to it.

Let χ be any irreducible character of G, and suppose that $\chi = ({g_1}^{\nu_1} \cdots {g_N}^{\nu_N})$. For $1 \leqq i \leqq N$, put $d_i = d(g_i)$ and $v_i = |\nu_i|$. Let $\lambda \in P_n$. Then we have

$$(1.5.2) \qquad \chi(u_\lambda) = \eta(\chi) \sum_{(\pi_1,\cdots,\pi_N)\in P_{(v_1,\cdots,v_N)}} \frac{1}{z_{\pi_1}\cdots z_{\pi_N}} \chi_{\pi_1}^{\nu_1}\cdots\chi_{\pi_N}^{\nu_N} Q^\lambda_{d_1\cdot\pi_1+\cdots+d_N\cdot\pi_N}(-q),$$

where $\eta(\chi) = \pm 1$ such that $\chi(u_\lambda) > 0$ if $\lambda = [1^n]$.

For $GL(n, q)$, the formula corresponding to (1.5.2) is well known. Ennola's formulation for a character theory of G is obtained from Green's character theory of $GL(n, q)$ formally by changing q everywhere by $-q$. So the proof of (1.5.2) can be obtained, for instance, from the calculations in [14, pp. 703–5].

1.6. Proof of Theorem 1. Let $\rho \in P_n$. Let $Q_\rho = Q_{\rho,U_n}$ be the class function on $G = U(n, q^2)$ given by $Q_\rho(u_\lambda) = Q^\lambda_\rho(-q)$, $\lambda \in P_n$, and $Q_\rho(g) = 0$ if g is not unipotent.

Let $\mu \in P_n$ and let χ be any irreducible character of G. Suppose that $\chi = ({g_1}^{\nu_1} \cdots {g_N}^{\nu_N})$, and, for $1 \leqq i \leqq N$, put $d_i = d(g_i)$ and $v_i = |\nu_i|$. Then, by (1.4.5) and (1.5.2), we have

$$(\gamma_\mu, \chi)_G = \frac{1}{|G|} \sum_{\substack{g\in G \\ \text{unipotent}}} \gamma_\mu(g)\chi(g^{-1})$$

(cf. as $\gamma_\mu = \mathrm{Ind}_{U_1}^G(\theta_\mu)$ and U_1 is a unipotent group, $\gamma_\mu(g) = 0$ if g is not unipotent)

$$= (-1)^{n_\mu+[n/2]} \sum_{\rho\in P_n} \frac{1}{z_\rho} \mathrm{sgn}(\rho)|c_\rho(-q)|X^\mu_\rho(-q)$$
$$\times\ \eta(\chi) \sum_{(\pi_1,\cdots,\pi_N)\in P_{(v_1,\cdots,v_N)}} \frac{1}{z_{\pi_1}\cdots z_{\pi_N}} \chi_{\pi_1}^{\nu_1}\cdots\chi_{\pi_N}^{\nu_N} (Q_\rho, Q_{d_1\cdot\pi_1+\cdots+d_N\cdot\pi_N})_G\,.$$

By (1.3.5), (1.2.2) and (1.2.1), the last expression is equal to:

$$(-1)^{n_\mu+[n/2]}\eta(\chi)$$
$$\times \sum_{\substack{(\pi_1,\cdots,\pi_N)\in P_{(v_1,\cdots,v_N)} \\ \rho=d_1\cdot\pi_1+\cdots+d_N\cdot\pi_N}} \frac{1}{z_\rho}\mathrm{sgn}(\rho)|c_\rho(-q)|X^\mu_\rho(-q)\cdot\frac{1}{z_{\pi_1}\cdots z_{\pi_N}}\chi_{\pi_1}^{\nu_1}\cdots\chi_{\pi_N}^{\nu_N}\cdot\frac{z_\rho}{|c_\rho(-q)|}$$
$$= (-1)^{n_\mu+[n/2]}\eta(\chi)(-1)^{\sum_{i=1}^N(d_i-1)v_i}$$
$$\times \sum_{(\pi_1,\cdots,\pi_N)\in P_{(v_1,\cdots,v_N)}} \frac{1}{z_{\pi_1}\cdots z_{\pi_N}}\chi_{\pi_1}^{\tilde\nu_1}\cdots\chi_{\pi_N}^{\tilde\nu_N} X^\mu_{d_1\cdot\pi_1+\cdots+d_N\cdot\pi_N}(-q)\,.$$

Thus, by Lemma 6, if we put

$$(1.6.1) \qquad \mu(\chi) = (d_1\cdot\tilde\nu_1)\cdot\cdots\cdot(d_N\cdot\tilde\nu_N)\,,$$

then the last expression is equal to 1 if $\mu = \mu(\chi)$ (in fact, to ± 1; but γ_μ and χ are actual characters, so that $(\gamma_\mu, \chi)_G \geqq 0$), and 0 if $\mu > \mu(\chi)$.

This proves Theorem 1.

By this proof, we find that

$$\eta(\chi) = (-1)^{n_\mu(\chi)+[n/2]+\sum_{i=1}^{N}(d_i-1)v_i}\,. \tag{1.6.2}$$

2. The case $p = 2$.

Let $G = U(n, q^2) = U_n(\mathbf{F}_q)$. Let $g = \{0\}$ be the 1-simplex with the root 0. For $\alpha \in P_n$, set

$$\rho_\alpha = (g^\alpha)\,. \tag{2.1}$$

Then the ρ_α, $\alpha \in P_n$, are exactly the unipotent characters of G.

THEOREM 2 ([17, Corollary]). *Let $\alpha \in P_n$. Let n' = the number of squares in the Young diagram of α which have an odd hook length minus the number of squares which have an even hook length. Then we have $m_{\mathbf{Q}}(\rho_\alpha) = 1$ if $[n'/2]$ is even. If $[n'/2]$ is odd, then we have $m_{\mathbf{R}}(\rho_\alpha) = m_{\mathbf{Q}_p}(\rho_\alpha) = 2$ and $m_{\mathbf{Q}_l}(\rho_\alpha) = 1$ for any prime number $l \neq p$.*

COROLLARY. *Assume that $p = 2$. Let μ be a partition of n, and let θ_μ be the irreducible character of U_1 defined in (1.4.3). Put $\alpha = \tilde{\mu}$, and let n' be the corresponding number defined in Theorem 2. Then we have $m_{\mathbf{Q}}(\theta_\mu) = 1$ if $[n'/2]$ is even. If $[n'/2]$ is odd, then we have $m_{\mathbf{R}}(\theta_\mu) = m_{\mathbf{Q}_2}(\theta_\mu) = 2$ and $m_{\mathbf{Q}_l}(\theta_\mu) = 1$ for any prime number $l \neq 2$.*

PROOF. By Theorem 1, we have $(\gamma_\mu, \rho_\alpha)_G = 1$. Hence $(\theta_\mu, \rho_\alpha \mid U_1)_{U_1} = 1$. We note that $\mathbf{Q}(\theta_\mu) = \mathbf{Q}(\rho_\alpha) = \mathbf{Q}$. Let B (resp. C) be the simple component of the group algebra $\mathbf{Q}[U_1]$ (resp. $\mathbf{Q}[G]$) associated with θ_μ (resp. ρ_α). Let $[B]$ (resp. $[C]$) be the class of B (resp. of C) in the Brauer group $\mathrm{Br}(\mathbf{Q})$ of $\mathbf{Q}$. Then, by a result of E. Witt (see, e.g., T. Yamada [20, Proposition 3.8]), we have $[B] = [C]$. Thus the assertion follows from Theorem 2.

THEOREM 3. *Assume that $p = 2$. Let χ be any irreducible character of $G = U(n, q^2)$, and suppose that $\chi = ({g_1}^{\nu_1} \cdots {g_N}^{\nu_N})$. Put $\mu = (d(g_1)\cdot\tilde{\nu}_1)\cdot \cdots \cdot (d(g_N)\cdot\tilde{\nu}_N)$. Then the local Schur indices of θ_μ can be determined by the corollary to Theorem 2. We have:*

(i) *If $m_{\mathbf{Q}}(\theta_\mu) = 1$, then $m_{\mathbf{Q}}(\chi) = 1$.*

(ii) *Suppose that $m_{\mathbf{Q}}(\theta_\mu) = 2$. Then we have $m_{\mathbf{Q}_l}(\chi) = 1$ for any prime number $l \neq 2$, we have $m_{\mathbf{R}}(\chi) = 1$ or 2 according as $\mathbf{R}(\chi) = \mathbf{C}$ or $\mathbf{R}$, respectively, and we have $m_{\mathbf{Q}_2}(\chi) = 1$ or 2 according as $[\mathbf{Q}_2(\chi) : \mathbf{Q}_2]$ is even or odd, respectively.*

PROOF. By Theorem 1, we have $(\gamma_\mu, \chi)_G = 1$, and $\gamma_\mu = (\theta_\mu)^G$. Suppose that $m_{\mathbf{Q}}(\theta_\mu) = 1$. Then θ_μ is realizable in $\mathbf{Q}$. Hence γ_μ is realizable in $\mathbf{Q}$. Therefore, by a property of the Schur index (see, e.g., Feit [3, (11.4), p. 62]), we have $m_{\mathbf{Q}}(\chi) = 1$.

Suppose that $m_{\mathbf{Q}}(\theta_\mu) = 2$. Then $m_{\mathbf{R}}(\theta_\mu) = m_{\mathbf{Q}_2}(\theta_\mu) = 2$ and $m_{\mathbf{Q}_l}(\theta_\mu) = 1$ for any prime number $l \neq 2$. Let l be any prime number $\neq 2$. Then θ_μ is realizable in $\mathbf{Q}_l$, so γ_μ is realizable in $\mathbf{Q}_l$. Hence we have $m_{\mathbf{Q}_l}(\chi) = 1$. Let D (resp. E) be the simple component of $\mathbf{R}(\chi)[G]$ (resp. of $\mathbf{R}(\chi)[U_1]$) associated with χ (resp. θ_μ). Then, as $(\theta_\mu, \chi \mid U_1)_{U_1} = 1$ and $\mathbf{R}(\chi)(\chi) = \mathbf{R}(\chi)(\theta_\mu) = \mathbf{R}(\chi)$, by the result of Witt, we have $[D] = [E]$ in $\mathrm{Br}(\mathbf{R}(\chi))$. If

$\mathbf{R}(\chi) = \mathbf{C}$, then $[D] = 0$ and $m_{\mathbf{R}}(\chi) = 1$. If $\mathbf{R}(\chi) = \mathbf{R}$, then E, hence D is similar to the quaternion algebra over $\mathbf{R}$, and $m_{\mathbf{R}}(\chi) = 2$.

Let M (resp. N) be the simple component of $\mathbf{Q}_2(\chi)[G]$ (resp. of $\mathbf{Q}_2(\chi)[U_1]$) associated with χ (resp. with θ_μ). Then, by the result of Witt, we have $[M] = [N]$ in $\mathrm{Br}(\mathbf{Q}_2(\chi))$. Let N_0 be the simple component of $\mathbf{Q}_2[U_1]$ associated with θ_μ. Then the Hasse invariant of N_0 is congruent modulo 1 to 1/2. As $N \simeq \mathbf{Q}_2(\chi) \bigotimes_{\mathbf{Q}_2} N_0$, the invariant of N is congruent modulo 1 to $[\mathbf{Q}_2(\chi) : \mathbf{Q}_2] \times 1/2$. Therefore $m_{\mathbf{Q}_2(\chi)}(\theta_\mu) = 1$ or 2 according as $[\mathbf{Q}_2(\chi) : \mathbf{Q}_2]$ is even or odd, respectively.

This completes the proof of Theorem 3.

REMARK. Let ι be the generator of the Galois group $G(\mathbf{C}/\mathbf{R})$ of $\mathbf{C}$ over $\mathbf{R}$. For an s-simplex $g = \{k, k(-q), k(-q)^2, \cdots, k(-q)^{s-1}\}$, let $-g = \{-k, -k(-q), -k(-q)^2, \cdots, -k(-q)^{s-1}\}$, an s-simplex. Then it is easy to see from [2] (cf. Appendix) that if $\chi = (\cdots g^{\nu(g)} \cdots)$ is an irreducible character of G, then $\chi^\iota = (\cdots (-g)^{\nu(g)} \cdots)$. Thus χ is real if and only if $\nu(g) = \nu(-g)$ for all $g \in S$.

Let ω be a primitive element of the subfield of $\bar{\mathbf{F}}_q$ with $q^{(2n)!}$ elements, and let σ be the generator of $G(\mathbf{Q}_p(\omega)/\mathbf{Q}_p)$ given by $\omega^\sigma = \omega^p$. We note that all the irreducible characters of $G = U(n, q^2)$ take their values in the field $\mathbf{Q}(\omega)$ (see Appendix). For an s-simplex g as above, we let $pg = \{pk, pk(-q), pk(-q)^2, \cdots, pk(-q)^{s-1}\}$. Then we see from [2] (cf. Appendix) that, for χ as above, $\chi^\sigma = (\cdots (pg)^{\nu(g)} \cdots)$. Let m be the least positive integer such that $\chi^{\sigma^m} = \chi$. Then $m = [\mathbf{Q}_2(\chi) : \mathbf{Q}_2]$.

3. The case $p \neq 2$.

Let $\mu = [d_1, d_2, \cdots, d_s]$ be a partition of n with $d_i > 0$, $1 \leqq i \leqq s$. We first assume that $\mu \neq [1^n]$. Let

$$w = \begin{bmatrix} w_1 & & & & \\ & w_2 & & 0 & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & w_s \end{bmatrix}, \qquad w_i = \overbrace{\begin{bmatrix} & & & & 1 \\ & 0 & & 1 & \\ & & \cdot & & \\ & & \cdot & & \\ & \cdot & & 0 & \\ 1 & & & & \end{bmatrix}}^{d_i}, \qquad 1 \leqq i \leqq s,$$

and let F be the endomorphism of $\mathbf{G} = GL_n(\bar{\mathbf{F}}_q)$ given by $F([g_{ij}]) = w^t[g_{ij}{}^q]^{-1}w^{-1}$. Then $\mathbf{G}^F$ is isomorphic to $U(n, q^2)$ [in fact, if $w = x^{-1}F(x)$, $x \in \mathbf{G}$, then $\operatorname{ad} x$ induces an isomorphism of $\mathbf{G}^F$ with $\mathbf{G}^{F'}$, where F' is the endomorphism of $\mathbf{G}$ given by $F'([g_{ij}]) = {}^t[g_{ij}{}^q]^{-1}$ (note that if we choose the "standard basis" then $\mathbf{G}^{F'} = U(n, q^2)$ (cf. [2, p. 4]))]. F acts on $\mathbf{L} = \operatorname{Lie}\mathbf{G}$ by $F([x_{ij}]) = -w^t[x_{ij}{}^q]w^{-1}$. Let ξ be a non-zero element of $\bar{\mathbf{F}}_q$ such that $\xi^q + \xi = 0$, and let

$$A = \begin{bmatrix} A_1 & & & & \\ & A_2 & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & & \cdot \\ & & & & A_s \end{bmatrix}, \qquad A_i = \overbrace{\begin{bmatrix} 0 & & & & & \\ \xi & 0 & & & 0 & \\ & \xi & 0 & & & \\ & & \xi & \cdot & & \\ & & & & \cdot & \\ & 0 & & & \cdot & \\ & & & & \xi & 0 \end{bmatrix}}^{d_i}, \qquad 1 \leqq i \leqq s .$$

Then $F(A) = A$. Let λ be the morphism of $\mathbf{F}_q{}^{\times}$ into $\mathbf{G}$ given by

$$\lambda(x) = \operatorname{diag}(x^{k_1^1}, \cdots, x^{k_{d_1}^1}, x^{k_1^2}, \cdots, x^{k_{d_2}^2}, \cdots, x^{k_1^s}, \cdots, x^{k_{d_s}^s}) \quad (x \in \mathbf{F}_q{}^{\times}),$$

where, for $1 \leqq i \leqq s$, $1 \leqq j \leqq d_i$, $k_j^i = 1 - d_i + 2(j-1)$. Then λ is the morphism associated with A (see 1.4). Let $\mathbf{L}(i)_A$, $\mathbf{u}_i$, $\mathbf{U}_i$ etc., be as in 1.4.

Assume that $p \neq 2$. First, we assume that all the d_i are odd. Let ρ be a fixed primitive element of $\mathbf{F}_{q^2}$. For $1 \leqq i \leqq s$, if $d_i = 2r+1$, then we put

$$m_i = \operatorname{diag}(\rho^{r(q^2-1)/(p-1)}, \rho^{(r-1)(q^2-1)/(p-1)}, \cdots, \rho^{(q^2-1)/(p-1)},$$
$$1, \rho^{-q(q^2-1)/(p-1)}, \rho^{-2q(q^2-1)/(p-1)}, \cdots, \rho^{-rq(q^2-1)/(p-1)}).$$

Let $m = \operatorname{diag}(m_1, m_2, \cdots, m_s)$. Then we have $F(m) = m$ and m normalizes each $\mathbf{L}(i)_A$. And we have

$$m^{-1}A^*m = \nu^{-1}A^*, \quad \nu = \rho^{(q^2-1)/(p-1)}. \tag{3.1}$$

Set $M = U_2\langle m\rangle$. We note that, as all the d_i are odd, $U_1 = U_2$, so $\theta_A = \xi_A$. We have, for $u \in U_2$,

$$\begin{aligned}
\theta_A^m(u) &= \xi_A(mum^{-1}) \\
&= \chi_0(\kappa([A^*, f(mum^{-1})])) && \text{(by (1.4.1))} \\
&= \chi_0(\kappa([A^*, mf(u)m^{-1}])) \\
&= \chi_0(\kappa([m^{-1}A^*m, f(u)])) \\
&= \chi_0(\kappa([\nu^{-1}A^*, f(u)])) && \text{(by (3.1))} \\
&= \chi_0(\nu^{-1}\kappa([A^*, f(u)])) \\
&= \chi_0(\kappa([A^*, f(u)]))^{\alpha} \\
&= \theta_A{}^{\alpha}(u),
\end{aligned}$$

where α is a certain generator of $G(\mathbf{Q}(\zeta_p)/\mathbf{Q})$ (ζ_p is a primitive p-th root of unity in $\mathbf{C}$). We note that, as $\mu \neq [1^n]$, by (1.4.4), we have $\mathbf{Q}(\theta_A) = \mathbf{Q}(\zeta_p) \neq \mathbf{Q}$. So it follows that $\theta_A{}^M$ is an irreducible character of M with values in $\mathbf{Q}$ (cf. $m^{p-1} = 1$). Moreover we have $m_{\mathbf{Q}}(\theta_A{}^M) = 1$. In fact, we see easily that the simple component of $\mathbf{Q}[M]$ associated with $\theta_A{}^M$ is isomorphic over $\mathbf{Q}$ to the cyclic algebra $(1, \mathbf{Q}(\zeta_p), \alpha)$ over $\mathbf{Q}$ ([20, Propositions 1.3, 3.4 and

3.5]), which clearly splits in $\mathbf{Q}$. Therefore $\theta_A{}^M$ is realizable in $\mathbf{Q}$, hence γ_A is realizable in $\mathbf{Q}$. If $\mu = [1^n]$, then γ_μ is the character of the regular representation of $G = \mathbf{G}^F$, so γ_μ is realizable in $\mathbf{Q}$.

Next, we assume that all the d_i are even. For $1 \leqq i \leqq s$, if $d_i = 2r$, then we put

$$\begin{aligned} m_i{}' = \operatorname{diag}(&\rho^{(q-1)/(p-1)+(r-1)(q^2-1)/(p-1)}, \\ &\rho^{(q-1)/(p-1)+(r-2)(q^2-1)/(p-1)}, \cdots, \rho^{(q-1)/(p-1)}, \rho^{-q(q-1)/(p-1)}, \\ &\rho^{-q(q-1)/(p-1)+(q^2-1)/(p-1)}, \cdots, \rho^{-q(q-1)/(p-1)+(r-1)(q^2-1)/(p-1)}). \end{aligned}$$

And let $m' = \operatorname{diag}(m_1{}', m_2{}', \cdots, m_s{}')$. Then $F(m') = m'$ and $m'^{-1}A^*m' = \nu^{-1}A^*$. We see that $c = m'^{p-1}$ is a generator of the centre C of G. We note that, as all the d_i are even, $U_1 = U_2$ and $\theta_A = \xi_A$. And we have $\theta_A{}^{m'} = \theta_A{}^\alpha$. Set $M' = U_2\langle m'\rangle$. Let ζ be a fixed primitive $(q+1)$-th root of unity in $\mathbf{C}$, and, for $1 \leqq j \leqq q+1$, let ϕ_j be the linear character of $C = \langle c\rangle$ defined by $\phi_j(c) = \zeta^j$, and set $\mu_j = (\theta_A\phi_j)^{M'}$. Then we see that the μ_j are mutually different irreducible characters of M' and $\theta_A{}^{M'} = \mu_1 + \mu_2 + \cdots + \mu_{q+1}$ (cf. [16, §3]). For $1 \leqq j \leqq q+1$, we have $\mathbf{Q}(\mu_j) = \mathbf{Q}(\zeta^j)$. For $1 \leqq j \leqq q+1$, let B_j be the simple component of $\mathbf{Q}(\zeta^j)[M']$ associated with μ_j. Then we see that (cf. [16, §3]), for $1 \leqq j \leqq q+1$, B_j is isomorphic over $\mathbf{Q}(\zeta^j)$ to the cyclic algebra $(\zeta^j, \mathbf{Q}(\zeta^j)(\zeta_p), \alpha_j)$ over $\mathbf{Q}(\zeta^j)$, where α_j is the extension of α to $\mathbf{Q}(\zeta^j)(\zeta_p)$ such that $\alpha_j(\zeta^j) = \zeta^j$. By Lemma 1 of [16], we see that, for $1 \leqq j \leqq q+1$, $j \neq (q+1)/2$, B_j splits in $\mathbf{Q}(\zeta^j)$, and, for $j = (q+1)/2$, B_j has the Hasse invariants $1/2 \bmod 1$ at ∞, p, and $0 \bmod 1$ at any other rational prime.

Thus we get

THEOREM 4. *Assume that $p \neq 2$. Let χ be any irreducible character of $G = U(n, q^2)$, and suppose that $\chi = (g_1{}^{\nu_1} \cdots g_N{}^{\nu_N})$. Let $\mu = (d(g_1)\cdot\tilde{\nu}_1)\cdot \cdots \cdot(d(g_N)\cdot\tilde{\nu}_N)$. Then, if all the parts of μ are odd, we have $m_{\mathbf{Q}}(\chi) = 1$. Assume that all the parts of μ are even. Let c be a generator of the centre of G. Then, if $\chi(c) \neq -\chi(1)$, we have $m_{\mathbf{Q}}(\chi) = 1$. If $\chi(c) = -\chi(1)$, then we have $m_{\mathbf{Q}_l}(\chi) = 1$ for any prime number $l \neq p$, we have $m_{\mathbf{R}}(\chi) = 2$ or 1 according as χ is real or not, respectively, and we have $m_{\mathbf{Q}_p} = 2$ or 1 according as $[\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$ is odd or even, respectively.*

PROOF. By Theorem 1, we have $(\gamma_\mu, \chi)_G = 1$. If all the parts of μ are odd, then γ_μ is realizable in $\mathbf{Q}$, so, by the property of the Schur index, we have $m_{\mathbf{Q}}(\chi) = 1$. Assume that all the parts of μ are even. Then we see from Schur's lemma that $(\chi \mid M', \mu_j)_{M'} = 1$ if and only if $\chi(c) = \zeta^j\chi(1)$ (we must change ζ by another primitive $(q+1)$-th root of unity if necessary), and, if this is the case, by the result of Witt, we have $m_E(\chi) = m_{E(\chi)}(\chi) = m_{E(\chi)}(\mu_j)$ for all fields E of characteristic zero. Therefore, if $j \neq (q+1)/2$, then we have $m_{\mathbf{Q}}(\chi) = 1$. Assume that $j = (q+1)/2$, that is, $\chi(c) = -\chi(1)$. Then we have $m_{\mathbf{R}(\chi)}(\mu_j) = 1$ or 2 according as $\mathbf{R}(\chi) = \mathbf{C}$ or $\mathbf{R}$, respectively, $m_{\mathbf{Q}_p(\chi)}(\mu_j) = 1$ or 2 according as $[\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$ is even or odd, respectively, and $m_{\mathbf{Q}_l(\chi)}(\mu_j) = 1$ for any prime number $l \neq p$.

REMARK. $\chi(c)$ is calculated in Proposition 1 below.

Let μ be a partition of n. We assume that some parts of μ are odd and some parts of μ are even. Then $\mathbf{Q}(\theta_\mu) = \mathbf{Q}(\zeta_p)$ (see (1.4.4)) (we are assuming that $p \neq 2$). Even in this case there is an element m'' of order $(q+1)(p-1)$ such that $\theta_\mu{}^{m''} = \theta_\mu{}^{\alpha}$ ($\langle\alpha\rangle = G(\mathbf{Q}(\zeta_p)/\mathbf{Q})$). Set $M'' = U_1\langle m''\rangle$. Then we have an irreducible decomposition $\theta_\mu{}^{M''} = \psi_1 + \cdots + \psi_{q+1}$. It seems likely that it is not so hard to determine the local Schur indices of each ψ_j. But $c'' = m''^{p-1}$ is not central, so it seems to be very difficult to determine the number j such that $(\chi \mid M'', \psi_j)_{M''} = 1$ for a given irreducible character χ of $G = U(n, q^2)$. So to treat this case is an open problem.

We say that a partition μ of n is involutive if the parts of μ can be arranged so that $\mu = [n_1, \cdots, n_s, n_{s+1}, n_s, \cdots, n_1]$. Then the following result is proved in [18]:

THEOREM 5 ([18, Theorem 4]). *Let $\chi = (g_1{}^{\nu_1} \cdots g_N{}^{\nu_N})$ be an irreducible character of $G = U(n, q^2)$. Let $\mu = (d(g_1) \cdot \tilde{\nu}_1) \cdot \cdots \cdot (d(g_N) \cdot \tilde{\nu}_N)$. Assume that μ is involutive, and suppose $\mu = [n_1, \cdots, n_s, n_{s+1}, n_s, \cdots, n_1]$. Then:*

(i) *If $p = 2$, or n is odd, or $n_{s+1} = 0$, then we have $m_{\mathbf{Q}}(\chi) = 1$.*

(ii) *Assume that $p \neq 2$, n is even, and $n_{s+1} \neq 0$. Let c be a generator of the centre of G. Then, if $\chi(c) \neq -\chi(1)$, we have $m_{\mathbf{Q}}(\chi) = 1$. Assume that $\chi(c) = -\chi(1)$. Then we have $m_{\mathbf{Q}_l}(\chi) = 1$ for any prime number $l \neq p$, we have $m_{\mathbf{R}}(\chi) = 2$ or 1 according as χ is real or not respectively, and $m_{\mathbf{Q}_p}(\chi) = 2$ or 1 according as $[\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$ is odd or even respectively.*

4. Proofs of Theorems C, D.

We first prove Theorem C. In the case where $p = 2$, the theorem follows from Theorem 3. So we assume that $p \neq 2$. Let us review some results of [16, §3]. Let U be a Sylow p-subgroup of $G = U(n, q^2)$. Let ζ_p be a fixed primitive p-th root of unity in C, and let α be a generator of $G(\mathbf{Q}(\zeta_p)/\mathbf{Q})$. Let ζ be a fixed primitive $(q+1)$-th root of unity in $\mathbf{C}$. If n is odd, then, for any linear character ϕ of U, ϕ^G is realizable in $\mathbf{Q}$. Suppose that n is even. Then there is an element m in the normalizer $N_G(U)$ of U in G such that $m^{(p-1)(q+1)} = 1$, $c = m^{p-1}$ is a generator of the centre C of G, and $\phi^m = \phi^\alpha$ for any linear character ϕ of U. Set $M = U\langle m\rangle$. Let ϕ be any non-principal linear character of U. For $0 \leqq j \leqq q$, let ϕ_j be the extension of ϕ to UC given by $\phi_j(c) = \zeta^j$, and let $\mu_j = \phi_j{}^M$. Then $\mu_0, \cdots, \mu_q$ are mutually different irreducible characters of M and $\phi^M = \mu_0 + \cdots + \mu_q$. For $0 \leqq j \leqq q$, put $k_j = \mathbf{Q}(\mu_j) = \mathbf{Q}(\zeta^j)$, and let B_j be the simple component of $k_j[M]$ associated with μ_j. Then, if $j \neq (q+1)/2$, B_j splits in k_j, and if $j = (q+1)/2$, B_j has non-zero Hasse invariants only at two places ∞, p of $k_j = \mathbf{Q}$: we have $m_{\mathbf{R}}(\mu_{(q+1)/2}) = m_{\mathbf{Q}_p}(\mu_{(q+1)/2}) = 2$ and $m_{\mathbf{Q}_l}(\mu_{(q+1)/2}) = 1$ for any prime number $l \neq p$.

Let E be a finite extension of $\mathbf{Q}_p$ such that $[E : \mathbf{Q}_p]$ is even. We show that ϕ^M is realizable in E. In fact, we note that ϕ^M is $\mathbf{Q}$-valued and that $m_E(\mu_{(q+1)/2}) = 1$, so $m_E(\mu_j) = 1$ for $0 \leqq j \leqq q$. Let $X_1, \cdots, X_s$ be all the $G(E(\zeta)/E)$-orbits on $\mu_0, \cdots, \mu_q$. Then, by a theorem of Schur (see, e.g., Feit [3, (11.4)]), for $1 \leqq i \leqq s$, $\psi_i = \sum_{\mu \in X_i} \mu$ is an irreducible E-character of M (that is, ψ_i is the character of s simple $E[M]$-module). Therefore $\phi^M = \psi_1 + \cdots + \psi_s$ is the character of some actual $E[M]$-module, that is, ϕ^M is realizable in E. Therefore $\phi^G = (\phi^M)^G$ is realizable in E.

To prove Theorem C (in the case when $p \neq 2$) we repeat the argument in the proof of the main theorem of [15]. Let χ be any irreducible character of G such that $[\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$ is even. Put $E = \mathbf{Q}_p(\chi)$. Then, by Theorem C of [14] and Ennola's conjecture, there is a unipotent element u of G such that $\chi(u)$ is a power of q up to ± 1 (also see Lemma 9 below). Let $\mathbf{G} = GL_n(\bar{\mathbf{F}}_q)$ and let F be the endomorphism of $\mathbf{G}$ given by $F([g_{ij}]) = {}^t[g_{ij}{}^q]^{-1}$. We assume that $G = \mathbf{G}^F$. Then u can be chosen so that there are a standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ and an F-stable Levi subgroup $\mathbf{H}$ of $\mathbf{P}$ such that u is a regular unipotent element of $\mathbf{H}$ (see [15, p. 362]). There are positive integers $n_1, \cdots, n_s$ such that $H = \mathbf{H}^F$ is isomorphic to the product $U(n_1, q^2) \times \cdots \times U(n_s, q^2)$ (see [15, p. 361]). Let U_H be the Sylow p-subgroup of H containing u. Let Λ be the set of all linear characters of U_H and let R be the set of all non-linear irreducible characters of U_H. As we have seen above, for any $\lambda \in \Lambda$, λ^G is realizable in E, and, by Theorem A$'$ of Lehrer [9], we have $\rho(u) = 0$ for all $\rho \in R$. Set

$$\chi \mid U_H = \sum_{\lambda \in \Lambda} a_\lambda \lambda + \sum_{\rho \in R} b_\rho \rho\,,$$

where the a_λ and the b_ρ are some non-negative integers. Then we have

$$\pm(\text{a power of } q) = \chi(u) = \sum_{\lambda \in \Lambda} a_\lambda \lambda(u) + 0\,.$$

Let $\lambda \in \Lambda$. Then, as λ^G is realizable in E, by the property of the Schur index, $m_E(\chi)$ divides $(\lambda^G, \chi)_G = (\lambda, \chi \mid U_H)_{U_H} = a_\lambda$. Thus we have an expression

$$\pm(\text{a power of } q)/m_E(\chi) = \sum_{\lambda \in \Lambda} (a_\lambda / m_E(\chi))\lambda(u)\,,$$

where the right hand side is an algebraic integer and the left hand side is a rational number. Hence $m_E(\chi)$ divides a power of q, odd. But, by Theorem A, $m_E(\chi)$ divides 2. Hence $m_E(\chi) = 1$.

This completes the proof of Theorem C.

We next prove Theorem D.

LEMMA 7. *Assume that $p = 2$. Let μ be a partition of n, and let θ_μ be as in* (1.4.3). *Let k be a finite extension of $\mathbf{Q}$ such that k contains a $(q+1)$-th root of unity $\neq 1$. Then $m_k(\theta_\mu) = 1$.*

PROOF. We know that $m_{\mathbf{R}}(\theta_\mu) = m_{\mathbf{Q}_2}(\theta_\mu) \leqq 2$ and $m_{\mathbf{Q}_l}(\theta_\mu) = 1$ for any prime number $l \neq 2$ (Corollary to Theorem 2). Let ζ be a $(q+1)$-th root of unity $\neq 1$ contained in k. Then, as $q+1$ is odd and > 2, k is not real, so we must have $m_{k_w}(\theta_\mu) = 1$ for any infinite place w of k (k_w is the completion of k at w). The Galois group $G(\mathbf{Q}_2(\zeta)/\mathbf{Q}_2)$ is generated by the automorphism $\sigma : \zeta \to \zeta^2$. Suppose that $q = 2^d$. Then $\zeta^{\sigma^d} = \zeta^q = \zeta^{-1}$, so σ^d is the involution $\zeta \to \zeta^{-1}$, an element of $G(\mathbf{Q}_2(\zeta)/\mathbf{Q}_2)$ of order 2. Thus $|G(\mathbf{Q}_2(\zeta)/\mathbf{Q}_2)| =$

$[\mathbf{Q}_2(\zeta) : \mathbf{Q}_2]$ must be even. Therefore, if w is any finite place of k lying above 2, then $[k_w : \mathbf{Q}_2]$ is even, so $m_{k_w}(\theta_\mu) = 1$. Therefore $m_k(\theta_\mu) = 1$.

Similarly we get

LEMMA 8. *Assume that $p \neq 2$. Let k be a finite extension of $\mathbf{Q}$ such that k contains a $(q+1)$-th root of unity $\neq \pm 1$. Then, for any linear character ϕ of U, ϕ^G is realizable in k.*

Theorem D now follows from Lemmas 7, 8. In fact, by using these lemmas, it suffices to repeat the argument in the proof of Theorem C.

REMARK. By the final remark in §2, for any irreducible character χ of G, we can determine whether $\mathbf{R}(\chi) = \mathbf{R}$ or $\mathbf{C}$ and whether $[\mathbf{Q}_p(\chi) : \mathbf{Q}_p]$ is odd or even by using Ennola's parametrization of χ. Thus in the case when $p \neq 2$ we can say that (in principle) we have sufficient conditions subject for that $m_{\mathbf{R}}(\chi) = 1$ and $m_{\mathbf{Q}_p}(\chi) = 1$.

The following fact may be of some use:

THEOREM 6. *Assume that $p \neq 2$. Let c be a generator of the centre of $G = U(n, q^2)$. Let χ be any irreducible character of G, and suppose that $\chi = ({g_1}^{\nu_1} \cdots {g_N}^{\nu_N})$. Let $\mu = (d(g_1)\cdot\nu_1)\cdot \cdots \cdot (d(g_N)\cdot\nu_N)$. Suppose either that all parts of μ are odd or that all parts of μ are even and $\chi(c) \neq -\chi(1)$. Then we have $m_{\mathbf{Q}}(\chi) = 1$.*

PROOF. Suppose that $\mu = [n_1, \cdots, n_s]$ with $n_i > 0$, $1 \leqq i \leqq s$. Let the notation be the same as in the proof of Theorem C in the case when $p \neq 2$. Then, by Lemma 9 below, we have $\chi(u) = \pm(p\text{-part of } \chi(1))$ and

$$(*) \qquad \chi(u) = \sum_{\lambda \in \Lambda} a_\lambda \lambda(u).$$

We have $U_H = V_1 \times \cdots \times V_s$, where, for $1 \leqq i \leqq s$, V_i is a Sylow p-subgroup of $U(n_i, q^2)$. Let $\lambda \in \Lambda$. Then we have $\lambda = \lambda_1 \times \cdots \times \lambda_s$, where, for $1 \leqq i \leqq s$, λ_i is a linear character of V_i, and we have

$$\lambda^H = {\lambda_1}^{U(n_1,q^2)} \times \cdots \times {\lambda_s}^{U(n_s,q^2)}.$$

Suppose that all the n_i are odd. Then each ${\lambda_i}^{U(n_i,q^2)}$ is realizable in $\mathbf{Q}$, hence $\lambda^G = (\lambda^H)^G$ is realizable in $\mathbf{Q}$. Therefore $m_{\mathbf{Q}}(\chi)$ divides all a_λ, so that $m_{\mathbf{Q}}(\chi) = 1$ (cf. $p \neq 2$).

We next suppose that all the n_i are even. Let α be a generator of $G(\mathbf{Q}(\zeta_p)/\mathbf{Q})$, where ζ_p is a primitive p-th root of unity. Let i be any integer such that $1 \leqq i \leqq s$. Then there is a semisimple element m_i in $N_{U(n_i,q^2)}(V_i)$, of order $(p-1)(q+1)$, such that ${\lambda_i}^{m_i} = {\lambda_i}^\alpha$ for any linear character λ_i of V_i and $c_i = {m_i}^{p-1}$ is a generator of the centre of $U(n_i, q^2)$. Let $m = \operatorname{diag}(m_1, \cdots, m_s)$. Then we may assume that $c = m^{p-1}$ is a generator of the centre of $G = U(n, q^2)$ by changing some of the m_i by their powers if necessary. Then we have $\lambda^m = \lambda^\alpha$ for all $\lambda \in \Lambda$ (α might be changed by its power if necessary). Let $\phi_0, \cdots, \phi_q$ be the characters of $C = \langle c \rangle$ defined by $\phi_j(c) = \zeta^j$, $0 \leqq j \leqq q$, where ζ is a previously fixed primitive $(q+1)$-th root of unity. Let ρ be any irreducible character of U_H. Let $N = U_H C$. Then we have $\rho^N = \rho\phi_0 + \cdots + \rho\phi_q$. If ρ is not linear, then, by Lehrer's result, we have $\rho^N(u) = (q+1)\rho(u) = 0$. Suppose that $\chi(c) = \zeta^j \chi(1)$ (cf. Schur's lemma). Then, by

Schur's lemma, we must have $a_\lambda = ((\lambda\phi_j)^G, \chi)_G = (\lambda\phi_j, \chi \mid N)_N$ for $\rho = \lambda \in \Lambda$. Thus we have (cf. (*))

$$\chi(u) = (\chi \mid N)(u) = \sum_{\lambda\in\Lambda} a_\lambda \cdot (\lambda\phi_j)(u)\,.$$

Suppose that $j \neq (q+1)/2$, i.e. $\chi(c) \neq -\chi(1)$. Set $M = U_H\langle m\rangle$. Let $\lambda \in \Lambda$. If $\lambda = 1$, then $1\phi_j$ is realizable in $k_j = \mathbf{Q}(\zeta^j)$, so $(1\phi_j)^G$ is realizable in k_j, and $m_{\mathbf{Q}}(\chi) = m_{k_j}(\chi)$ divides a_1. If $\lambda \neq 1$, then $(\lambda\phi_j)^M$ is an irreducible character of M which is realizable in k_j (cf. [16, §3]). So, if $\lambda \neq 1$, $(\lambda\phi_j)^G$ is realizable in k_j, and $m_{\mathbf{Q}}(\chi) = m_{k_j}(\chi)$ divides a_λ. Put $m' = m_{\mathbf{Q}}(\chi)$. Then, in the expression

$$\pm(\text{a power of } q)/m' = \sum_{\lambda\in\Lambda}(a_\lambda/m')(\lambda\phi_j)(u)\,,$$

the right hand side is an algebraic integer and the left hand side is a rational number. Hence m' divides a power of q, odd, hence, by Theorem A, $m' = 1$.

This completes the proof of Theorem 6.

5. Some other sufficient conditions.

The results of this section is logically contained in some of the results of the previous sections (see §4, Remark below Lemma 8), but they are more practical, so will be useful. We shall omit the detailed calculations (cf. Appendix).

Let ε be an element of $\mathbf{F}_{q^2}$ of order $q+1$, and let ζ be a certain primitive $(q+1)$-th root of unity in $\mathbf{C}$. Let n be a positive integer, and let M be a positive integer. Let $l_1, \cdots, l_M$ be integers such that $l_i \not\equiv l_j \pmod{q+1}$ for $1 \leqq i \neq j \leqq M$, and let $\lambda_1, \cdots, \lambda_M$ be partitions of some natural numbers such that $|\lambda_1| + \cdots + |\lambda_M| = n$. Let c be the class of $G = U(n, q^2)$ parametrized by $((t-\varepsilon^{l_1})^{\lambda_1}\cdots(t-\varepsilon^{l_M})^{\lambda_M})$ (t is a variable over $\bar{\mathbf{F}}_q$) (cf. Appendix).

Let χ be any irreducible character of G, and suppose that $\chi = ({g_1}^{\nu_1}\cdots {g_N}^{\nu_N})$. For $1 \leqq i \leqq N$, put $d_i = d(g_i)$ and $v_i = |\nu_i|$. Set

$$\begin{aligned}(5.1)\qquad A(\chi, c) = \Big\{&(v_{11}, \cdots, v_{1M}; v_{21}, \cdots, v_{2M}; \cdots; v_{N1}, \cdots, v_{NM}) \;\Big|\\ &v_{ij} \in \mathbf{Z},\ v_{ij} \geqq 0\ (1 \leqq i \leqq N, 1 \leqq j \leqq M),\\ &\sum_{j=1}^{M} v_{ij} = d_i v_i\ (1 \leqq i \leqq N),\ d_i \mid v_{ij}\ (1 \leqq i \leqq N,\\ &1 \leqq j \leqq M),\ \sum_{i=1}^{N} v_{ij} = |\lambda_j|\ (1 \leqq j \leqq M)\Big\}\,,\end{aligned}$$

and, for $(v_{ij}) \in A(\chi, c)$, put

$$(5.2)\qquad e_i((v_{ij})) = k_i \sum_{j=1}^{M} l_j v_{ij}/d_i \quad (1 \leqq i \leqq N)\,,$$

where, for $1 \leqq i \leqq N$, k_i is a root of the simplex g_i. For $1 \leqq i \leqq N$, put $\eta_i = \eta(({g_i}^{\nu_i}))$ (cf. Appendix, (A.5.3)). Let $(v_{ij}) \in A(\chi, c)$. Fix j, $1 \leqq j \leqq M$, let $(\xi^j) = (\xi_{1j}, \cdots, \xi_{Nj}) \in$

$P_{(v_{1j}/d_1,\cdots,v_{Nj}/d_N)}$, and put $\eta(\xi^j) = \eta((g_1{}^{\nu_{1j}}\cdots g_N{}^{\nu_{Nj}}))\eta((g_1{}^{\xi_{1j}}))\cdots\eta((g_N{}^{\xi_{Nj}}))$. Then, by a rather long calculation, we get

$$\begin{aligned}\chi(c) =&\eta(\chi)\eta_1\cdots\eta_N \sum_{(v_{ij})\in A(\chi,c)} \zeta^{\sum_{i=1}^N e_i((v_{ij}))} \\ &\times \sum_{(\xi_{ij})\in\prod_{i,j} P_{v_{ij}/d_i}} c^{\nu_1}_{\xi_{11}\cdots\xi_{1M}}\cdots c^{\nu_N}_{\xi_{N1}\cdots\xi_{NM}} \\ &\times \prod_{j=1}^M \eta(\xi^j)(g_1{}^{\xi_{1j}}\cdots g_N{}^{\xi_{Nj}})(u_{\lambda_j}).\end{aligned} \tag{5.3}$$

PROPOSITION 1. *Let χ be any irreducible character of $G = U(n, q^2)$, and suppose that $\chi = (g_1{}^{\nu_1}\cdots g_N{}^{\nu_N})$. For $1 \leqq i \leqq N$, let k_i be a root of the simplex g_i and put $v_i = |\nu_i|$. Let l be an integer and let λ be a partition of n. Let c be the class $((t-\varepsilon^l)^\lambda)$ of G and let u_λ be a unipotent element of type λ. Then we have*

$$\chi(c) = \zeta^{l\sum_{i=1}^N k_i v_i}\chi(u_\lambda).$$

Here ζ is a certain primitive $(q+1)$-th root of unity in $\mathbf{C}$.

PROOF. We have $A(\chi, c) = \{(d_1 v_1, \cdots, d_N v_N)\}$ and the assertion follows from (5.3).

REMARK. In Proposition 1, we let $l = 1$ and $\lambda = [1^n]$. Then the element x in c is a generator of the centre of G. And we have

$$\chi(x) = \zeta^{\sum_{i=1}^N k_i v_i}\chi(1).$$

Thus, in the case when $p \neq 2$, we have $\chi(x) = -\chi(1)$ if and only if $(q+1)/2 \mid \sum_{i=1}^N k_i v_i$ and $q+1 \nmid \sum_{i=1}^N k_i v_i$.

PROPOSITION 2. *Let χ be any irreducible character of $G = U(n, q^2)$, and suppose that $\chi = (g_1{}^{\nu_1}\cdots g_N{}^{\nu_N})$. For $1 \leqq i \leqq N$, put $d_i = d(g_i)$, $v_i = |\nu_i|$, and let k_i be a root of the simplex g_i. Assume that, for $1 \leqq i \leqq N$, $\nu_i = [v'_{i1}, v'_{i2}, \cdots, v'_{is_i}]$ with $v'_{i1} \geqq v'_{i2} \geqq \cdots \geqq v'_{is_i} > 0$. Put $M = \mathrm{Max}\{s_1, \cdots, s_N\}$. For $1 \leqq j \leqq M$, put $m_j = d_1 v'_{1j} + d_2 v'_{2j} + \cdots + d_N v'_{Nj}$ (if $j > s_i$, we set $v'_{ij} = 0$). Let $l_1, \cdots, l_M$ be integers such that $l_i \not\equiv l_j \pmod{q+1}$ for $1 \leqq i \neq j \leqq M$. Let c be the class $((t-\varepsilon^{l_1})^{[m_1]}(t-\varepsilon^{l_2})^{[m_2]}\cdots(t-\varepsilon^{l_M})^{[m_M]})$. Then we have*

$$\chi(c) = \pm\zeta^{\sum_{i=1}^N k_i \sum_{j=1}^M l_j v'_{ij}}.$$

LEMMA 9. *Let χ be any irreducible character of $G = U(n, q^2)$, and suppose that $\chi = (g_1{}^{\nu_1}\cdots g_N{}^{\nu_N})$. Let $\mu = (d(g_1)\cdot\nu_1)\cdot\cdots\cdot(d(g_N)\cdot\nu_N)$, and let λ be a partition of n. Then we have $\chi(u_\lambda) = \pm(p$-part of $\chi(1))$ if $\lambda = \mu$, and $= 0$ if $\lambda > \mu$.*

This lemma can be proved by a calculation similar to that in the proof of Theorem C of [14].

PROOF OF PROPOSITION 2. Let $(v_{ij}) \in A(\chi, c)$, and let $(\xi_{ij}) \in \prod_{i,j} P_{v_{ij}/d_i}$. Then, by Lemma 9, we have $(g_1{}^{\xi_{1j}}\cdots g_N{}^{\xi_{Nj}})(u_{[m_j]}) = 0$ if $[m_j] > (d_1\cdot\xi_{1j})\cdot\cdots\cdot(d_N\cdot\xi_{Nj})$

$(1 \leqq j \leqq M)$. Let us fix j $(1 \leqq j \leqq M)$, and suppose that $[m_j] = (d_1 \cdot \xi_{1j}) \cdot \cdots \cdot (d_N \cdot \xi_{Nj})$. Then we have $\xi_{ij} = [v_{ij}/d_i]$, $1 \leqq i \leqq N$.

Next, we fix i $(1 \leqq i \leqq N)$. Suppose that $c^{\nu_i}_{\xi_{i1}\cdots\xi_{iM}} \neq 0$ in the expression (5.3). Then, by Lemma 1, we must have $\xi_{i1} + \cdots + \xi_{iM} = [v_{i1}/d_i, \cdots, v_{iM}/d_i] \leqq \nu_i$. We have:

$$\begin{aligned} m_1 &= d_1 v'_{11} + d_2 v'_{21} + \cdots + d_N v'_{N1} \\ &\geqq d_1 \cdot \frac{v_{11}}{d_1} + d_2 \cdot \frac{v_{21}}{d_2} + \cdots + d_N \cdot \frac{v_{N1}}{d_N} \\ &= v_{11} + v_{21} + \cdots + v_{N1} = m_1 . \end{aligned}$$

This forces that $v_{i1}/d_i = v'_{i1}$, $1 \leqq i \leqq N$. We note that, for $1 \leqq i \leqq N$, $v_{i1}/d_i = \mathrm{Max}\{v_{i1}/d_i, \cdots, v_{iM}/d_i\}$. So we have:

$$\begin{aligned} m_2 &= d_1 v'_{12} + d_2 v'_{22} + \cdots + d_N v'_{N2} \\ &\geqq d_1 \cdot \frac{v_{12}}{d_1} + d_2 \cdot \frac{v_{22}}{d_2} + \cdots + d_N \cdot \frac{v_{N2}}{d_N} = m_2 , \end{aligned}$$

which forces that $v'_{i2} = v_{i2}/d_i$, $1 \leqq i \leqq N$, and, for $1 \leqq i \leqq N$, $v_{i2}/d_i = \mathrm{Max}\{v_{i2}/d_i, v_{i3}/d_i, \cdots, v_{iM}/d_i\}$. By repeating similar considerations, we conclude that $[v_{i1}, \cdots, v_{iM}] = d_i \cdot \nu_i$, $1 \leqq i \leqq N$, and $\xi_{ij} = [v'_{ij}]$, $1 \leqq i \leqq N$, $1 \leqq j \leqq M$. Thus we have $\xi_{i1} + \cdots + \xi_{iM} = \nu_i$, hence, by Lemma 1, we have $c^{\nu_i}_{\xi_{i1}\cdots\xi_{iM}} = 1$, $1 \leqq i \leqq N$.

The assertion in Proposition 2 now follows from (5.3) by using Lemma 9.

By Theorem D and Propositions 1, 2, we get the following two facts:

THEOREM 7. *Let χ be any irreducible character of $G = U(n, q^2)$, and suppose that $\chi = (g_1{}^{\nu_1} \cdots g_N{}^{\nu_N})$. For $1 \leqq i \leqq N$, let k_i be a root of the simplex g_i and put $v_i = |\nu_i|$. If $p = 2$ (resp. $p \neq 2$), assume that $q + 1 \nmid \sum_{i=1}^N k_i v_i$ (resp. $(q+1)/2 \nmid \sum_{i=1}^N k_i v_i$). Then $m_{\mathbf{Q}}(\chi) = 1$.*

THEOREM 8. *Let χ be any irreducible character of G, and suppose that $\chi = (g_1{}^{\nu_1} \cdots g_N{}^{\nu_N})$. For $1 \leqq i \leqq N$, let k_i be a root of g_i and suppose that $\nu_i = [v'_{i1}, v'_{i2}, \cdots, v'_{is_i}]$ with $v'_{i1} \geqq v'_{i2} \geqq \cdots \geqq v'_{is_i} > 0$. Let $M = \mathrm{Max}\{s_1, s_2, \cdots, s_N\}$. Suppose that there are integers $l_1, \cdots, l_M$ such that $l_i \not\equiv l_j \pmod{q+1}$ for $1 \leqq i \neq j \leqq M$ and that $q + 1 \nmid \sum_{i=1}^N k_i \sum_{j=1}^M l_j v'_{ij}$ (resp. $(q+1)/2 \nmid \sum_{i=1}^N k_i \sum_{j=1}^M l_j v'_{ij}$) if $p = 2$ (resp. $p \neq 2$) (we set $v'_{ij} = 0$ if $j > s_i$). Then we have $m_{\mathbf{Q}}(\chi) = 1$.*

Appendix.

In this appendix we review V. Ennola's formulation for the character theory of $U(n, q^2)$ ([2]). For each positive integer n, we let $G_n = U(n, q^2)$. In his formulation, Ennola replaced q in Greens' character theory for $GL(n, q)$ [6] everywhere by $-q$.

A.1. Conjugacy classes of G_n. Let t be a variable over $\bar{\mathbf{F}}_q$. For a monic polynomial $f(t) = t^d + a_1 t^{d-1} + \cdots + a_d$ over $\mathbf{F}_{q^2}$ with $a_d \neq 0$, we set $\tilde{f}(t) = a_d{}^{-1}(a_d{}^q t^d + a_{d-1}{}^q t^{d-1} + \cdots + 1)$. Then we say that a monic polynomial $g(t)$ over $\mathbf{F}_{q^2}$ is U-irreducible if either $g(t)$

is irreducible over $\mathbf{F}_{q^2}$ and $g(t) = \tilde{g}(t)$ or $g(t) = f(t)\tilde{f}(t)$ where $f(t)$ is an irreducible polynomial over $\mathbf{F}_{q^2}$ such that $f(t) \neq \tilde{f}(t)$. (Thus the polynomial $t - \varepsilon^l$ is U-irreducible.)

Let V be the set of all U-irreducible polynomials over $\mathbf{F}_{q^2}$, excepting the polynomial t. Write $d(f)$ for the degree of $f \in V$. Then the conjugacy classes of G_n can be parametrized by the functions $\nu : V \to P$ satisfying

$$\sum_{f \in V} |\nu(f)| d(f) = n\,.$$

Let Y be the set of all such functions ν; if the class c of G_n corresponds to $\nu \in Y$, then we write

$$c = (\cdots f^{\nu(f)} \cdots) = (f_1{}^{\nu_1} \cdots f_N{}^{\nu_N})\,,$$

where $f_1, \cdots, f_N$ are all the $f \in V$ such that $\nu(f) \neq 0$ and, for $1 \leqq i \leqq N$, $\nu_i = \nu(f_i)$.

A.2. Dual classes. Let X be as in 1.5. For $\nu \in X$, the symbol

$$e = (\cdots g^{\nu(g)} \cdots)$$

will be called the dual class of G_n corresponding to ν. As we have stated in 1.5, and as we shall see below, the dual classes of G_n parametrized the irreducible characters of G_n. (In 1.5, we identified a dual class of G_n with the irreducible character of G_n corresponding to it.)

A.3. Substitutions. Let $\rho = [1^{r_1} 2^{r_2} \cdots n^{r_n}]$ be a partition of n. Set

$$X^\rho = \{x_{11}, \cdots, x_{1r_1}, x_{21}, \cdots, x_{2r_2}, \cdots, x_{n1}, \cdots, x_{nr_n}\}\,,$$

where the $x_{ij} = x_{ij}{}^\rho$ are n independent variables (if $r_i = 0$ for some i, then, for such i, the x_{ij} do not occur in X^ρ). For x in X^ρ, if $x = x_{ij}$, then we write $d(x) = i$. We say that a mapping $\alpha : X^\rho \to V$ is a substitution of X^ρ if $d(\alpha(x)) \mid d(x)$ for all $x \in X^\rho$. For such an α, for $f \in V$, we let

$$\rho(\alpha, f) = [1^{r_1(\alpha,f)} 2^{r_2(\alpha,f)} 3^{r_3(\alpha,f)} \cdots]\,,$$

where, for $i = 1, 2, 3, \cdots$,

$$r_i(\alpha, f) = |\{x \in X^\rho \mid \alpha(x) = f, d(x) = i d(f)\}|\,.$$

We say that two substitutions α, α' of X^ρ are equivalent if $\rho(\alpha, f) = \rho(\alpha', f)$ for all $f \in V$. The equivalent class m of a substitution α of X^ρ will be called the mode of α, and we write $\rho(m, f) = \rho(\alpha, f)$, $f \in V$. A substitution of X^ρ into a class $c = (\cdots f^{\nu_c(g)} \cdots)$ of G_n is a substitution α of X^ρ satisfying

$$|\rho(\alpha, f)| = |\nu_c(f)|\,, \quad f \in V\,.$$

Then we can consider the mode of a substitution of X^ρ into c.

Let

$$Y^\rho = \{y_{11}, \cdots, y_{1r_1}, y_{21}, \cdots, y_{2r_2}, \cdots, y_{n1}, \cdots, y_{nr_n}\}$$

be the set of n new variables $y_{ij} = y_{ij}{}^\rho$. Then, as in the case of X^ρ, we can consider the mode of a substitution of Y^ρ into a dual class of G_n.

A.4. Basic uniform functions. Let K be a finite field with $q^{(2n)!}$ elements (and we consider $\mathbf{F}_q$ as a subfield of K), and we fix an isomorphism Θ of $K^\times$ into $\mathbf{C}^\times$.

If d is a positive integer, then, for an integer k and a non-zero element ξ of K, we let

$$S_d(k:\xi) = \sum_{i=0}^{d-1} \Theta^{k(-q)^i}(\xi). \tag{A.4.1}$$

Let $\rho = [1^{r_1}2^{r_2}\cdots n^{r_n}]$ be a partition of n. Then, for a vector $h^\rho = (h_{11}, \cdots, h_{1r_1}, h_{21}, \cdots, h_{2r_2}, \cdots, h_{n1}, \cdots, h_{nr_n})$ of integers h_{ij} (if $r_i = 0$ for some i, then, for such i, the h_{ij} do not occur in h^ρ), we define a function $B_\rho(h^\rho)$ on the set of vectors $\xi^\rho = (\xi_{11}, \cdots, \xi_{1r_1}, \xi_{21}, \cdots, \xi_{2r_2}, \cdots, \xi_{n1}, \cdots, \xi_{nr_n})$ of non-zero elements ξ_{ij} of K (if $r_i = 0$ for some i, then, for such i, the ξ_{ij} do not occur in ξ^ρ) by

$$B_\rho(h^\rho)(\xi^\rho) = \prod_{d=1}^{n} \left\{ \sum_{1',\cdots,r_d'} S_d(h_{d1}:\xi_{d1'})\cdots S_d(h_{dr_d}:\xi_{dr_d'}) \right\}, \tag{A.4.2}$$

where, for $1 \leqq d \leqq n$, the sum is taken over all the permutations $1', \cdots, r_d'$ of $1, \cdots, r_d$ (if $r_d = 0$, we put $\sum_{1',\cdots,r_d'} {}^* = 1$).

If $c = (\cdots f^{\nu_c(f)} \cdots)$ is a class of G_n, and if m is the mode of a substitution α of X^ρ into c, then we denote by $X^\rho m$ the vector $\xi^\rho = (\xi_{11}, \cdots, \xi_{1r_1}, \xi_{21}, \cdots, \xi_{2r_2}, \cdots, \xi_{n1}, \cdots, \xi_{nr_n})$, where, for $1 \leqq i \leqq n$, $1 \leqq j \leqq r_i$, ξ_{ij} is a root of $\alpha(x_{ij}) (\in V)$.

Now, for a vector $h^\rho = (h_{11}, \cdots, h_{1r_1}, h_{21}, \cdots, h_{2r_2}, \cdots, h_{n1}, \cdots, h_{nr_n})$ of n integers h_{ij}, we define a class function $B^\rho(h^\rho)$ of G_n by

$$B^\rho(h^\rho)(c) = \sum_m Q(m, c) B_\rho(h^\rho)(\chi^\rho m), \tag{A.4.3}$$

where the sum is taken over all the modes m of substitutions of X^ρ into a class $c = (\cdots f^{\nu_c(f)} \cdots)$ of G_n and

$$Q(m, c) = \prod_{f \in V} \frac{1}{z_{\rho(m,f)}} Q^{\nu_c(f)}_{\rho(m,f)}((-q)^{d(f)}). \tag{A.4.4}$$

We note that $B^\rho(h^\rho)$ coincides with the Deligne-Lusztig virtual character R_T^θ of G_n ([1]; see [8, p. 203]).

A.5. The irreducible characters of G_n. Let $\nu \in X$, and let $e = (\cdots g^{\nu(g)} \cdots)$ be the corresponding dual class of G_n.

If $\rho = [1^{r_1}2^{r_2}\cdots n^{r_n}]$ is a partition of n, and if m is the mode of a substitution α of Y^ρ into e, then we denote by $h^\rho m$ the vector $(h_{11}, \cdots, h_{1r_1}, h_{21}, \cdots, h_{2r_2}, \cdots, h_{n1}, \cdots, h_{nr_n})$ of integers h_{ij} defined, k_{ij} being a root of $\alpha(y_{ij})$, by

$$h_{ij} = k_{ij} \cdot \frac{(-q)^i - 1}{(-q)^{d(\alpha(y_{ij}))} - 1}, \qquad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq r_i \tag{A.5.1}$$

(if $r_i = 0$ for some i, then, for such i, we do not consider h_{ij}'s), and we let

$$\chi(m, e) = \prod_{g \in S} \frac{1}{z_{\rho(m,g)}} \chi_{\rho(m,g)}^{\nu(g)} . \tag{A.5.2}$$

Then we define a class function $\chi_\nu = \chi_e$ of G_n by

$$\chi_\nu = \eta \sum_{\rho \in P_n} \sum_m \chi(m, e) B^\rho(h^\rho m) , \tag{A.5.3}$$

where, for a partition ρ of n, the second sum is taken over all the modes m of substitutions of Y^ρ into the dual class e, and $\eta = \pm 1$ so that the value of the right hand side at the class $\{1_n\}$ of G_n is positive (we write $\eta = \eta(\chi_\nu)$).

As we have stated in 1.5, the χ_ν, $\nu \in X$, are precisely the irreducible characters of G_n (Ennola conjecture [2, p. 11]).

A.6. o-products. Let $n, n_1, \cdots, n_N$ be positive integers such that $n = n_1 + \cdots + n_N$. If $c = (\cdots f^{\nu_c(f)} \cdots)$ is a class of G_n, and, for $1 \leqq i \leqq N$, $c_i = (\cdots f^{\nu_i(f)} \cdots)$ is a class of G_{n_i}, then we let

$$g_{c_1 \cdots c_N}^{c} = \prod_{f \in V} g_{\nu_1(f) \cdots \nu_N(f)}^{\nu_c(f)}((-q)^{d(f)}) . \tag{A.6.1}$$

Let $\chi_\nu = (\cdots g^{\nu(g)} \cdots) = (g_1{}^{\nu_1} \cdots g_N{}^{\nu_N})$ be an irreducible character of G_n. For $1 \leqq i \leqq N$, put $n_i = d(g_i)|\nu_i|$ and let $\chi_i = (g_i{}^{\nu_i})$, and irreducible character of G_{n_i}. Then we have

$$\chi_\nu(c) = \eta_\nu \sum_{c_1, \cdots, c_N} g_{c_1 \cdots c_N}^{c} \chi_1(c_1) \cdots \chi_N(c_N) , \tag{A.6.2}$$

where the sum is taken over all rows $c_1, \cdots, c_N$ of classes respectively of $G_{n_1}, \cdots, G_{n_N}$ and $\eta_\nu = (-1)^s$, $s = n(n-1)/2 - \sum_{i=1}^N n_i(n_i - 1)/2$.

As to the proof of (5.3), we first treat the case when $\chi = (g^\nu)$, and, using (A.6.2), we next treat the general case.

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Present Address:
IWAMIZAWA CAMPUS, HOKKAIDO UNIVERSITY OF EDUCATION,
MIDORIGAOKA, IWAMIZAWA, HOKKAIDO, 068–0835 JAPAN.