TOKYO J. MATH.
VOL. 23, NO. 1, 2000

Structure of Ideals and Isomorphisms of C^*-crossed Products by Single Homeomorphism

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(Communicated by Y. Ito)

1. Introduction.

Let $\Sigma = (X, \sigma)$ be a topological dynamical system where X is a compact Hausdorff space and σ is a homeomorphism. We denote by α the automorphism induced by σ on the algebra $C(X)$ of all continuous functions on X, namely $\alpha(f)(x) = f(\sigma^{(-1)}x)$. Let $A(\Sigma)$ be the associated transformation group C^*-algebra, that is, the C^*-crossed product with respect to α, regarding it as an action of the integer group $\mathbf{Z}$. We call this algebra a homeomorphism C^*-algebra.

Now one of the main problems about the interplay between topological dynamical systems and C^*-algebras is to determine relations between two dynamical systems when their associated homeomorphism C^*-algebras are isomorphic with each other (General isomorphism problem). A more restrictive problem is to settle the relationship between dynamical systems when an isomorphism between those C^*-algebras keeps their distinguished subalgebras of continuous functions (Restricted isomorphism problem). In case of Cantor minimal systems, we now know full aspects of isomorphism problems due to recent remarkable presentations by T. Giordano, I. Putnam and C. Skau ([6], [7]). With their results, we have then recognized that contrary to the case of measurable dynamical systems there appears a serious gap (even for minimal systems) from general isomorphisms to restricted isomorphisms.

In the author's joint work [3] with M. Boyle, we have solved the latter problem in the best possible way. However, we have not been making so much progress towards the general isomorphism problem. On the other hand, there are isomorphism theorems proved for some limited classes of dynamical systems such as rotations, Denjoy homeomorphisms etc. ([11], [12]), in which no obstruction appears. In fact, we do not know in general how the obstruction between two types of isomorphisms appears, and whether or not it disappears in an elementary connected space as it has been shown to be the case mentioned above.

Received May 28, 1998

Since the global structure of ideals of C^*-algebras is not changed by isomorphisms, it is the purpose of this paper to investigate the properties of ideals of $A(\Sigma)$ and then to analyse the isomorphisms themselves.

There has been extensive literature concerning structure of ideals of general C^*-crossed products and that of transformation group C^*-algebras, but results presented there are not applicable to our situation, particularly when we treat topological dynamical systems admitting periodic points.

In the following we shall consider first the approximation in the universal C^*-crossed product by the integer group Z(definition below) in connection with generations of ideals. We then introduce the notion of well behaving ideals and give characterizations of them (Theorem 2).

In the last section, we shall discuss properties of topological dynamical systems preserved by isomorphisms of relevant homeomorphism C^*-algebras and then good isomorphisms (standard) which preserve those well behaving ideals.

2. Approximation in the universal C^*-crossed product by the integer group Z.

Let A be a unital C^*-algebra acting on a Hilbert space H with an automorphism α. Let $A \times_\alpha Z$ be the C^*-crossed product with respect to the automorphism α (regarding it as an action of Z) with the generating unitary δ and the canonical projection of norm one E : $A \times_\alpha Z \to A$. Denote by $\{a(n)\}$ the Fourier coefficients of an element a of $A \times_\alpha Z$. Then the norm convergent property of the expansion of a, $a = \sum_{n\in Z} a(n)\delta^n$, is somewhat misleading (as is the case of a von Neumann crossed product with respect to strong topology), and this certainly does not hold. We have however the result stating that the generalized Cesàro mean $\sigma_n(a)$ converges to a in norm ([5, Theorem VIII. 2.2]). Since this result is quite useful we shall present first this type of approximation theorem in a more general form including the case of Cesàro mean. Moreover, in connection with out problem of isomorphisms among homeomorphism C^*-algebras we consider the approximation as a result in the universal C^*-crossed product by Z formulated in the following way.

Let

$$K = l_2 \otimes H = l_2(Z, H),$$

and consider the unitary representation v_t of the torus T where for each point t of the torus T the unitary operator v_t on K is defined as

$$v_t\xi(n) = e^{2\pi i n t}\xi(n).$$

Denote by λ the shift unitary operator on K, that is, $(\lambda\xi)(n) = \xi(n-1)$. We write the one parameter automorphism groups of $B(K)$ induced by the adjoints of v_t by $\hat{\omega}_t$. As is well known, the restriction of this action to each C^*-crossed product $A \times_\alpha Z$ is called the dual action of α, usually written as $\hat{\alpha}_t$.

Now Let $B(Z)$ be the C^*-algebra in $B(K)$ consisting of all elements on which the action $\hat{\omega}_t(a)$ is norm continuous. This is a quite big irreducible C^*-algebra on K absorbing all C^*-crossed products of a single automorphism. Let $B(\hat{\omega})$ be the fixed point algebra of the action $\hat{\omega}$. We define the projection of norm one E_Z form $B(Z)$ to $B(\hat{\omega})$ by

$$E_Z(a) = \int_0^1 \hat{\omega}_t(a)dt\,.$$

We then set the generalized n-th Fourier coefficient of an element a in $B(Z)$ as $a(n) = E_Z(a\lambda^{*n})$. Note that

$$\hat{\omega}_t(\lambda) = e^{2\pi it}\lambda \quad \text{and} \quad E_Z(\lambda^n) = 0 \quad \forall n \neq 0\,.$$

Henceforth we regard this algebra as the universal C^*-crossed product by the integer group Z.

Next recall that a sequence of real valued continuous functions $\{k_n(t)\}$ on the torus T is called a summability kernel if they satisfy the following three conditions:

(a) $\int_T k_n(t)dt = 1$,

(b) $\int_T |k_n(t)|dt \leqslant C$ (constant).

(c) For every $0 < \delta < 1$,

$$\lim_{n\to\infty}\int_\delta^{1-\delta} |k_n(t)|dt = 0\,.$$

Well known summability kernels are Fejér kernel,

$$K_n(t) = \sum_{-n}^{n}\left(1 - \frac{|j|}{n+1}\right)e^{2\pi ijt}$$

de la Vallée Poussin kernel,

$$V_n(t) = 2K_{2n-1}(t) - K_{n-1}(t)\,,$$

and Jackson kernel etc, which are trigonometric polynomials. On the other hand, parameters of summability kernels need not be natural numbers in general. Whenever families of continuous functions satisfy the above three conditions with respects to the attached parameters, we can apply the same arguments. Therefore, we can regard the Poisson kernel $P(r,t)$ as a summability kernel with continuous parameter r. In this case $P_r(t)$ satisfies the condition (c) as $r \to 1$. This kernel is however not consisting of trigonometric polynomials. The Dirichlet kernel $\{D_n(t)\}$ is not a summability kernel because it does not satisfy the third condition, and this shows why we can not obtain the norm convergence of the sum $\sum_{-\infty}^{\infty} a(n)\delta^n$.

Let B be a Banach space and consider the space of all B-valued continuous functions on T, $C(T, B)$. We define the convolution $k_n \star F$ in $C(T, B)$ by

$$k_n \star F(t) = \int_T k_n(s)F(t-s)ds\,.$$

One then easily sees that the convolution is also a B-valued continuous function. We assert here the Banach space version of the following classical approximation theorem in Fourier analysis.

PROPOSITION 1. *For any summability kernel $\{k_n\}$ and a continuous function $F(t)$ in $C(T, B)$, the convolution $k_n \star F(t)$ converges uniformly to $F(t)$ in B.*

The proof of this result is just a linear modification of the one given in the classical Fourier analysis, and we leave the readers its verification.

Define the n-th Fourier coefficient $\hat{F}(n)$ of F by

$$\hat{F}(n) = \int_T F(t)e^{-2\pi int}dt.$$

Then if $k_n(t)$ is a polynomial of a form,

$$k_n(t) = \sum_{-l_n}^{l_n} c_j e^{2\pi ijt},$$

we have

$$k_n \star F(t) = \sum_{-l_n}^{l_n} c_j \hat{F}(j) e^{2\pi ijt}.$$

Hence the above result says that the function $F(t)$ is uniformly approximated in norm by the above trigonometric polynomials.

We now apply this result to the algebra $B(Z)$ taking this algebra as the above Banach space B with the continous function $\hat{\omega}_t(a)$ for an element a of $B(Z)$. Write this function as $\tilde{a}(t)$. We have then

$$a(n)\lambda^n = \int_T \hat{\omega}_t(a\lambda^{*n})dt\lambda^n = \int_T \hat{\omega}_t(a)e^{-2\pi int}dt = \hat{\tilde{a}}(n).$$

Therefore we obtain the following approximation theorem in $B(Z)$.

THEOREM 1. *Let $\{k_n(t)\}$ be a summability kernel on the torus T. Then an element a in $B(Z)$ is approximated in norm by the sequence $k_n \star \tilde{a}(0)$. In particular if the kernel consists of trigonometric polynomials of the form*

$$k_n(t) = \sum_{-l_n}^{l_n} c_j e^{2\pi ijt},$$

a is approximated by the generalized polynomials of λ with the form

$$k_n \star \tilde{a}(0) = \sum_{-l_n}^{l_n} c_j a(j)\lambda^j.$$

Hence $B(Z)$ is linearly spanned by $\{\lambda^n\}$ in norm over the fixed point algebra $B(\hat{\omega})$.

Thus, though we do not assume at first any crossed product structure for $B(Z)$ we are able to deduce the fact that it is linearly generated by generalized polynomials of λ whose coefficients are specifically defined from the Fourier coefficients of the elements to which they converge.

Now take a crossed product $A \times_\alpha Z$ regarded as a C^*-subalgebra of $B(Z)$. It is then clear that the canonical projection E in $A \times_\alpha Z$ is just the restriction of E_Z and $A = B(\hat{\omega}) \cap A \times_\alpha Z$.

Hence the Fourier coefficients of an element a in $A \times_\alpha Z$ is nothing but those defined as an element of $B(Z)$.

Therefore from the above theorem we can derive usual conclusions on the unicity of the generalized Fourier coefficients, faithfullness of the projection etc. in a quite C^*-algebraic manner.

In a global sense we know that the crossed product is a norm span of those generalized polynomials of the generating unitary δ with coefficients coming from elements of A, but the advantage of the above approximation theorem lies in the fact that for the approximation of a fixed element we can refer to its Fourier coefficients for those approximation polynomials, even in several ways depending on which summability kernels to use. Among them the Cesàro mean for Fejér kernel,

$$\sigma_n(a) = K_n \star \tilde{a}(0) = \sum_{-n}^{n} \left(1 - \frac{|j|}{n+1}\right) a(j)\delta^j ,$$

is most elementary.

3. Structure of ideals of $A(\Sigma)$.

Henceforth we stick to a dynamical system $\Sigma = (X, \sigma)$, where X is an arbitrary compact Hausdorff space. We emphasize the fact that the space X need not be metrizable.

In the following we shall use the following notations:
$Aper(\sigma)$ = set of all aperiodic points,
$Per(\sigma)$ = set of all periodic points.
For a positive integer n,

$$Per^n(\sigma) = \{x \in X \mid \sigma^n(x) = x\},$$

$Per_n(\sigma)$ = set of all n-periodic points.

We write $O_\sigma(x)$ for the orbit of x by the homeomorphism σ or $O(x)$ if no confusion occurs.

For the associated C^*-crossed product $A(\Sigma)$, we denote by δ and E the generating unitary of $A(\Sigma)$ and the canonical projection of norm one to the subalgebra $C(X)$. For an element a of $A(\Sigma)$, $a(k)$ means the k-th Fourier coefficient of a. A representation of $A(\Sigma)$ is written as $\tilde{\pi} = \pi \times u$ where π is a representation of $C(X)$ and u the generating unitary element of $\tilde{\pi}(A(\Sigma))$ such that $\tilde{\pi}(\delta) = u$.

For basic results concerning the interplay between topological dynamics and C^*-theory used here we mainly refer the reader to the author's articles [13] or [14].

Now take an aperiodic point x. We write $P(\bar{x})$ the kernel of the irreducible representation associated to a unique pure state extension of the point evaluation μ_x at the point x. This ideal depends only on the orbit of x. For a periodic point y in X, we write $P(\bar{y}, \lambda)$ the kernel of the irreducible representation associated to the pure state extension $\varphi(y, \lambda)$ of the point evaluation μ_y for a parameter λ in the torus T [14, Proposition 4.3]. The intersection of all such ideals is written as $Q(\bar{y})$. We note first that the family $\{P(\bar{y}, \lambda) \mid y \in Per(\sigma), \lambda \in T\}$ exhausts all primitive ideals of $A(\Sigma)$ which are kernels of finite dimensional irreducible representations [13, Proposition 4.1.7]. On the other hand, as far as the infinite dimensional irreducible representations are concerned a primitive ideal need not to be the kernel of an irreducible representation induced by a point, that is, in our setting the family $\{P(\bar{x}) \mid x \in Aper(\sigma)\}$ does not exhaust the primitive ideals of infinite dimensional irreducible representations of $A(\Sigma)$ unless X is metrizable.

We, however, still have the following

PROPOSITION 2. *Every ideal of $A(\Sigma)$ is the intersection of those primitive ideals of $P(\bar{x}_\alpha)$ and $P(\bar{y}_\beta, \lambda)$ where x_α, y_β and λ are ranging over some sets of aperiodic points, periodic points and points of the torus, respectively.*

This fact has been mentioned already in [1] (and [16, Proposition 4.5] without proof). Since we do not impose any countability condition on the space X, the result is not so trivial and depends heavily on a particular nature of the images of infinite dimensional irreducible representations of $A(\Sigma)$ explained below together with the fact that any finite dimensional irreducible representation of $A(\Sigma)$ comes from a periodic point in X. As the reference [1] is not easily available, we give here the proof again.

PROOF. It suffices to show that any primitive ideal P of $A(\Sigma)$ is a intersection of those specified ideals. As mentioned above, this holds if P is the kernel of a finite dimensional irreducible representation. In order to treat infinite dimensional case we have to recall first the following elementary facts about covariant representations induced from closed invariant subsets of X. Namely, let S be a closed invariant subset and $\rho_s : f \to f \mid S$ be the restriction map from $C(X)$ to $C(S)$. Denote the induced automorphism of $C(S)$ by α_s. Then this pair $\{\rho_s, \alpha_s\}$ becomes a covariant representation of $A(\Sigma)$, hence it gives rise to the canonical homomorphism $\rho_s \times \delta_s$ from $A(\Sigma)$ to $A(\Sigma_s)$ where δ_s means the generating unitary in $A(\Sigma_s)$. Let P_s be the kernel of this homomorphism, then it coincides with the ideal generated by the kernel of S, $k(S)$, of $C(X)$. We write this as $P_s = J(k(S))$. It follows that an element a of $A(\Sigma)$ belongs to P_s if and only if every Fourier coefficient of a vanishes on S. Now let P be the kernel of the infinite dimensional irreducible representation $\tilde{\pi} = \pi \times u$ where π is a representation of $C(X)$. Here the kernel of π is written as $k(S)$ for some closed invariant subset S. There exists then a faithful projection of norm one from $\tilde{\pi}(A(\Sigma))$ to $\pi(C(X))$ so that the image can be canonically identified with the C^*-crossed product $C(S) \times_{\alpha_S} Z$ with respect to the restricted action of σ to S together with that projection E_S [14, Corollary 5.1 B]. Therefore the ideal P is expressed as the intersection of the primitive ideals of those induced irreducible representations coming from the points of S.

Next we consider the ideals of $A(\Sigma)$ by means of their images under the canonical projection E. Let I be a closed ideal of the algebra, then the image $E(I)$ becomes an ideal of $C(X)$ (not necessarily closed) by the module property of E. One might assume here that $E(I)$ would be a proper ideal of $C(X)$, but in $A(\Sigma)$ it may not be the case. In fact, let $P(\bar{y}, 1)$ be the kernel of the n-dimensional irreducible representation $\tilde{\pi} = \pi \times u$ for a periodic point

y with the parameter 1, that is, $\tilde{\pi}(\delta) = u$ and $u^n = 1$. Then by definition the element $1 - \delta^n$ belongs to the ideal and $E(1 - \delta^n) = 1$. Hence $E(P(\bar{y}, 1)) = C(X)$. We regard this kind of ideal as the worst behaving one and look for the class of some better behaving ideals. One may regard an ideal I a little better one if $E(I)$ is a proper ideal of $C(X)$. We further consider the following ideal.

In the theorem the inclusion of the assertion (3) is suggested by A. Kishimoto.

THEOREM 2. *Let $\Sigma = (X, \sigma)$ be a dynamical system in X. Then the following assertions are equivalent:*

(1) *I is an intersection of some family of $P(\bar{x})$ and $Q(\bar{y})$,*

(2) *$E(I) \subset I$ and $E(I)$ is a closed invariant ideal of $C(X)$,*

(3) *I is invariant by the dual action $\hat{\alpha}$,*

(4) *I is linearly spanned in norm by $\{\delta^n\}$ over $E(I)$,*

(5) *The quotient algebra $A(\Sigma)/I$ is canonically isomorphic to the C^*-crossed product $q(C(X)) \times_{\alpha_I} Z$ with respect to the induced automorphism α_I of $q(C(X))$ in such a way that*

$$q \circ E(a) = E_I \circ q(a)$$

where q and E_I are the quotient homomorphism and the canonical projection in $q(C(X)) \times_{\alpha I} Z$, respectively.

In particular, when the dynamical system is free, that is, with no periodic points, then there is a one to one correspondence between the set of closed ideals of $A(\Sigma)$ and the set of closed invariant subsets of X.

PROOF. We note first that the ideals $P(\bar{x})$ and $Q(\bar{y})$ satisfy the assertion (2) by [14, Proposition 5.2] hence we have the implication (1) $\Rightarrow$ (2).

Assume the assertion (2). Then one sided inclusion is clear for (4) and the other inclusion is obtained by Theorem 1 by using Casàro mean. (One may of course refer here the old Zeller-Meier's result [17, Proposition 5.10] but we want to emphasize the above important aspect of the crossed products by Z).

Next the assertion (4) clearly implies (3) by the properties of dual actions, and the assertion (3) leads to (2) by the definition of the projection E.

The assertion (4) $\Rightarrow$ (5). Define the map ε_I by $\varepsilon_I(q(a)) = q(E(a))$. Then by the assumption, this map is well defined and one may easily verify that it is a projection of norm one from $A(\Sigma)/I$ to $q(C(X))$ satisfying the relation

$$\varepsilon_I \circ q = q \circ E\,.$$

Now since the quotient algebra $A(\Sigma)/I$ is generated by $q(C(X))$ and $q(\delta)$, there exists a homomorphism Φ from the crossed product $q(C(X)) \times_{\alpha_I} Z$ to $A(\Sigma)/I$ such that $\Phi(\delta_I) = q(\delta)$ where δ_I stands for the generating unitary of the crossed product. Moreover, the above property of the projection ε_I implies the relation,

$$\varepsilon_I \circ \Phi = \Phi \circ E_I\,.$$

Here E_I is the faithful canonical projection of the crossed product $q(C(X)) \times_{\alpha_I} Z$ and Φ is naturally faithful on $q(C(X))$. Hence Φ is an isomorphism.

The assertion (5) $\Rightarrow$ (1). The kernel of the quotient map q on $C(X)$ is written as $k(S)$ for an invariant closed subset S of X. Let S' be the spectrum of $q(C(X))$ and σ'_s be the homeomorphism defined by the induced automorphism α_I on $q(C(X))$. We can then identify the covariant system $\{C(S), \alpha_s\}$ with $\{C(S'), \alpha_I\}$ as well as the dynamical systems $\Sigma_s = (S, \sigma \mid S)$ with $\Sigma'_I = (S', \sigma_{S'})$. Therefore with this identification the map q is regarded as the homomorphism $\rho_s \times \delta_s$ from $A(\Sigma)$ to $A(\Sigma_s) = A(\Sigma)/I$. It follows that the ideal I is the intersection of all kernels of irreducible representations of $A(\Sigma)$ coming from those points of S. Hence we have the conclusion (1).

The statement of the second half is clear because in this case, by Proposition 2, every closed ideal of $A(\Sigma)$ satisfies the equivalent conditions stated in the first half.

This completes all proofs.

We notice that this gives another background for the classical equivalence between simplicity of $A(\Sigma)$ and minimality of the dynamical system Σ.

REMARK. Actually all the equivalences from (2) to (5) are valid for an arbitrary crossed product $A \times_\alpha Z$, but we are interested in those properties only from the point of view of their relationships in the algebra $A(\Sigma)$.

Next, let $\overline{E(I)}$ be the closure of $E(I)$ and write it as the kernel $k(S)$ in $C(X)$ for a closed invariant subset S of X. Let $P_1, P_2, \cdots, P_n$ be an n-tuple of ideals associated with periodic points $y_1, y_2, \cdots, y_n$ whose orbits are disjoint from each other and moreover from S, too. The following ideal is then a prototype of those ideals which do not satisfy the conditions of the above Theorem. Namely we have

PROPOSITION 3. *If every P_i does not coincide with the ideal $Q(\overline{y_i})$, then the ideal*

$$I = J(k(S)) \cap P_1 \cap P_2 \cap \cdots \cap P_n$$

does not satisfy the condition, $E(I) \subset I$.

PROOF. Let K be the union of all orbits $O(y_i)$, then I clearly contains the ideal

$$J(k(S)) \cap J(k(K)) = J(k(S \cup K)).$$

Take a point y_i and fix. From the assumption, $P_i \supsetneqq Q(\overline{y_i})$, there exists an element a of P_i such that $E(a)$ does not vanish on the orbit of y_i. On the other hand, we can also find a function f on X such that it vanishes on S and the union of other orbits of y_k whereas $f \mid O(y_i) = 1$. Then the element fa belongs to I and $E(fa)$ does not vanish on the orbit $O(y_i)$. Now suppose I satisfied the conditions of the Theorem, say I had the form $J(k(R))$ for a closed invariant subset R. The set R is then contained in $S \cup K$, but the above argument shows that R is disjoint from each orbit $O(y_i)$. Hence, $R = S$ and $I = J(k(S))$. On the other hand, since S and K are disjoint we see that the ideal $J(k(S)) + J(k(K))$ is dense in $A(\Sigma)$, hence it coincides with $A(\Sigma)$. Therefore, if $I = J(k(S))$ we reach the contradiction,

$$P_1 \cap P_2 \cap \cdots \cap P_n = A(\Sigma)\,.$$

This completes the proof.

Actually the above arguments show that, in this case, for each orbit $O(y_i)$ there exists an element a_i in $P = P_1 \cap P_2 \cap \cdots \cap P_n$ such that $E(a_i)$ becomes 1 on $O(y_i)$ and vanishes on other orbits. Hence P contains the element a for which $E(a)$ takes 1 on the set K. Since all continuous functions vanishing on K belong to the above ideal, this means that $E(P)$ contains the constant functions and $E(P) = C(X)$. Thus, when $S = \phi$ in the proposition the ideal I shows a prototype of those ideals whose images of the projection exhaust the whole algebra $C(X)$.

The reason that we might not obtain an exclusive description of an ideal I for which $E(I)$ is not contained in I stems from the following situation. We have an example of a topologically free dynamical system in which there exists a countable set $\{y_n\}$ of periodic points without isolated points and ideals $\{P_n\}$ with $P_n \supsetneqq Q(\bar{y}_n)$ but never-the-less we have

$$\bigcap_{n=1}^{\infty} P_n = \bigcap_{n=1}^{\infty} Q(\overline{y_n})\,.$$

Here, a dynamical system is said to be topologically free if the set $Aper(\sigma)$ is dense in X. Then the C^*-algebra $A(\Sigma)$ has in this case nice properties as a crossed product described in [14, Theorem 5.4]. Before going to show our example, however, we need the following observation.

LEMMA 1. *The map*

$$\Phi : Per_k(\sigma) \times T \longrightarrow \{\varphi(x, \lambda) \mid x \in Per_k(\sigma), \lambda \in T\}$$

is a homeomorphism with respect to the w^*-topology in the pure state space.

PROOF. Suppose a net $\{(y_\alpha, \lambda_\alpha)\}$ converges to a point (y_0, λ_0). Since each $\varphi(y_\alpha, \lambda_\alpha)$ is a pure state extension of the point evaluation μ_{y_α}, $\varphi(y_\alpha, \lambda_\alpha)(f) = f(y_\alpha)$ converges to $f(y_0) = \varphi(y_0, \lambda_0)$ for every continuous function f. On the other hand, we have, by the definition of the parameter for pure state extensions, that

$$\varphi(y_\alpha, \lambda_\alpha)(\delta^{nk}) = \lambda_\alpha^n \longrightarrow \lambda_0^n = \varphi(y_0, \lambda_0)(\delta^{nk})\,.$$

Moreover, the values of pure states of other powers of the unitary δ are all zero [14, Proposition 4.3] or [16, Theorem 3.1]. Now since

$$\varphi(y, \lambda)(f\delta^n) = f(y)\varphi(y, \lambda)(\delta^n)$$

by [14, Lemma 4.2], we see that the net $\{\varphi(y_\alpha, \lambda_\alpha)\}$ converges to $\varphi(y_0, \lambda_0)$ in the w^*-topology.

The converse continuity may be easily seen from the above arguments.

When we lack the condition for the period, we can not expect this kind of result.

[Example] Let $\Sigma = (T^2, \sigma)$ be the topologically free dynamical system in the two dimensional torus T^2 defined by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ described in [14]. Denote a point in T^2 by (s, t) and consider the segment $(1/2, T)$. Let $\{y_n\}$ be the set of all rational points of this segment between 0 and 1/3. For each point y_n, let P_n be the intersection of all primitive ideals $P(\overline{y_n}, \lambda)$ where λ is ranging over the interval $[0, 1 - 1/n]$. Then each P_n contains strictly the ideal $Q(\overline{y_n})$. Moreover, by the above lemma every pure state extension $\varphi(y_n, \lambda)$ can be weakly approximated by a subsequence $\{\varphi(y_{n_k}, \lambda_{n_k})\}$ where $n_k \neq n$ and $\lambda_{n_k} \in [0, 1 - 1/n_k]$. Therefore the intersection of all P_m is contained in the ideal $P(\overline{y_n}, \lambda)$ for every integer n and parameter λ, hence

$$\bigcap_{n=1}^{\infty} P_n = \bigcap_{n=1}^{\infty} Q(\overline{y_n}) \,.$$

4. Algebraic invariants of topological dynamical systems and isomorphisms.

So far, our motivation to investigate the structure of ideals of $A(\Sigma)$ lies in its relationship to the general isomorphism problem between homeomorphism C^*-algebras. Therefore it would be natural to consider the following definition.

DEFINITION 1. *Let $\Sigma_1 = (X, \sigma)$ and $\Sigma_2 = (Y, \tau)$ be two dynamical systems with associated C^*-algebras $A(\Sigma_1)$ and $A(\Sigma_2)$. We say that a property of Σ_1 is an algebraic invariant if the other dynamical system Σ_2 has the same property when $A(\Sigma_2)$ is isomorphic with $A(\Sigma_1)$.*

Typical examples of such properties of dynamical systems are minimality and topological transitivity. Namely they are kept by isomorphisms through the simplicity and primeness of those associated C^*-algebras. We notice that the above definition does not simply mean algebraic characterization of relevant properties. For instance, we have a nice equivalent algebraic assertion of topological freeness of Σ_1 cited before as the maximality of the commutative C^*-subalgebra $C(X)$, but then it does not trivially imply the maximality of the subalgebra $C(Y)$. In fact, location of the algebra $C(X)$ in the C^*-crossed product $A(\Sigma)$ is deeply connected with the gap between general isomorphisms and restricted isomorphisms. Thus the following characterization of topological freeness is quite meaningful.

PROPOSITION 4. *The dynamical system Σ is topologically free if and only if the C^*-algebra $A(\Sigma)$ has sufficiently many infinite dimensional irreducible representations. Consequently, topological freeness of a dynamical system is an algebraic invariant.*

PROOF. If Σ is topologically free, the intersection of all kernels of infinite dimensional irreducible representations induced by aperiodic points of in X becomes zero by [14, Corollary 5.1B and Proposition 5.2].

Conversely, take a non-zero continuous function f on X. By the assumption, there exists an infinite dimensional irreducible representation $\tilde{\pi} = \pi \times u$ with $\pi(f) \neq 0$. Let $\Sigma_\pi = (X_\pi, \sigma_\pi)$ be the dynamical system induced by the representation $\tilde{\pi}$ ([14, p. 26]). Then

Σ_π becomes topologically free ([14, Corollary 5.1B]), and $\pi(f)$ does not vanish on the set $Aper(\sigma_\pi)$. Since $\pi(f)$ on X_π is identified with the restriction of f on the invariant subset X_π, this means that f does not vanish on the set $Aper(\sigma)$. Hence, Σ is topologically free.

The above result says that actually in $A(\Sigma)$ the existence of sufficiently many infinite dimensional irreducible representations is equivalent to the existence of sufficiently many of those representations induced from aperiodic points in X, although there is a big difference between these two families in general.

It would be better to mention here the following fact.

LEMMA 2. *The kernel of an infinite dimensional irreducible representation of* $A(\Sigma)$ *satisfies the condition of Theorem* 2.

PROOF. In this case the induced topological dynamical system becomes topologically free, hence by [14, Theorem 5.1] the kernel satisfies the condition.

Besides the above three properties we illustrate below other algebraic invariants of topological dynamical systems.

(1) Σ is free ([14, Proposition 4.5]),

(2) $Per(\sigma) = X$, and (3) $Per(\sigma)$ is dense in X, ([14, Theorem 4.6]).

When X is metrizable;

(4) $C(\sigma) = Per(\sigma)$ where $C(\sigma)$ is the set of all recurrent points of Σ, and

(5) $C(\sigma)\backslash Per(\sigma)$ is dense in X ([2, Corollary 2.1 and Corollary 2.2]).

(6) $X(\sigma) = X$ where $X(\sigma)$ is the set of all chain recurrent points of Σ ([10, Theorem 9]).

These properties are preserved by any isomorphism among homeomorphism C^*-algebras. There may be, however, kinds of badly behaving isomorphisms, and we consider the following.

DEFINITION 2. *We say that an isomorphism between* $A(\Sigma_1)$ *and* $A(\Sigma_2)$ *is standard if it preserves the class of ideals described in Theorem* 2.

For an isomorphism, kernels of infinite dimensional irreducible representations are always preserved, hence the problem whether the isomorphism is standard or not is actually the problem whether it keeps the ideal of the form $Q(\bar{x})$ for every periodic point x. In this connection, we note first that there exists a pair of compact connected manifolds (X, Y) which are not homeomorphic to each other but their product spaces with torus are homeomorphic, that is, $X \times T \approx Y \times T$. Hence if we consider the trivial dynamical systems in these compact manifolds, we have an isomorphism

$$A(\Sigma_1) = C(X \times T) \simeq C(Y \times T) = A(\Sigma_2)\,.$$

But no isomorphism between these C^*-algebras keeps the ideals $Q(\bar{y})$ for all points of X because if there exists such an isomorphism, then it would inuce a homeomorphism between X and Y. Here an isomorphism brings certainly unitarily equivalent irreducible representations to equivalent pairs. But classes of finite dimensional irreducible representations have two parameters, namely orbits and numbers from the torus, and the trouble arises from the circumstances where we can not tell how these two kinds of parameters change after each isomorphism.

On the other hand, as mentioned in Theorem 2, if a dynamical system is free all isomorphisms for $A(\Sigma)$ are standard. Futhermore we have.

PROPOSITION 5. *In the dynamical system Σ if the set $Per(\sigma)$ is at most countable every isomorphism through $A(\Sigma)$ is standard.*

PROOF. Suppose Φ be an isomorphism from $A(\Sigma)$ to another homeomorphism C^*-algebra $A(\Sigma_1)$ for $\Sigma_1 = (Y, \tau)$. In order to show that Φ is a standard isomorphism, it is enough to show by (1) of Theorem 2 that it keeps the ideals of the form $Q(\bar{x})$ for periodic points. Let $Q(\bar{y})$ be such an ideal in $A(\Sigma_1)$ for a periodic point y with period p. We assert that the inverse image $I = \Phi^{-1}(Q(\bar{y}))$ has the same form. Note first that the quotient algebra $A(\Sigma_1)/Q(\bar{y})$ is regarded as the homeomorphism algebra on the orbit $Q_\tau(y)$ and it is canonically isomorphic to the algebra of all M_p-valued continuous functions on the torus T ([16, Proposition 4]). Hence, the dual of that algebra is homeomorphic to T, a compact connected space. On the other hand, dual of the corresponding quotient algebra, $A(\Sigma)/I$, is by [9, Theorem A] a compact subset Π of the product space $(Per_p(\sigma)/\sim) \times T$ as a part of the p-dimensional dual of $A(\Sigma)$. Hence Π is written as the disjoint sum of at most countable number of closed sets,

$$F_n = \{(\bar{x}_n, \lambda) \in \Pi\}.$$

Hence, by Sierpinski's theorem we have that

$$\Pi = F_{n_0} \quad \text{for some } n_0.$$

Thus,

$$T' = \{\lambda \mid (\bar{x}_{n_0}, \lambda) \in \Pi\}$$

becomes a compact subset of T, which is homeomorphic to T. It follows that $T' = T$ and I has the form $Q(\bar{x}_{n_0})$.

From the above arguments we also see that the set of all ideals of the form $Q(\bar{y})$ in $A(\Sigma_1)$ is at most countable, hence the set $Per(\tau)$ is at most countable, too. Thus we conclude that Φ also keeps the form of the ideal $Q(\bar{x})$ for a periodic point x in X. This completes the proof.

We notice that most of examples of dynamical systems in manifolds satisfy the above condition (thus becoming topologically free dynamicl systems). The author does not know whether or not an isomorphism between homeomorphism C^*-algebras of topologically free dynamical systems is always a standard isomorphism.

So far, we come to know that for most of those reasonable dynamical systems all isomorphisms between associated C^*-algebaras are relatively better behaving ones, but unfortunately this fact does not mean that we can perturb them towards extremely well-behaving isomorphisms, that is, restricted isomorphisms.

For a topologically free dynamical system, we can give a characterization of a restricted isomorphism as the one which almost commutes with dual actions. This will be shown by the arguments based on the short exact sequence concerning the normalizer of the topological full group and the automorphism group of $A(\Sigma)$ preserving $C(X)$. We shall discuss this elsewhere with other results about full groups.

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