TOKYO J. MATH.
VOL. 23, NO. 2, 2000

# Characterization of Singular Integral Kernel Space and $T(1)$ Theorem

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(communicated by T. Nagano)

**Abstract.** In this paper, one uses a discrete representation to characterize two singular integral kernel spaces with two others simple Banach, and to improve a Meyer's result: David and Journé's $T(1)$ theorem under a weaker kernel condition.

## 1. Introduction.

According to L. Schwartz's kernel theorem, a linear continuous mapping $T : D \to D'$ corresponds to a kernel distribution $K(x, y)$ in the sense that $\langle Tf, g\rangle = \iint K(x, y)g(x)f(y)dxdy$. The Calderón-Zygmund school pays attention to the kernel $K(x, y)$, which satisfies the following pointwise conditions:

$$|K(x, y)| \leq \frac{C}{|x-y|^n}. \tag{1.1}$$

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{C|x-x'|^\gamma}{|x-y|^{n+\gamma}}, \tag{1.2}$$

$$\forall 0 < \gamma \leq 1, \quad |x-x'| \leq \frac{1}{2}|x-y|.$$

And the following weak boundedness condition:

$$\left|\iint K(x, y)g(x)f(y)dxdy\right| \tag{1.3}$$
$$\leq C|Q|(\|f\|_\infty + |Q|^{1/n}\|f'\|_\infty)(\|g\|_\infty + |Q|^{1/n}\|g'\|_\infty),$$

for the arbitrary ball $Q$, and $f(x), g(x) \in C_0^1(Q)$.

One denotes $K(x, y) \in S_\gamma$ or $T \in OpS_\gamma$. If $T \in OpS_\gamma$, and $T$ can extend to a bounded operator on $L^2$, one calls that $T \in OpF_\gamma$. Then celebrated David and Journé's $T(1)$ Theorem (cf. [2]) can be written as follows.

Received January 6, 1999
Revised January 11, 2000
Supported by a foundation from the Education Ministry of China for young scholar back from abroad.

THEOREM 1. *Suppose that $T \in OpS_\gamma$. Then the operator $T$ extends to a bounded operator on $L^2$ if and only if*

$$T(1) \in BMO, \quad T^t(1) \in BMO. \tag{1.4}$$

The kernel conditions were further weakened by Meyer (cf. [3]), who replaced the pointwise estimates (1.1) and (1.2) by the integral estimates:

$$\sup_{r>0} \int_{r\le|x-y|\le 2r} (|K(x,y)| + |K(y,x)|)dy \le C. \tag{1.5}$$

$$\sup_{\substack{r>0\\ |u|+|v|\le r}} \int_{2^k r\le|x-y|\le 2^{k+1}r} \{|K(x+u,y+v) - K(x,y)| + |K(y+u,x+v) - K(y,x)|\}dy \le \varepsilon(k), \tag{1.6}$$

where $k = 1, 2, 3, \cdots$, and $\varepsilon(k)$ satisfy:

$$\sum_{k=1}^{\infty} k\varepsilon(k) < \infty. \tag{1.7}$$

The purpose of this paper is to use a wavelets representation characterize $OpS_\gamma$ and $OpF_\gamma$ with two others Banach spaces, and to prove the $T(1)$ theorem under Condition (2.7), which is weaker than (1.7).

## 2. Wavelets basis and main theorems.

Denotes $\varphi(x)$ be the father of Daubechies wavelets, denotes $\psi(x)$ be the mother of Daubechies, and denotes $\Phi^{(0)}(x) = \varphi(x)$, $\Phi^{(1)}(x) = \psi(x)$. $\forall\lambda = (\varepsilon, j, k)$, $\varepsilon \in \{0,1\}^n$, $j \in Z$, $k \in Z^n$, $\forall x \in R^n$, denotes $\Phi^{(\varepsilon)}(x) = \prod_{i=1}^n \Phi^{(\varepsilon_i)}(x_i)$, $\Phi_\lambda(x) = \Phi_{j,k}^{(\varepsilon)}(x) = 2^{nj/2}\prod_{i=1}^n \Phi^{(\varepsilon_i)}(2^j x_i - k_i)$, denotes $\tilde{\Lambda} = \{(\varepsilon, j, k), \varepsilon \in \{0,1\}^n \setminus \{0\}, j \in Z, k \in Z^n\}$, then $\{\Phi_\lambda(x)\}_{\lambda\in\tilde{\Lambda}}$ becomes an unconditional basis for most of function spaces. One hopes that $\{\Phi_\lambda(x)\Phi_{\lambda'}(y)\}_{(\lambda,\lambda')\in\tilde{\Lambda}\times\tilde{\Lambda}}$ can analyse a usual operator. In fact, Meyer has analysed $OpM_\gamma = \{T \in OpF_\gamma, T(1) = T^t(1) = 0\}$ with this basis, but if $T(1) \ne 0$ or $T^t(1) \ne 0$, the matrix under the basis $\{\Phi_\lambda(x)\Phi_{\lambda'}(y)\}_{(\lambda,\lambda')\in\tilde{\Lambda}\times\tilde{\Lambda}}$ does not give a useful information for an operator. Here one uses a basis related to the Beylkin, Coifman and Rokhlin (B-C-R) algorithm, cf. [1]. Let $\Lambda = \{\lambda = (\varepsilon, \varepsilon', j, k, l), \varepsilon, \varepsilon' \in \{0,1\}^n, |\varepsilon| + |\varepsilon'| \ne 0, j \in Z, k, l \in Z^n\}$, for all $\lambda \in \Lambda$, let $\Phi_\lambda(x,y) = \Phi_{j,k,l}^{(\varepsilon,\varepsilon')}(x,y) = 2^{jn}\Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l)$, one denotes:

$$a_\lambda = a_{j,k,l}^{(\varepsilon,\varepsilon')} = \iint K(x,y)\Phi_{j,k,l}^{(\varepsilon,\varepsilon')}(x,y)dxdy. \tag{2.1}$$

The B-C-R algorithm says that:

$$K(x,y) = \sum_{\lambda\in\Lambda} a_\lambda \Phi_\lambda(x,y). \tag{2.2}$$

In fact, $\{\Phi_\lambda(x, y)\}_{\lambda\in\Lambda}$ is an orthonormal basis in $L^2(R^n \times R^n)$. For $T \in OpS_\gamma$, (2.1) can be defined, then (2.2) is true in the sense of distribution. Hence $\{a_\lambda\}_{\lambda\in\Lambda}$ becomes a new representation for operator $T$.

Let $\varphi(x) \in C_0^m([-M, M])$, $\psi(x) \in C_0^m([-M, M])$, one presents the following definition:

DEFINITION 1. *$T \in OpNS_\gamma$, if*

$$|a_\lambda| \le \frac{C_{M,\gamma}}{(1+|k-l|)^{n+\gamma}}, \quad \forall\lambda = (\varepsilon, \varepsilon', j, k, l) \in \Lambda. \tag{2.3}$$

Then one has the following theorems:

THEOREM 2. *$T \in OpS_\gamma \Leftrightarrow T \in OpNS_\gamma$.*

For $j \in Z, k \in Z^n$, denote $Q(j, k) = \prod_{i=1}^n [k_i 2^{-j}, (k_i + 1)2^{-j}]$; denote $\Omega$ be the set of all the dyadic cube. For $\lambda = (\varepsilon, j, k) \in \tilde{\Lambda}$, let

$$\alpha_\lambda = \alpha_{j,k}^\varepsilon = \sum_l a_{j,k,l}^{(\varepsilon,0)} \quad \text{and} \quad \beta_\lambda = \beta_{j,k}^\varepsilon = \sum_l a_{j,k,l}^{(0,\varepsilon)}, \tag{2.4}$$

let

$$C^\alpha = \sup_{Q\subset\Omega} |Q|^{-1} \sum_{Q(j,k)\subset Q} 2^{-j}|\alpha_{j,k}^\varepsilon|^2, \quad C^\beta = \sup_{Q\subset\Omega} |Q|^{-1} \sum_{Q(j,k)\subset Q} 2^{-j}|\beta_{j,k}^\varepsilon|^2,$$

one presents the following definition:

DEFINITION 2. *$T \in OpNF_\gamma$, if $T \in OpNS_\gamma$ and $T$ satisfies the following condition:*

$$C^\alpha + C^\beta < +\infty. \tag{2.5}$$

Then one has:

THEOREM 3. *$T \in OpF_\gamma \Leftrightarrow T \in OpNF_\gamma$.*

In fact, the conditions (1.1) and (1.2) can be weakened. One denotes:

$$A(R) = \sup_{\varepsilon,\varepsilon',j,k} \sum_{l:2^{R-1}\le|k-l|<2^R} \{|a_{j,k,l}^{(\varepsilon,\varepsilon')}| + |a_{j,l,k}^{(\varepsilon,\varepsilon')}|\}. \tag{2.6}$$

THEOREM 4. *If $T$ satisfies the conditions* (1.3), (1.4) *and the following condition:*

$$\sum_{R=2}^{\infty} R^{1/2} A(R) < \infty, \tag{2.7}$$

*then $T$ is continuous from $L^2$ to $L^2$.*

When one proves this theorem, one can choose the Haar wavelets. If one chooses the Haar wavelets, then the Condition (2.7) is a slightly weaker than the following condition:

$$\sum_{k=1}^{\infty} k^{1/2}\varepsilon(k) < \infty. \tag{2.8}$$

Hence one has improved the Meyer's result.

$\forall x \in R$, let $\Phi^{(0)}(x) = \chi(x)$ denote the characteristic function of interval $[0,1)$ and $\Phi^{(1)}(x) = h(x) = \chi(2x) - \chi(2x+1)$. Then $\{\Phi^{(\varepsilon)}_{j,k}(x)\}_{(\varepsilon,j,k)\in\tilde{\Lambda}}$ is a Haar wavelets basis. $\forall \varepsilon \in \{0,1\}^n \setminus \{0\}$, let $i_\varepsilon$ be the smallest index $i$ such that $\varepsilon_i \neq 0$, let $\tilde{\Phi}^{(\varepsilon)}(x) = \prod_{i=1}^{-1+i_\varepsilon} \Phi^{(\varepsilon_i)}(x_i)\chi(2x_{i_\varepsilon})\prod_{i=1+i_\varepsilon}^{n} \Phi^{(\varepsilon_i)}(x_i)$, let $\tilde{Q}^{\varepsilon}_{j,k} = \{x = (x_1, \cdots, x_n), x_i \in [2^{-j}k_i, 2^{-j}(k_i+1))$, if $i \neq i_\varepsilon$; $x_i \in [2^{-j}k_i, 2^{-j}(k_i+1/2))$, if $i = i_\varepsilon\}$, let $\tilde{x}_\varepsilon = (x_1, \cdots, x_n)$ where $x_i = 0$, if $i \neq i_\varepsilon$; $x_i = 1$, if $i = i_\varepsilon$, then one has:

THEOREM 5. $A(R) \leq 3\varepsilon(R-1)$.

PROOF. If $\varepsilon \neq 0$, then one has:

$$\sum_{l:2^{R-1}\leq|k-l|<2^R} |a^{(\varepsilon,\varepsilon')}_{j,k,l}| = \sum_{l:2^{R-1}\leq|k-l|<2^R} 2^{jn}\left|\iint K(x,y)\Phi^{(\varepsilon)}(2^jx-k)\Phi^{(\varepsilon')}(2^jy-l)dxdy\right|.$$

Here one uses the Haar basis, then one has:

$$\begin{aligned}
&\sum_{l:2^{R-1}\leq|k-l|<2^R} |a^{(\varepsilon,\varepsilon')}_{j,k,l}| \\
&\leq \sum_{l:2^{R-1}\leq|k-l|<2^R} 2^{jn}\left|\iint (K(x,y) - K(x+2^{-1-j}\tilde{x}_\varepsilon, y))\tilde{\Phi}^{(\varepsilon)}(2^jx-k)\right. \\
&\quad \left.\times\, \Phi^{(\varepsilon')}(2^jy-l)dxdy\right| \\
&\leq \sum_{l:2^{R-1}\leq|k-l|<2^R} 2^{jn}\iint |K(x,y) - K(x+2^{-1-j}\tilde{x}_\varepsilon, y)|\,|\tilde{\Phi}^{(\varepsilon)}(2^jx-k)| \\
&\quad \times |\Phi^{(0)}(2^jy-l)|dxdy \\
&\leq \sup_{j,k}\sup_{x\in\tilde{Q}^{\varepsilon}_{j,k}} \int_{2^{R-2-j}\leq|x-y|\leq 2^{R+1-j}} |K(x,y) - K(x+2^{-1-j}\tilde{x}_\varepsilon, y)|dy \\
&\leq \sup_{x\in R^n} \int_{2^{R-2-j}\leq|x-y|\leq 2^{R+1-j}} \{|K(x,y) - K(x+2^{-1-j}\tilde{x}_\varepsilon, y)|\}dy\,.
\end{aligned}$$

One decomposes the integral for $y$ into three parts: $2^{R-2-j} \leq |x-y| \leq 2^{R-1-j}$, $2^{R-1-j} \leq |x-y| \leq 2^{R-j}$ and $2^{R-j} \leq |x-y| \leq 2^{R+1-j}$. According to the definition of $\varepsilon(R)$, one has: $\sum_{l:2^{R-1}|k-l|<2^R} |a^{(\varepsilon,\varepsilon')}_{j,k,l}| \leq 3\varepsilon(R-1)$. Similarly, one can obtain other estimates and end the proof of Theorem 5.

In the following section, one proves some important lemmas; in the Section 4, one proves Theorem 2; and in the last section, one proves the first Theorem 4, and ends with the proof of Theorem 3.

## 3. Some important lemmas.

Denotes $B(x_0, R)$ a ball with the centre $x_0$ and the radius $R$. $\forall\varepsilon \in \{0,1\}^n \setminus \{0\}$, $\forall\mu(x)$, let $I_\varepsilon\mu(x) = \int_{-\infty}^{x i_\varepsilon} \mu(x_1, \cdots, x_{-1+i_\varepsilon}, y_1, x_{1+i_\varepsilon}, \cdots, x_n)dy_1$, let $\mu_{j,k}(x) \in l^1 \cap l^\infty$, $\nu_{j,k}(x) \in$

$l^\infty$, Supp $\nu_{j,k}(x) \subset B(0, M)$ and $I_\varepsilon \nu_{j,k}(x) \in l^\infty$, Supp $I_\varepsilon \nu_{j,k}(x) \subset B(0, M)$, one denotes $K_{\mu,\nu}(x, y) = \sum_{j,k} 2^{jn} \mu_{j,k}(2^j x - k)\nu_{j,k}(2^j y - k)$. Then one has:

LEMMA 1. *$\forall f(x) \in C_0^1(B(x_0, R))$, $g(x) \in l^\infty(B(x_0, R))$, one has*:

$$\left| \iint K_{\mu,\nu}(x, y) g(x) f(y) dx dy \right| \le C R^n (\|f\|_\infty + R\|f'\|_\infty)\|g\|_\infty .$$

PROOF. Let $[a]$ denote the integer part of $a$. Then one has:

$$\begin{aligned}
&\left| \iint K_{\mu,\nu}(x, y) g(x) f(y) dx dy \right| \\
&\quad = \left| \iint \sum_{j,k} 2^{jn} \mu_{j,k}(2^j x - k)\nu_{j,k}(2^j y - k) g(x) f(y) dx dy \right| \\
&\quad \le \iint \sum_{j \le -[\log_2 R]} \sum_k 2^{jn} |\mu_{j,k}(2^j x - k)| |\nu_{j,k}(2^j y - k)| |g(x)| |f(y)| dx dy \\
&\qquad + \left| \iint \sum_{j > -[\log_2 R]} \sum_k 2^{jn} \mu_{j,k}(2^j x - k)\nu_{j,k}(2^j y - k) g(x) f(y) dx dy \right| \\
&\quad = I_1 + I_2 .
\end{aligned}$$

As for $I_1$, one has:

$$I_1 \le C \sum_{j \le -[\log_2 R]} 2^{jn} \|f\|_\infty \|g\|_\infty R^{2n} \le C R^n \|f\|_\infty \|g\|_\infty .$$

As for $I_2$, one has:

$$\begin{aligned}
I_2 &\le \left| \iint \sum_{j > -[\log_2 R]} \sum_k 2^{j(n-1)} \mu_{j,k}(2^j x - k)(I_\varepsilon \nu_{j,k})(2^j y - k) g(x) f'(y) dx dy \right| \\
&\le C \sum_{j > -[\log_2 R]} 2^{-j} \sup \int 2^{jn} |\mu_{j,k}(2^j x - k)| dx R^n \|f'\|_\infty \|g\|_\infty \\
&\le C R^{n+1} \|f'\|_\infty \|g\|_\infty .
\end{aligned}$$

This ends the proof of Lemma 1.

LEMMA 2. (i) *If $T$ satisfies the weak boundedness condition* (1.3), *then one has*:

(3.1) $$|a_{j,k,l}^{(\varepsilon,\varepsilon')}| \le C , \quad \forall(\varepsilon, \varepsilon', j, k, l) \in \Lambda \quad \textit{and} \quad |k - l| \le 8^n M^n .$$

(ii) *If $T$ satisfies the condition* (1.2), *then one has*:

(3.2) $$|a_{j,k,l}^{(\varepsilon,\varepsilon')}| \le \frac{C}{(1 + |k - l|)^{n+\gamma}} , \quad \forall(\varepsilon, \varepsilon', j, k, l) \in \Lambda \quad \textit{and} \quad |k - l| > 8^n M^n .$$

PROOF. Because that (3.1) is evident, one proves (3.2). For $\varepsilon \in \{0, 1\}^n \setminus \{0\}$, one has $\int \Phi^{(\varepsilon)}(2^j x - k) dx = 0$, hence one has:

$$a_{j,k,l}^{(\varepsilon,\varepsilon')} = 2^{jn} \iint (K(x, y) - K(2^{-j}k, y))\Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l) dx dy .$$

Using the condition (1.2), one gets:

$$|a_{j,k,l}^{(\varepsilon,\varepsilon')}| \le C2^{jn} \iint \frac{|x-2^{-j}k|^{\gamma}}{|x-y|^{n+\gamma}} |\Phi^{(\varepsilon)}(2^j x - k)|\, |\Phi^{(\varepsilon')}(2^j y - l)| dxdy\,.$$

Since that $\operatorname{Supp} \Phi^{(\varepsilon)}(x) \subset [-M, M]^n$ and $|k-l| > 8^n M^n$, then one has: $|x-y| \sim 2^{-j}|k-l|$. So one gets:

$$\begin{aligned} |a_{j,k,l}^{(\varepsilon,\varepsilon')}| &\le \frac{C2^{2jn}}{(1+|k-l|)^{n+\gamma}} \iint |2^j x - k|^{\gamma} |\Phi^{(\varepsilon)}(2^j x - k)|\, |\Phi^{(\varepsilon')}(2^j y - l)| dxdy \\ &\le \frac{C}{(1+|k-l|)^{n+\gamma}}\,. \end{aligned}$$

If $\varepsilon = 0$, then $\varepsilon' \ne 0$, one uses the same reason, one gets the estimation for $\varepsilon = 0$. □

LEMMA 3. *The following two conditions are equivalent:*

$$\sum_{(\varepsilon,j,k)\in\tilde{\Lambda}} a_{j,k}^{\varepsilon} \Phi^{(\varepsilon)}(2^j x - k) \in BMO\,. \tag{3.3}$$

$$C^a = \sup_{Q\in\Omega} |Q|^{-1} \sum_{Q_{j,k}\subset Q} 2^{-j} |a_{j,k}^{\varepsilon}|^2 < +\infty\,. \tag{3.4}$$

One can find the proof of this Lemma in Chapter 5 of tome 1 in [4].

The following important lemma is famous in the proof of the famous $T(1)$ theorem:

LEMMA 4. *If $\{a_{j,k}^{\varepsilon}\}_{(\varepsilon,j,k)\in\tilde{\Lambda}}$ satisfies the condition (3.3), then $K_a(x,y) = \sum_{(\varepsilon,j,k)\in\tilde{\Lambda}} 2^{nj} a_{j,k}^{\varepsilon} \Phi^{(\varepsilon)}(2^j x - k)\Phi^{(0)}(2^j y - k)$ defines an operator $T^a$, which is continuous from $L^2$ to $L^2$.*

One can find the proof of this Lemma in Chapter 8 of tome 2 in [4].

Let $\sum_k \{|a(k,l)| + |a(l,k)|\} \le M_a$, then one has the following Schur Lemma:

LEMMA 5.

$$\sum_l \left| \sum_k a(k,l) a(k) \right|^2 \le M_a^2 \sum_k |a(k)|^2\,.$$

PROOF. In fact, one has: $\sum_l |\sum_k a(k,l)a(k)|^2 \le \sum_l \{\sum_k |a(k,l)|\,|a(k)|^2 \sum_k |a(k,l)|\}$. Then one applys the condition: $\sum_k\{|a(k,l)| + |a(l,k)|\} \le M_a$, one gets the conclusion of this Lemma. □

Finally, one presents another important Lemma. Let $\{a_{k,l}\}_{(k,l)\in Z^n\times Z^n}$, $\{b_{k,l}\}_{(k,l)\in Z^n\times Z^n}$ be two series of numbers such that: $a_{k,l}$ satisfies the following condition: $a_{k,l} = 0$, if $|k-l| \ge 2^R$, $\sum_l a_{k,l} = 0$, $\forall k \in Z^n$, $\sum_l\{|a_{k,l}| + |a_{l,k}|\} \le A(R)$; and $b_{k,l}$ satisfies the following condition: $\sum_l\{|b_{k,l}| + |b_{l,k}|\} \le B(R)$. Let $\Phi(x) \in l^{\infty}$, and $\operatorname{Supp} \Phi_a(x) \subset B(0,M)$, $\Phi_b(x) \in C_0^1(B(0,M))$, then one has the following lemma:

LEMMA 6.

$$\left|\int \sum_{k,l} a_{k,l} a_k \Phi_a(x-l) \sum_{k',l'} b_{k',l'} b_{k'} \Phi_b(2^j x - l') dx\right|$$
$$\leq C 2^{j(1-n/2)+R} A(R) B(R) \left(\sum_k |a_k|^2\right)^{1/2} \left(\sum_k |b_k|^2\right)^{1/2}, \quad \forall j \leq -R.$$

PROOF.

$$\left|\int \sum_{k,l} a_{k,l} a_k \Phi_a(x-l) \sum_{k',l'} b_{k',l'} b_{k'} \Phi_b(2^j x - l') dx\right|$$
$$= \left|\int \sum_k \sum_{0<|m|<2^R} a_{k,k+m} a_k (\Phi_a(x-k+m) - \Phi_a(x-k)) \sum_{k',l'} b_{k',l'} b_{k'} \Phi_b(2^j x - l') dx\right|$$
$$= \left|\int \sum_k \sum_{0<|m|<2^R} a_{k,k+m} a_k \Phi_a(x-k) \sum_{k',l'} b_{k',l'} b_{k'} \right.$$
$$\left. \times (\Phi_b(2^j x - 2^j m - l') - \Phi_b(2^j x - l')) dx\right|$$
$$\leq \int \sum_k \sum_{0<|m|<2^R} |a_{k,k+m}||a_k||\Phi_a(x-k)| \sup_{0<|m|<2^R} \sum_{k',l'} |b_{k',l'}|\,|b_{k'}|$$
$$\times |\Phi_b(2^j x - 2^j m - l') - \Phi_b(2^j x - l')| dx$$
$$\leq \left\| \sum_k \sum_{0<|m|<2^R} |a_{k,k+m}||a_k||\Phi_a(x-k)| \right\|_{L^2} \left\| \sup_{0<|m|<2^R} \sum_{k',l'} |b_{k',l'}|\,|b_{k'}| \right.$$
$$\left. \times |\Phi_b(2^j x - 2^j m - l') - \Phi_b(2^j x - l')| \right\|_{L^2}.$$

Since $\Phi_a(x) \in l^\infty$, and $\operatorname{Supp} \Phi_a(x) \subset B(0, M)$, then

$$\left\| \sum_k \sum_{0<|m|<2^R} |a_{k,k+m}||a_k||\Phi_a(x-k)| \right\|_{L^2} \leq C \left\{ \sum_k \left( \sum_{0<|m|<2^R} |a_{k,k+m}||a_k| \right)^2 \right\}^{1/2}.$$

One applys Lemma 5, one gets:

$$\left\| \sum_k \sum_{0<|m|<2^R} |a_{k,k+m}||a_k||\Phi_a(x-k)| \right\|_{L^2} \leq C A(R) \left( \sum_k |a_k|^2 \right)^{1/2}.$$

Since $\Phi_b(x) \in C_0^1(B(0, M))$, then there exists a function $\tilde{\Phi}_b(x) \in l^\infty$, and $\operatorname{Supp} \tilde{\Phi}_b(x) \subset B(0, 2M)$, such that

$$\sup_{0<|m|<2^R} |\Phi_b(x - 2^j m) - \Phi_b(x)| \leq 2^{j+R} \tilde{\Phi}_b(x).$$

Then one has:

$$\left\| \sup_{0<|m|<2^R} \sum_{k',l'} |b_{k',l'}\|b_{k'}\|\Phi_b(2^j x - 2^j m - l') - \Phi_b(2^j x - l')| \right\|_{L^2}$$
$$\leq 2^{j+R} \left\| \sup_{0<|m|<2^R} \sum_{k',l'} |b_{k',l'}\|b_{k'}\|\tilde{\Phi}_b(2^j x - l')| \right\|_{L^2} .$$

One applys the same reason as above, one gets:

$$\left\| \sup_{0<|m|<2^R} \sum_{k',l'} |b_{k',l'}\|b_{k'}\|\Phi_b(2^j x - 2^j m - l') - \Phi_b(2^j x - l')| \right\|_{L^2}$$
$$\leq C 2^{j(1-n/2)+R} B(R) \left( \sum_k |b_k|^2 \right)^{1/2} .$$

This ends the proof of the Lemma. □

REMARK 1. If one chooses $\Phi_a(x) = \Phi_b(x) = \chi(x)$, this lemma is also true.

## 4. Characterization of $OpS_\gamma$.

One returns to the proof of Theorem 2.

One applys Lemma 1, one gets that: $T \in OpS_\gamma$ implies that $T \in OpNS_\gamma$.

We shall show that $T \in OpS_\gamma$ implies that $T \in OpNS_\gamma$. Let $K_{\varepsilon,\varepsilon'}(x, y) = \sum_{(\varepsilon,\varepsilon',j,k,l)\in\Lambda} 2^{jn} a_{j,k,l}^{(\varepsilon,\varepsilon')} \Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l)$, according to the Lemma 2, then $T \in OpNS_\gamma$ implies that $K_{\varepsilon,\varepsilon'}(x, y)$ satisfies the condition (1.3). Let $K(x, y) = \sum_{(\varepsilon,\varepsilon',j,k,l)\in\Lambda} 2^{jn} a_{j,k,l}^{(\varepsilon,\varepsilon')} \Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l)$, one proves $T \in OpNS_\gamma$ implies the conditions (1.1) and (1.2). In fact, one has:

$$\begin{aligned}
|K(x, y)| &= \left| \sum_{(\varepsilon,\varepsilon',j,k,l)\in\Lambda} 2^{jn} a_{j,k,l}^{(\varepsilon,\varepsilon')} \Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l) \right| \\
&\leq \sum_{\substack{(\varepsilon,\varepsilon',j,k,l)\in\Lambda \\ j\geq -\log_2|x-y|}} 2^{jn} |a_{j,k,l}^{(\varepsilon,\varepsilon')}\|\Phi^{(\varepsilon)}(2^j x - k)\|\Phi^{(\varepsilon')}(2^j y - l)| \\
&\quad + \sum_{\substack{(\varepsilon,\varepsilon',j,k,l)\in\Lambda \\ j<-\log_2|x-y|}} 2^{jn} |a_{j,k,l}^{(\varepsilon,\varepsilon')}\|\Phi^{(\varepsilon)}(2^j x - k)\|\Phi^{(\varepsilon')}(2^j y - l)| \\
&= I_1 + I_2 .
\end{aligned}$$

As for $I_1$, one applys the fact: if $\Phi^{(\varepsilon)}(x) \neq 0$, then $|x| \leq 8^n M^n$, one gets: if $j \geq -\log_2|x - y|$, and $\Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l) \neq 0$, then $1/(1 + |k - l|) \sim 1/(2^j|x - y|)$. Hence, one has:

$$I_1 \leq \sum_{j\geq -\log_2|x-y|} \frac{C 2^{jn}}{(2^j|x - y|)^{n+\gamma}} \leq \sum_{j\geq -\log_2|x-y|} \frac{C 2^{-j\gamma}}{|x - y|^{n+\gamma}} \leq \frac{C}{|x - y|^n} .$$

As for $I_2$, for each $j \in Z$, there exist at most a finite number of $k$ and a finite number of $l$ such that $\Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l) \neq 0$, then one has:

$$I_2 \leq \sum_{j<-\log_2|x-y|} C2^{jn} \leq \frac{C}{|x-y|^n}.$$

Finally one proves (1.2). For $|x-x'| \leq \frac{1}{2}|x-y|$, one denotes:

$$\begin{aligned} I_1 = &\Bigg|\sum_{\substack{(\varepsilon,\varepsilon',j,k,l)\in\Lambda \\ j\geq -\log_2|x-x'|}} 2^{jn} a_{j,k,l}^{(\varepsilon,\varepsilon')} \Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l)\Bigg| \\ &+ \Bigg|\sum_{\substack{(\varepsilon,\varepsilon',j,k,l)\in\Lambda \\ j\geq -\log_2|x-x'|}} 2^{jn} a_{j,k,l}^{(\varepsilon,\varepsilon')} \Phi^{(\varepsilon)}(2^j x' - k)\Phi^{(\varepsilon')}(2^j y - l)\Bigg|, \end{aligned}$$

$$I_2 = \Bigg|\sum_{\substack{(\varepsilon,\varepsilon',j,k,l)\in\Lambda \\ -\log_2|x-y|\leq j<-\log_2|x-x'|}} 2^{jn} a_{j,k,l}^{(\varepsilon,\varepsilon')} (\Phi^{(\varepsilon)}(2^j x - k) - \Phi^{(\varepsilon)}(2^j x' - k))\Phi^{(\varepsilon')}(2^j y - l)\Bigg|,$$

$$I_3 = \Bigg|\sum_{\substack{(\varepsilon,\varepsilon',j,k,l)\in\Lambda \\ j<-\log_2|x-y|}} 2^{jn} a_{j,k,l}^{(\varepsilon,\varepsilon')} (\Phi^{(\varepsilon)}(2^j x - k) - \Phi^{(\varepsilon)}(2^j x' - k))\Phi^{(\varepsilon')}(2^j y - l)\Bigg|.$$

Then one has: $|K(x,y) - K(x',y)| \leq I_1 + I_2 + I_3$.

As for $I_1$, one applys the fact: if $\Phi^{(\varepsilon)}(x) \neq 0$, then $|x| \leq 8^n M^n$, one gets: if $j \geq -\log_2|x-x'|$, and $\Phi^{(\varepsilon)}(2^j x - k)\Phi^{(\varepsilon')}(2^j y - l) \neq 0$, then $1/(1+|k-l|) \leq C/(2^j|x-y|)$; and if $j \geq -\log_2|x-x'|$, and $\Phi^{(\varepsilon)}(2^j x' - k)\Phi^{(\varepsilon')}(2^j y - l) \neq 0$, then $1/(1+|k-l|) \leq 1/(2^j|x'-y|)$; one has:

$$\begin{aligned} I_1 &\leq \sum_{j\geq -\log_2|x-x'|} \left\{\frac{C2^{jn}}{(2^j|x-y|)^{n+\gamma}} + \frac{C2^{jn}}{(2^j|x'-y|)^{n+\gamma}}\right\} \\ &\leq C \sum_{j\geq -\log_2|x-x'|} 2^{-j\gamma}\left\{\frac{1}{|x-y|^{n+\gamma}} + \frac{1}{|x'-y|^{n+\gamma}}\right\} \\ &\leq \frac{C|x-x'|^\gamma}{|x-y|^{n+\gamma}}. \end{aligned}$$

As for $I_2$, one applys the fact: $|a_{j,k,l}^{(\varepsilon,\varepsilon')}| \leq C/((1+|k-l|)^{n+\gamma}) \leq C/((2^j|x-y|)^{n+\gamma})$, and $|\Phi^{(\varepsilon)}(2^j x - k) - \Phi^{(\varepsilon)}(2^j x' - k)| \leq C2^j|x-x'|$, one gets:

$$I_2 \leq \sum_{\substack{(\varepsilon,\varepsilon',j,k,l)\in\Lambda \\ -\log_2|x-y|\leq j<-\log_2|x-x'|}} 2^{jn}|a_{j,k,l}^{(\varepsilon,\varepsilon')}|\,|\Phi^{(\varepsilon)}(2^j x - k) - \Phi^{(\varepsilon)}(2^j x' - k)|\,|\Phi^{(\varepsilon')}(2^j y - l)|$$

$$\leq \sum_{-\log_2|x-y|\leq j<-\log_2|x-x'|} 2^{jn}\frac{C}{(2^j|x-y|)^{n+\gamma}}\cdot C2^j|x-x'|$$
$$\leq C\sum_{-\log_2|x-y|\leq j<-\log_2|x-x'|} 2^{j(1-\gamma)}\frac{|x-x'|}{|x-y|^{n+\gamma}}$$
$$\leq \frac{C|x-x'|^{\gamma}}{|x-y|^{n+\gamma}}.$$

As for $I_3$, for each $j\in Z$, there exist at most a finite number of $k$ and a finite number of $l$ such that $(\Phi^{(\varepsilon)}(2^jx-k)-\Phi^{(\varepsilon)}(2^jx'-k))\Phi^{(\varepsilon')}(2^jy-l)\neq 0$, one applys then $|\Phi^{(\varepsilon)}(2^jx-k)-\Phi^{(\varepsilon)}(2^jx'-k)|\leq C2^j|x-x'|$, one gets:

$$I_3\leq \sum_{\substack{(\varepsilon,\varepsilon',j,k,l)\in\Lambda\\ j<-\log_2|x-y|}} 2^{jn}|a^{(\varepsilon,\varepsilon')}_{j,k,l}|\,|\Phi^{(\varepsilon)}(2^jx-k)-\Phi^{(\varepsilon)}(2^jx'-k)|\,|\Phi^{(\varepsilon')}(2^jy-l)|$$
$$\leq \sum_{j<-\log_2|x-y|} C2^{j(n+1)}|x-x'|$$
$$\leq \frac{C|x-x'|}{|x-y|^{n+1}}$$
$$\leq \frac{C|x-x'|^{\gamma}}{|x-y|^{n+\gamma}}.\quad\square$$

## 5. $T(1)$ Theorem under a weaker kernel condition.

First, one proves Theorem 4. $\forall\varepsilon\in\{0,1\}^n$, one denotes $\mu(\varepsilon)=1$, if $\varepsilon=0$; $\mu(\varepsilon)=0$, if $\varepsilon\neq 0$; and one denotes $a^{(\varepsilon,\varepsilon',R)}_{j,k}=\mu(\varepsilon)\sum_{2^{R-1}\leq|k-m|<2^R}a^{(\varepsilon,\varepsilon')}_{j,m,k}+\mu(\varepsilon')\sum_{2^{R-1}\leq|k-m|<2^R}a^{(\varepsilon,\varepsilon')}_{j,k,m}$. Let $\{\alpha^{\varepsilon}_{j,k}\}_{(\varepsilon,j,k)\in\tilde\Lambda}$ and $\{\beta^{\varepsilon}_{j,k}\}_{(\varepsilon,j,k)\in\tilde\Lambda}$ be defined as in (2.4), one decomposes $T$ into a series of operators $T^{(\varepsilon,\varepsilon')}_R$ $(R\geq 0,\ (\varepsilon,\varepsilon')\in\{0,1\}^{2n}\setminus\{0\})$, where their kernel distribution $K^{(\varepsilon,\varepsilon')}_R(x,y)$ satisfies the following conditions:

(i) $R=0:\quad K^{(\varepsilon,\varepsilon')}_0(x,y)=\sum_{j,k}a^{(\varepsilon,\varepsilon')}_{j,k,k}\Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,k}(y)$, if $|\varepsilon|\,|\varepsilon'|\neq 0$;

$$K^{(\varepsilon,0)}_0(x,y)=\sum_{j,k}\alpha^{\varepsilon}_{j,k}\Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(0)}_{j,k}(y),$$
$$K^{(0,\varepsilon')}_0(x,y)=\sum_{j,k}\beta^{\varepsilon'}_{j,k}\Phi^{(0)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,k}(y).$$

(ii) $R>0:$
$$K^{(\varepsilon,\varepsilon')}_R(x,y)=\sum_j\sum_{k,l}\tilde a^{(\varepsilon,\varepsilon',R)}_{j,k,l}\Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,l}(y)$$
$$=\sum_j\sum_{2^{R-1}\leq|k-l|<2^R}a^{(\varepsilon,\varepsilon')}_{j,k,l}\Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,l}(y)-\sum_{j,k}a^{(\varepsilon,\varepsilon',R)}_{j,k}\Phi^{(\varepsilon)}_{j,k}(x)\Phi^{(\varepsilon')}_{j,k}(y).$$

If $T$ satisfies the condition (1.3), one applys Lemma 2, one has: $A(1) \leq C$. To prove Theorem 4, it is enough to prove that: $\forall(\varepsilon, \varepsilon') \in \{0, 1\}^{2n} \setminus \{0\}$, each $T_0^{(\varepsilon,\varepsilon')}$ is continuous from $L^2$ to $L^2$; and $\forall R \geq 1$, $(\varepsilon, \varepsilon') \in \{0, 1\}^{2n} \setminus \{0\}$, $\|T_R^{(\varepsilon,\varepsilon')}\|_{L^2 \to L^2} \leq CR^{1/2}A(R)$.

First, one considers the case where $R = 0$.

(i) Since that $T$ satisfies the condition (1.3), one applys Lemma 2, then one has $|a_{j,k,k}^{(\varepsilon,\varepsilon')}| \leq C$. Hence one has: if $R = 0$, $|\varepsilon|\,|\varepsilon'| \neq 0$, $\|T_0^{(\varepsilon,\varepsilon')} f\|_{L^2} \leq C\|f\|_{L^2}$.

(ii) Furthermore, $\{a_{j,k,l}^{(\varepsilon,\varepsilon')}\}_{(\varepsilon,\varepsilon',j,k,l)\in\Lambda}$ is a representation for an operator $T$, then one has $T1 = \sum_{(\varepsilon,j,k)\in\tilde{\Lambda}} \alpha_{j,k}^{\varepsilon} \Phi^{(\varepsilon)}(2^j x - k)$ and $T^t 1 = \sum_{(\varepsilon,j,k)\in\tilde{\Lambda}} \beta_{j,k}^{\varepsilon} \Phi^{(\varepsilon)}(2^j x - k)$.

Since $T$ satisfies the condition (1.4), one applys the Lemma 4, if $|\varepsilon|\,|\varepsilon'| = 0$, then $\|T_0^{(\varepsilon,0)} f\|_{L^2} \leq CC^{\alpha}\|f\|_{L^2}$, $\|T_0^{(0,\varepsilon')} f\|_{L^2} \leq CC^{\beta}\|f\|_{L^2}$.

Secondly, one considers the case where $R > 0$ and $|\varepsilon|\,|\varepsilon'| \neq 0$. One applys Lemma 5, one gets: $\|T_R^{(\varepsilon,\varepsilon')} f\|_{L^2} \leq CA(R)\|f\|_{L^2}$.

Finally, one considers the case where $R > 0$ and $|\varepsilon|\,|\varepsilon'| = 0$. One proves that: if $R > 0$ and $|\varepsilon|\,|\varepsilon'| = 0$, $\|T_R^{(\varepsilon,\varepsilon')} f\|_{L^2} \leq CR^{1/2}A(R)\|f\|_{L^2}$. By similarity, one supposes $\varepsilon = 0$.

In fact, $\forall f(x) = \sum_{\lambda\in\tilde{\Lambda}} a_\lambda \Phi_\lambda(x) \in L^2$, one has:

$$\begin{aligned}
\|T_R^{(0,\varepsilon')} f\|_{L^2}^2 &= \|\sum \tilde{a}_{j,k,l}^{(0,\varepsilon',R)} a_{j,l}^{(\varepsilon')} \Phi_{j,k}^{(0)}(x)\|_{L^2}^2 \\
&\leq \sum_{|j-j'|\leq 2R} \left| \int \sum \tilde{a}_{j,k,l}^{(0,\varepsilon',R)} a_{j,l}^{(\varepsilon')} \Phi_{j,k}^{(0)}(x) \sum \tilde{a}_{j',k',l'}^{(0,\varepsilon',R)} a_{j',l'}^{(\varepsilon')} \Phi_{j',k'}^{(0)}(x)dx \right| \\
&\quad + \sum_{|j-j'|>2R} \left| \int \sum \tilde{a}_{j,k,l}^{(0,\varepsilon',R)} a_{j,l}^{(\varepsilon')} \Phi_{j,k}^{(0)}(x) \sum \tilde{a}_{j',k',l'}^{(0,\varepsilon',R)} a_{j',l'}^{(\varepsilon')} \Phi_{j',k'}^{(0)}(x)dx \right| \\
&= I_1 + I_2 .
\end{aligned}$$

As for $I_1$, one has:

$$\begin{aligned}
I_1 &\leq \sum_{|j-j'|\leq 2R} \|\sum \tilde{a}_{j,k,l}^{(0,\varepsilon',R)} a_{j,l}^{(\varepsilon')} \Phi_{j,k}^{(0)}(x)\|_{L^2} \|\sum \tilde{a}_{j',k',l'}^{(0,\varepsilon',R)} a_{j',l'}^{(\varepsilon')} \Phi_{j',k'}^{(0)}(x)\|_{L^2} \\
&\leq \sum_{|j-j'|\leq 2R} \left(\sum_k \left|\sum_l \tilde{a}_{j,k,l}^{(0,\varepsilon',R)} a_{j,l}^{(\varepsilon')}\right|^2\right)^{1/2} \left(\sum_{k'} \left|\sum_{l'} \tilde{a}_{j',k',l'}^{(0,\varepsilon',R)} a_{j',l'}^{(\varepsilon')}\right|^2\right)^{1/2} .
\end{aligned}$$

One applys then Lemma 5, one gets: $I_1 \leq CRA^2(R)\|f\|_{L^2}^2$.

As for $I_2$, one applys then Lemma 6, one gets:

$$I_2 \leq C \sum_{|j-j'|>2R} 2^{R-|j-j'|} A^2(R)\|f\|_{L^2}^2 \leq CA^2(R)\|f\|_{L^2}^2 . \quad \square$$

Finally, one proves Theorem 3. In fact, if $T \in OpNF_\gamma$, then $T \in OpNS_\gamma$. One applys Theorem 2, $T \in OpS_\gamma$. Now, one proves that $T$ can extend to a bounded operator on $L^2$. If $T \in OpNF_\gamma$, then $T$ satisfies the condition (2.5). One applys Lemma 3, $T$ satisfies the condition (1.4). Furthermore, if $T \in OpNF_\gamma$, then $T \in OpNS_\gamma$. That is to say: $T$ satisfies the condition (1.3) and $A(R) \le C2^{-R\gamma}$. Hence $T$ satisfies the condition of Theorem 4, $T$ can extend to a bounded operator on $L^2$.

If $T \in OpF_\gamma$, then $T \in OpS_\gamma$, so $T \in OpNS_\gamma$. Now, one proves $T$ satisfies the condition (2.5). One applys Lemma 3, one proves that $T$ satisfies the condition (1.4). One proves that there exists a constant $C$ such that: for all the ball $B = B(x_0, R)$, there exists a constant $C_{x_0,R}$, such that $\int_{B(x_0,R)} |T1 - C_{x_0,R}|^2 dx \le CR^n$. Let $\alpha_B(x) \in C_0^\infty(B(x_0, 4R))$, $|\alpha_B(x)| \le 1$, if $x \in B(x_0, 2R)$, $\alpha_B(x) = 1$. Let $\beta_B(x) = 1 - \alpha_B(x)$, one has: $\int_{B(x_0,R)} |T1 - T\beta_B(x_0)|^2 dx \le 2\int_{B(x_0,R)} |T\alpha_B(x)|^2 dx + 2\int_{B(x_0,R)} |T\beta_B(x) - T\beta_B(x_0)|^2 dx$.

Since that $T$ is a bounded operator on $L^2$, then $\int_{B(x_0,R)} |T\alpha_B(x)|^2 dx \le CR^n$. Furthermore $T \in OpS_\gamma$, then $\int_{B(x_0,R)} |T\beta_B(x) - T\beta_B(x_0)|^2 dx \le CR^n$. Then $T1 \in BMO$. One applys the same reason, $T^t 1 \in BMO$. □

## References

[1] G. Beylkin, R. Coifman and V. Rokhlin, Fast wavelet transformation and numerical algorithm I, Comm. Pure. Appl. Math. **44** (1991), 141–183.

[2] G. David and J. L. Journé, A boundedness criterion for generalized Calderón-Zygmund operators, Ann. of Math. **120** (1984), 371–397.

[3] Y. Meyer, La minimalité de l'espace de Besov$\dot{B}_1^{0,1}$ et la continuité des opérateurs définis par des integrales singulières, Monografias de Matematicas, **4**, Universidad autónoma de Madrid.

[4] Y. Meyer, *Ondelettes et opérateurs I, II*, Hermann (1990–1991).

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