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# Time-Space Estimates of Solutions to General Semilinear Parabolic Equations

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**Abstract.** We study the Cauchy problem and the initial boundary value problem (IBVP) for nonlinear parabolic equations in $C_b([0,T);L^p)$ and $L^q(0,T;L^p)$. We give a unified method to construct local mild solutions of the Cauchy problem or IBVP for a class of nonlinear parabolic equations in $C_b([0,T);L^p)$ or $L^q(0,T;L^p)$ by introducing admissible triplet, generalized admissible triplet and establishing time space estimates for the solutions to the linear parabolic equations. Moreover, using our method, we also obtain the existence of global small solutions to the nonlinear parabolic equations.

## 1. Introduction.

In this paper we study the following nonlinear parabolic equation

$$u_t + Au = F(u, \partial u, \cdots, \partial^{2m-1}u)\,, \quad A = \sum_{|\alpha|\le 2m} a_\alpha(x)\partial^\alpha \tag{1.1}$$

subject to

$$u(0) = \varphi(x)\,, \quad x \in \mathbf{R}^n\,, \tag{1.2}$$

or

$$\begin{cases} u(0) = \varphi(x)\,, & x \in \Omega \\ \left.\dfrac{\partial^j u}{\partial \nu^j}\right|_{\partial\Omega} = 0\,, & j \le m-1\,, \end{cases} \tag{1.3}$$

where $\partial^k = \{\partial^\alpha, |\alpha| = k\}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, $C_\alpha$ or $C_j$ denote real (or complex) constants, $\Omega \subset \mathbf{R}^n$ is a bounded smooth domain with boundary $\partial\Omega$, $\nu$ denotes the unit outward normal to $\partial\Omega$. $u(x,t) \equiv u(t)$ is a real (or complex) valued function defined on $[0,\infty) \times \mathbf{R}^n$ or $[0,\infty) \times \Omega$. $a_\alpha(x) \in C_b^\infty(\mathbf{R}^n)$ or $C_b^\infty(\Omega)$ such that

$$(-1)^m \mathrm{Re} \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha \ge C_0|\xi|^{2m}\,, \quad C_0 > 0\,, \quad x \in \mathbf{R}^n \quad \text{or } x \in \bar{\Omega}\,. \tag{1.4}$$

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The nonlinear term has the form of

$$F(u, \partial u, \cdots, \partial^{2m-1}u) = \sum_{|\beta|\le 2m-1} C_\beta \partial^\beta f_\beta(u), \tag{1.5}$$

satisfying $f_\beta \in C^{|\beta|}(\mathbf{R}, \mathbf{R})$ and

$$\begin{gathered} |f_\beta(u) - f_\beta(v)| \le \lambda(1 + |u|^{\theta_\beta} + |v|^{\theta_\beta})|u - v|, \quad f_\beta(0) = 0, \\ \theta_\beta = b_\beta\left(1 - \frac{|\beta|}{2m}\right), \quad |\beta| \le 2m - 1, \end{gathered} \tag{1.6}$$

where $b_\beta \in \mathbf{R}^+$. Moreover, when $\Omega = \mathbf{R}^n$, let

$$(-\Delta)^{\frac{j}{2}} v = \mathcal{F}^{-1}|\xi|^j \mathcal{F} v .$$

Then we can also deal with the following nonlinear term

$$F(u, \partial u, \cdots, \partial^\alpha u) = \sum_{j=0}^{2m-1} C_j(-\Delta)^{\frac{j}{2}} f_j(u), \tag{1.5'}$$

where $f_j \in C^j(\mathbf{R}, \mathbf{R})$ satisfy

$$\begin{gathered} |f_j(u) - f_j(v)| \le \lambda(1 + |u|^{\theta_j} + |v|^{\theta_j})|u - v|, \quad f_j(0) = 0, \\ \theta_j = b_j\left(1 - \frac{j}{2m}\right), \quad j \le 2m - 1, \quad \theta_j \in \mathbf{R}^+ . \end{gathered} \tag{1.6'}$$

Hence if we replace $\theta_\beta$ with $\theta_j$, the similar results in Theorem 2.1–2.3 are still valid, see Section 2.

The main purpose of this paper is to give a unified method to treat the Cauchy problem and the IBV problem for general parabolic equations. This paper consists of three parts: First, we shall introduce the admissible triplet and generalized admissible triplet, and estiablish a series of time space estimates for the solutions to linear parabolic equations based on the $L^p$ $L^r$ estimates of analytic semigroups and the harmonic analysis method. One can find these time space estimates are different from the time-space estimates of the wave or dispersive wave equation. Of course, some basic estimates are due to Giga and Weissler [6, 8, 10, 11, 17–19], but we give some new time-space estimates which are suitable for the study of (1.1), (1.2) and (1.1), (1.3). Second, We study the local existence and uniqueness of solutions to (1.1), (1.2) and (1.1), (1.3) in $C$-spaces and $L$-spaces using the time-space estimates and other analytical techniques. Finally, we obtain the existence of global small solutions of (1.1), (1.2) and (1.1), (1.3) provided that

$$\begin{gathered} |f_\beta(u) - f_\beta(v)| \le \lambda(|u|^{\theta_\beta} + |v|^{\theta_\beta})|u - v|, \quad f_\beta(0) = 0, \\ \theta_\beta = b_\beta\left(1 - \frac{|\beta|}{2m}\right), \quad b_\beta > \frac{2m}{n} . \end{gathered} \tag{1.7}$$

When $m = 1$, $A = -\Delta$, $F(u, \partial u) \triangleq \lambda |u|^{b_0} u$, the Cauchy problem (1.1), (1.2) and IBVP (1.1), (1.3) were extensively studied by many authors, see [5], [6], [17–19]. Their main results can be stated as

(i) For $\varphi(x) \in L^r(\Omega)$ or $L^r(\mathbf{R}^n)$, there is a local solution $u(t) \in \mathcal{C}_b(0, T; L^p)$ to (1.1), (1.2) and (1.1), (1.3), where $p \geq r > 1$ satisfies suitable conditions which are dependent on the nonlinear term $f(u)$. When $p = r$ is chosen as the critical exponent ($p = r = nb_0/2 > 1$), the solution $u(t)$ can be extended to a global solution provided that $\|\varphi\|_r$ is sufficiently small [5, 6, 17, 19].

(ii) For $\varphi(x) \in L^r$, $p \geq r$, $q > r$, $1/q = (n/2)(1/r - 1/p) \geq 0$, there exists at most a solution $u(t) \in L^q(0, T; L^p)$ to (1.1), (1.2) and (1.1), (1.3). In particular, when $r$ is equal to the critical exponent, $T = \infty$ provided that $\|\varphi\|_r$ is sufficiently small. Of course, the uniqueness of solutions in $L^q(0, T; L^p)$ or $\mathcal{C}_b([0, T), L^p)$ is also valid under suitable conditions for $p, q, r$.

Recently Ginibre and Velo in [7] studied the Cauchy problem in local spaces for the complex Ginzburg-Landau equation ($m = 1$) and pointed out that there are two kinds of spaces suitable for the study of the Cauchy problem and IBVP of parabolic equations. The first one refers to using the space such as $t^{1/q} u(t) \in \mathcal{C}(I; L^p)$ for suitable $p, q, r$ to study (1.1), (1.2) or (1.1), (1.3) with data in $L^r$, we call it as $\mathcal{C}$ space theory; another one refers to using function space $L^q(I; L^p)$ to study (1.1), (1.2) or (1.1), (1.3) with $L^r$ data, we usually call it as $L$ space theory. Both kinds of spaces have their advantages. $\mathcal{C}$ space theory exploits the regularizing property of the equation and yields in particular the fact that the solutions are more regular than the initial data for $t > 0$, thereby yielding better result of existences. On the contrary, $L$-space theory describes translation invariant regularity in time and allows at each time $t$ for singularities that are dimensionally equivalent to those allowed by $\mathcal{C}$-spaces at $t = 0$, thereby yielding better result of uniqueness results. As for the results of uniqueness, see Remark 2.2 (iii).

We find that the relations of exponents in the statements and proofs of the main results in [6, 7, 10, 11] are very complicated and difficult for the reader to understand them. To overcome this difficulty we introduce the definition of admissible triplet, generalized admissible triplet and give a unified method to deal with the Cauchy problem and IBVP for general parabolic equations. One easily sees that our results will improve and extend the known results even in simple cases, see [5, 6, 7, 10, 11, 17–19]. Some ideas in this paper were inspired by the work in [6, 7, 18, 10–11].

This paper is organized as follows. In section 2, we shall give our main results and some remarks. Section 3 is devoted to establishing a series of time-space estimates for the solutions to the linear parabolic equations, which are necessary for the study of (1.1), (1.2) and (1.1), (1.3). In section 4, we first give some necessary nonlinear estimates by time-space estimates, then we study the local existence and uniqueness of the problem (1.1), (1.2) and (1.1), (1.3) in $\mathcal{C}$-spaces and in $L$-spaces. In section 5, we establish the existence of global small solutions for the Cauchy problem (1.1), (1.2) and IBVP (1.1), (1.2) by time-space estimates and choosing sutiable work-spaces.

We conclude this introduction with several notations. For $1 \le p \le \infty$, $L^p$ denotes the standard Lebegue space with norm $\|\cdot\|_p$. $C_b([0,T);X)$ denotes the space of bounded continuous functions which define on $[0,T)$ and take value in Banach space $X$ with norm $\max_{0\le t<T}\|\cdot\|_X$. $L^q([0,T);L^p)$ denotes the time-space Lebesgue space with norm $\|\cdot\|_{p,q,T} = (\int_0^T \|\cdot\|_p^q dt)^{1/q}$. We shall denote by $\mathcal{L}(X;Y)$ the space of all bounded linear operators from the Banach space $X$ to another Banach space $Y$.

## 2. Main results.

It is well known that solving (1.1), (1.2) or (1.1), (1.3) in the spaces $C(I;L^p)$ and $L^p(I;L^p)$ is equivalent to solving the following abstract Cauchy problem of evolution equation

$$\begin{cases} u_t + Au = F(u, \partial u, \cdots, \partial^{2m-1}u), & x \in \Omega \text{ or } \mathbf{R}^n, \quad t \in [0,T) \\ u(0) = \varphi(x), & \varphi \in D(A), \end{cases} \tag{2.1}$$

where $D(A) = W^{2m,p}(\mathbf{R}^n)$ if $\Omega = \mathbf{R}^n$, $D(A) = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ if $\Omega \subset \mathbf{R}^n$ is a bounded smooth domain. One casily see that $A$ generates and analytic semigroup. Without loss of generality, we assume that

$$\mathcal{R}(\lambda, A)\big| \le \frac{M}{|\lambda|+1}, \quad \mathrm{Re}\,\lambda \le 0. \tag{2.2}$$

If not so, we put $v = c^{-\omega t}u$ and the respective operator become $\mathcal{A} = \omega + A$ for suitable $\omega > 0$. It can be easily seen that $\mathcal{A}$ satisfies (2.2), see [4, 12, 13]. By this way, we have $0 \in \rho(A)$ and an equivalent norm of Sobolev space $W^{2m,p}$ as

$$\|u\|_{W^{2m,p}} = \|Au\|_p .$$

As usual, we study (2.1) via the corresponding integral equation

$$\begin{aligned} u &= e^{-tA}\varphi(x) + \int_0^5 e^{-(t-s)A}F(u, \partial u, \cdots, \partial^{2m-1}u)ds \\ &\triangleq e^{-tA}\varphi(x) + \mathcal{J}u . \end{aligned} \tag{2.3}$$

Usually we define the solution in function space $C_b(I;L^p)$ or $L^q(I;L^p)$ as a mild solution of (2.1). We shall construct the mild solution only in these spaces because it can be easily verified that the mild solutions are differentiable in $t$ and strong solutions of (2.1) based on the regularity of analytic semigroup $e^{-At}$. Moreover, $L^p$-theory, $C^\alpha$-theory and the bootstrapping method imply the mild solution $u$ of (2.3) is just classical solutions of (2.1) under some smooth conditions on nonlinear functions $F(u, \partial u, \cdots, \partial^{2m-1}u)$.

For the sake of convenience, we introduce the admissible triplet and generalized admissible triplet with respect to a 2m-order parabolic operator before we state our main results.

DEFINITION 2.1. We call $(p,q,r)$ an admissible triplet if

$$\frac{1}{q} = \frac{n}{2m}\left(\frac{1}{r} - \frac{1}{p}\right), \tag{2.4}$$

where

$$1 < r \le p < \begin{cases} \dfrac{nr}{n-2m}, & n > 2m, \\ \infty, & n \le 2m. \end{cases} \tag{2.5}$$

DEFINITION 2.2. We call $(p, q, r)$ a generalized admissible triplet if

$$\frac{1}{q} = \frac{n}{2m}\left(\frac{1}{r} - \frac{1}{p}\right), \tag{2.6}$$

where

$$1 < r \le p < \begin{cases} \dfrac{nr}{n-2mr}, & n > 2mr, \\ \infty, & n \le 2mr. \end{cases} \tag{2.7}$$

REMARK 2.1. (i) One can easily find that $q$ is unique determined by $p$ and $r$. Usually we write $q = q(p, r)$.

(ii) It is easy to see that $r < q \le \infty$ if $(p, q, r)$ is an admissible triplet.

(iii) It is easy to see that $1 < q \le \infty$ if $(p, q, r)$ is a generalized admissible triplet.

Our main results can be expressed as follows:

THEOREM 2.1 (Existence and uniqueness). *Let* $F(u, \partial u, \cdots, \partial^{2m-1}u)$ *satisfy* (1.5) *and* (1.6). *Put* $r_0 = \max_{|\beta| \le 2m-1}\{n\theta_\beta/(2m - |\beta|)\}$. *Assume* $r \ge r_0$ *when* $r_0 > 1$ *or* $r > 1$ *when* $r_0 \le 1$. *Let* $\varphi(x) \in L^r$ *and* $(p, q, r)$ *be any admissible triplet. We further assume that*

$$r > \frac{1 + 2\max\limits_{|\beta| \le 2m-1} \theta_\beta}{1 + \max\limits_{|\beta| \le 2m-1} \theta_\beta}, \tag{2.8}$$

*when* $p \le 1 + \max_{|\beta| \le 2m-1} \theta_\beta$. *Then there exist the maximal interval* $[0, T^*)$ *and the unique solution* $u(t)$ *satisfying* (1.1), (1.2) *or* (1.1), (1.3) *with* $t^{1/q}u(t) \in L^\infty((0, T^*); L^p)$. □

(Properties of solution) Let $(p, q, r)$ be any admissible triplet with

$$1 + \max_{|\beta| \le 2m-1} \theta_\beta < p < r\left(1 + \max_{|\beta| \le 2m-1} \theta_\beta\right). \tag{2.9}$$

Then the solution which is obtained in the first part is of the following regularity:

(i) $t^{1/q}u(t) \in C([0, T^*); L^p)$ for all $p \ge r$.

(ii) For all $p > r$, $u(t)$ satisfies

$$t^{\frac{1}{q}} \|u(t)\|_r \to 0, \quad \text{when} \quad t \to 0. \tag{2.10}$$

(iii) $u(t) \in C((0, T^*); L^r \cap L^\infty)$.

(iv) If $T^* < \infty$, then

$$\lim_{t \to T^*} \|u(t)\|_p = \infty, \quad r \le p \le \infty, \quad p > r_0.$$

In addition, we have

$$\|u(t)\|_p \ge \min_{|\beta| \le 2m-1}\{C/(T^* - t)^{\frac{2m-|\beta|}{2m\theta_\beta} - \frac{n}{2mp}}\}. \tag{2.11}$$

THEOREM 2.2 (Existence and uniqueness). *Let* $F(u, \partial u, \cdots, \partial^{2m-1}u)$ *satisfy* (1.5) *and* (1.6). *Put* $r_0 = \max_{|\beta|\le 2m-1}\{n\theta_\beta/(2m-|\beta|)\}$. *Let* $r \ge r_0$ *when* $r_0 > 1$ *or* $r > 1$ *when* $r_0 \le 1$. *Let* $\varphi(x) \in L^r$ *and* $(p, q, r)$ *be any one admissible triplet. We further assume that* (2.8) *is satisfied when* $p \le 1 + \max_{|\beta|\le 2m-1}\theta_\beta$. *Then there exist the maximal* $T^* > 0$ *and a unique function* $u(t) \in L^q((0, T^*); L^p)$ *satisfying* (1.1), (1.2) *or* (1.1), (1.3).

(Properties of the solution) Let $\varphi(x) \in L^r$, $u(t)$ be the solution which is obtained in Theorem 2.1 in the first part. Then we have

(i) For any admissible triplet $(p, q, r)$ satisfying (2.9), $u(t) \in L^q((0, T^*); L^p)$ is the unique solution of (1.1), (1.2) or (1.1), (1.3).

(ii) If $T^* < \infty$, we have

$$\lim_{\varepsilon\to 0}\|u(t); L^q([T^*/2, T^*-\varepsilon; L^p)\| = \infty, \quad r \le p < \infty, \quad p > r_0. \tag{2.12}$$

REMARK 2.2. (i) In view of the theory of abstract evolution equations, we can obtain local well-posedness for (1.1), (1.2) or (1.1), (1.3) in $\mathcal{C}([0, T); L^p) \cap \mathcal{C}^1((0, T); L^p) \cap \mathcal{C}((0, T); E_\gamma)$ for general nonlinear functions $F(u, \partial u, \cdots, \partial^{2m-1}u)$, where

$$E_\gamma = \{u \in D(A^\gamma) : \|A^\gamma u\|_p < \infty, \|\cdot\|_{E_\gamma} \triangleq \|A^\gamma u\|_p\}$$

where $0 < \gamma < 1$. In fact, if the nonlinear function $F(u, \partial u, \cdots, \partial^{2m-1}u)$ is defined on an open subset U of $E_\gamma$ $(0 < \gamma < 1)$ taking values in $L^p$ and is local Lip continuous with respect to $(u, \partial u, \cdots, \partial^{2m-1}u)$, i.e. for each point $(u, \partial u, \cdots, \partial^{2m-1}u) \in U$, there exists a neighborhood $V \subset U$, a constant $L = L(V) > 0$ such that

$$\begin{aligned}&\|F(u, \partial u, \cdots, \partial^{2m-1}u) - F(v, \partial v, \cdots, \partial^{2m-1}v)\|_p \\ &\quad\le L(V)\|A^\gamma(u-v)\|_p = L(V)\|u-v\|_{E_\gamma}, \quad \gamma = \frac{2m-1}{2m}, \\ &\quad\forall (u, \partial u, \cdots, \partial^{2m-1}u), \quad (v, \partial v, \cdots, \partial^{2m-1}v) \in V.\end{aligned} \tag{2.13}$$

Then the problem (1.1), (1.2) and (1.1), (1.3) have a unique local solution $u(t) \in \mathcal{C}([0, T); L^p) \cap \mathcal{C}^1((0, T); L^p) \cap \mathcal{C}((0, T); E_\gamma)$. When $p > n$, it is not difficult to verify (2.13). But, when $p \le n$, in order to obtain the local existence one has to assume that the nonlinear function $F(u, \partial u, \cdots, \partial^{2m-1}u)$ satisfies additional conditions for verifying (2.13). Of course it is difficult to obtain other properties of the solution such as the rate of blow-up for the solution, see [4, 13] for the details.

(ii) When $A = (-\Delta)$ or $-(\mu + i\nu)\Delta$, $\mu > 0$ and $F = f(u)$ (e.g. $\lambda|u|^{b_0}u$), the problems (1.1), (1.2) and (1.1), (1.3) turn out to be the Cauchy problem and IBVP for the heat equation and the Ginzburg Laudan equation. Our results extend the known results and give a unified way to discribe and prove the main results, see [5, 6, 10, 11, 17–19].

(iii) As for the result of uniqueness, I obtained the uniqueness of solution to (1.1), (1.2) or (1.1), (1.3) for any adimissible triplet $(p, q, r)$ with (2.9), the referee recommend the papers [2] and [9] to me and help me to get the result of uniqueness for any admissible triplet $(p, q, r)$. In fact, when $(n-2m)\max_{|\beta|\le 2m-1}\theta_\beta > 2m$, one easily sees that (2.9) is always valid, because in this case we have $(r, 2r/(n-2m)) \subset (1+\max_{|\beta|\leq 2m-1}\theta_\beta, r(1+\max_{|\beta|\leq 2m-1}\theta_\beta))$. But, when $(n-2m)\max_{|\beta|\leq 2m-1}\theta_\beta \leq 2m$, (2.9) is not always valid. In view of the result of Ginibre and Velo in [7] one easily obtain the uniqueness of solution to (1.1), (1.2) or (1.1), (1.3) for any admissible triplet $(p, q, r)$ with $p < r(1+\max_{|\beta|\leq 2m-1}\theta_\beta)$. For usual admissible triplet $(p, q, r)$, we conclude the existence and uniqueness for (1.1), (1.2) and (1.1), (1.3) in $C$-spaces and in $L$-spaces. Therefore this paper improves the result of uniqueness in [7]. But, the uniqueness in this paper is not the best result. e.g. when $F(u, \partial u, \cdots, \partial^{2m-1}u) = \nabla\cdot\vec{f}(u)$, $\vec{f}(u) = (f_1(u), \cdots, f_n(u))$ and $|f_j'(u)| \leq C|u|^b$, $f_j(0) = 0$, $j+1, \cdots, n$. For $r_0 = nb/(2m-1) > 1$, $\varphi(x) \in L^{r_0}$, let $(p, q, r_0)$ satisfy

$$\frac{1}{q} = \frac{n}{2m}\left(\frac{1}{r_0} - \frac{1}{p}\right), \quad q \geq 2.$$

In view of methods in [9], we easily conclude that there is the unique solution $u(t)$ to (1.1), (1.2) or (1.1), (1.3) in $C([0,T]; L^{r_0}) \cap L^q((0,T); L^p)$. One easily sees that this result is better than the result of uniqueness in this paper for this special case, see [9] for the detail proof. For general nonlinear term $F(u, \partial u, \cdots, \partial^{2m-1}u)$, it may be a interesting problem that how to prove the uniqueness of solution to (1.1), (1.2) or (1.1), (1.3) in $C([0,T]; L^r) \cap L^q((0,T); L^p)$ for any generalized admissible triplet $(p, q, r)$ with $r \geq r_0 = \max_{|\beta|\leq 2m-1}\{u\theta_\beta/(2m-|\beta|)\} > 1$ and

$$r_0 \leq p < \begin{cases} \dfrac{nr}{n-rm}, & n > rm, \\ \infty, & n \leq rm. \end{cases}$$

(iv) When the nonlinear function $F(u, \partial u, \cdots, \partial^{2m-1}u)$ satisfies (1.7) and

$$1 < r = \frac{n\theta_\beta}{2m-|\beta|} \overset{\Delta}{=} \frac{nb_\beta}{2m} \equiv \text{constant}, \quad \text{for all } |\beta| \leq 2m-1,$$

that is, $F(u, \partial u, \cdots, \partial^{2m-1}u)$ is of critical growth with respect to all variables, we obtain the existence of global small solutions to (1.1), (1.2) and (1.1), (1.3). This is an immedate result from the proof of Theorem 2.1 and Theorem 2.2, see Section 4 of [6, 10, 11].

For the case of usual super-critical growth, it is not an obvious result that the global small solutions to (1.1), (1.2) and (1.1), (1.3) exist. We shall prove the existence of global small solution to (1.1), (1.2) and (1.1), (1.3) by time-space estimates and the contraction mapping principle.

Before the statement of Theorem 2.3, we first introduce some notations. Let $\bar{b} = \min_{|\beta|\leq 2m-1} b_\beta$, $\tilde{b} = \max_{|\beta|\leq 2m-1} b_\beta$, $\bar{r} = n\bar{b}/2m$, $\tilde{r} = n\tilde{b}/2m$. In view of Hölder's inequality, (1.7) implies

$$|f_\beta'(u)| \leq C(|u|^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)} + |u|^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}), \quad |\beta| \leq 2m-1, \tag{2.14}$$

and $2m/n < \bar{b} \leq \tilde{b} < \infty$.

THEOREM 2.3. (i) *Let $(\bar{p}, \bar{q}, \bar{r})$ and $(\tilde{p}, \tilde{q}, \tilde{r})$ be admissible triplets such that*

$$\begin{cases} (\bar{b}+1) < \bar{p} < \bar{r}(1+\bar{b}), \\ (\tilde{b}+1) < \tilde{p} < \tilde{r}(1+\tilde{b}), \end{cases} \tag{2.15}$$

$$\frac{\bar{r}}{\bar{p}} = \frac{\tilde{r}}{\tilde{p}}. \tag{2.16}$$

*Then, there exists a $\delta > 0$ such that if $\|\varphi(x)\|_{L^{\bar{r}} \cap L^{\tilde{r}}} < \delta$, (1.1), (1.2) or (1.1), (1.3) has a unique global small solution $u(t) \in L^{\bar{q}}([0, \infty); L^{\bar{p}}) \cap L^{\tilde{q}}([0, \infty); L^{\tilde{p}})$.*

(ii) *Let $(\hat{p}, \hat{q}, \hat{r})$ be any admissible triplet with $\bar{r} \le \hat{r} \le \tilde{r}$ and*

$$\frac{\bar{r}}{\bar{p}} = \frac{\hat{r}}{\hat{p}} = \frac{\tilde{r}}{\tilde{p}}. \tag{2.17}$$

*then the solution $u(t)$ belongs to $L^{\hat{q}}([0, \infty); L^{\hat{p}})$.*

(iii) *Let $(p, q, \tilde{r}) = (p, q(p, \tilde{r}), \tilde{r})$ or $(p, q, \bar{r}) = (p, q(p, \bar{r}), \bar{r})$ be any admissible triplet, then the solution $u(t)$ belongs to $L^q([0, \infty); L^p)$.*

(iv) *Let $(\hat{p}, \hat{q}, \hat{r})$ be any generalized admissible triplet such that $\bar{r} \le \hat{r} \le \tilde{r}$, then the solution $u(t)$ belongs to $L^{\hat{q}}([0, \infty); L^{\hat{p}})$.*

REMARK 2.3. (i) one can easily see that there exist admissible triplets $(\bar{p}, \bar{q}, \bar{r})$ and $(\tilde{p}, \tilde{q}, \tilde{r})$ satisfying (2.15) and (2.16). In fact, noticing that

$$\begin{cases} \max\left(1, \dfrac{1+\bar{b}}{\bar{r}}\right) \le \dfrac{\bar{p}}{\bar{r}} < (1+\bar{b}) \\ \max\left(1, \dfrac{1+\tilde{b}}{\tilde{r}}\right) \le \dfrac{\tilde{p}}{\tilde{r}} < (1+\tilde{b}), \end{cases}$$

$$\left[\frac{2m(1+\bar{b})}{n\bar{b}}, 1+\bar{b}\right) \subset \left[\frac{2m(1+\tilde{b})}{n\tilde{b}}, 1+\tilde{b}\right),$$

so we can easily choose admissible triplets $(\bar{p}, \bar{q}, \bar{r})$ and $(\tilde{p}, \tilde{q}, \tilde{r})$ satisfying (2.15) and (2.16). Moreover, for any admissible triplet $(p, q, r)$ with $r = nb/2m > 1$, we can find an admissible triplet $(p_0, q_0, r)$ such that

$$\frac{\bar{r}}{\bar{p}} = \frac{\tilde{r}}{\tilde{p}} = \frac{r}{p_0}. \tag{2.18}$$

(ii) Under the condition (2.16), (2.15) can be replaced by

$$(\bar{b}+1) < \bar{p} < \bar{r}(1+\bar{b}). \tag{2.15$'$}$$

In fact, it is easy to see that

$$\frac{\tilde{b}}{\tilde{p}} = \frac{\bar{b}}{\bar{p}} < \frac{\bar{b}}{1+\bar{b}} < \frac{\tilde{b}}{1+\tilde{b}}, \quad \frac{\tilde{b}}{\tilde{p}} = \frac{\bar{b}}{\bar{p}} > \frac{1}{1+\bar{b}} > \frac{1}{1+\tilde{b}}.$$

Hence we have

$$(\tilde{b}+1) < \tilde{p} < \tilde{r}(1+\tilde{b}).$$

Moreover, (2.17) in (ii) implies

$$\left(1+\frac{2m}{n}\hat{r}\right) < \hat{p} < \hat{r}\left(1+\frac{2m}{n}\hat{r}\right). \tag{2.19}$$

On the other hand, one may find that $(n-2m)\bar{b} > 2m$ by the Remark 2.2 (iii), so we can remove the condition (2.15). Hence the conditions (2.15) is only needed for the case $(n-2m)\bar{b} \le 2m$.

(iii) As a consequence of Theorem 2.3, the existence of global small solutions in $L$-spaces for nonlinear heat equation and nonlinear Ginzburg Laudan equations is obtained.

## 3. Time-space estimates for solution to linear parabolic equations.

We now consider the abstract Cauchy problem of the following linear parabolic equation

$$\begin{cases} u_t + Au = f(x,t), & t \in [0,T), \quad 0 < T \le \infty \\ u(0) = \varphi(x), & \varphi \in D(A), \end{cases} \tag{3.1}$$

where $D(A) = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ or $W^{2m,p}(\mathbf{R}^n)$. It is well known that

$$u(t) = e^{-At}\varphi + \int_0^t e^{-(t-s)A} f(x,t)ds \triangleq e^{-At}\varphi + Gf(x,t) \tag{3.2}$$

solves (3.1). In view of (2.2), one easily sees that

$$\|\partial^\beta e^{-At}\varphi\|_p \le C\|A^{\frac{|\beta|}{2m}} e^{-At}\varphi\|_p, \quad 1 < p < \infty, \quad \varphi \in D(A), \tag{3.3}$$

$$\begin{aligned} \|A^{\frac{\gamma}{2m}} e^{-At}\varphi\|_p \le & Ct^{-\frac{\gamma}{2m}-\frac{n}{2m}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\varphi\|_r, \quad t > 0, \\ & 0 \le \frac{\gamma}{2m} < 1, \quad p \ge r, \quad \varphi \in D(A), \end{aligned} \tag{3.4}$$

by the Hörmander-Mikhlin mutiplier theory, estimates of analytic semigroups and Young's inequality, see [4, 6, 8, 10–12, 14, 16] for the details. Due to Marcinkiewicz's interpolation theorem[14] and (3.4), we have the following lemma, the proof of which can be found in [6, 11].

LEMMA 3.1. *Let $(p,q,r)$ be any admissible triplet, $\varphi(x) \in L^r$, then $e^{-At}\varphi \in L^q([0,\infty); L^p) \cap C_b([0,\infty); L^r)$ with*

$$\|e^{-At}\varphi\|_{L^q(I;L^p)} \le C\|\varphi\|_r, \quad I = [0,\infty) \quad \text{or} \quad I \subset [0,\infty). \tag{3.5}$$

*where $C$ is constant independent of $\varphi(x)$.*

To prove our main results we need a series of nonlinear estimates. For this propose we first establish time space estimates for solutions to the linear parabolic equation.

PROPOSITION 3.2. (i) *Let $r \ge na/(2m-\gamma) > 1$, or $r > 1$ when $na/(2m-\gamma) \le 1$. $(p,q,r)$ be any generalized admissible triplet with*

$$(a+1) < p < r(a+1). \tag{3.6}$$

*Assume* $f \in L^{\frac{q}{a+1}}([0,T]; L^{\frac{p}{a+1}}$. *Then* $A^{\frac{\gamma}{2m}} Gf(x,t) \in L^q([0,T]; L^p)$ *and*

$$\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^q([0,T];L^p)} \le CT^{1-\frac{\gamma}{2m}-\frac{na}{2mr}} \|f\|_{L^{\frac{q}{a+1}}([0,T];L^{\frac{p}{a+1}})}$$
$$0 \le \gamma < 2m, \tag{3.7}$$

*where C is a constant independent of* $f(x,t)$ *and* $T$.

(ii) *Let* $r > (2a+1)/(a+1)$ *and* $r \ge na/(2m-\gamma)$, $(p_1, q_1, r)$ *be any generalized admissible triplet with* $p_1 \le a+1$. *Then, there exists a generalized admissible triplet* $(p,q,r)$ *with* (3.6), *such that* $A^{\gamma/2m} Gf(x,t) \in L^{q_1}([0,T]; L^{p_1})$ *and*

$$\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{q_1}([0,T];L^{p_1})} \le CT^{1-\frac{\gamma}{2m}-\frac{na}{2mr}} \|f\|^{\frac{a}{a+1}}_{L^{\frac{p}{a+1}}([0,T];L^{\frac{q}{a+1}})}$$
$$\times \left\| |f|^{\frac{1}{a+1}} \right\|_{L^{q_1}([0,T];L^{p_1})}, \quad 0 \le \gamma < 2m. \tag{3.8}$$

(iii) *Let* $r \ge na/(2m-\gamma) > 1$ *or* $r > 1$ *when* $na/(2m-\gamma) \le 1$. *Let* $(p_2, q_2, r)$ *be any generalized admissible triplet with* $p_2 \ge r(a+1)$. *Then, there exists a generalized admissible triplet* $(p,q,r)$ *with* (3.6), *such that* $A^{\frac{\gamma}{2m}} Gf(x,t) \in L^{q_2}([0,T]; L^{p_2})$ *and*

$$\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{q_2}([0,T];L^{p_2})} \le CT^{1-\frac{\gamma}{2m}-\frac{na}{2mr}} \|f\|^{\frac{a}{a+1}}_{L^{\frac{p}{a+1}}([0,T];L^{\frac{q}{a+1}})}$$
$$\times \left\| |f|^{\frac{1}{a+1}} \right\|_{L^{q_2}([0,T];L^{p_2})}, \quad 0 \le \gamma < 2m. \tag{3.9}$$

PROOF. We first prove (i). One can easily see that (3.6) implies $q > (1+a)$, so we have

$$\|A^{\frac{\gamma}{2m}} GF(x,t)\|_{L^q([0,T];L^p)} \le C \left\| \int_0^t |t-s|^{-\frac{\gamma}{2m}-\frac{n}{2m}\left(\frac{a+1}{p}-\frac{1}{p}\right)} \|f(x,s)\|_{\frac{p}{a+1}} ds \right\|_q$$
$$\le CT^{1-\frac{\gamma}{2m}-\frac{na}{2mr}} \|f(x,t)\|_{L^{\frac{q}{a+1}}([0,T];L^{\frac{p}{a+1}})}, \tag{3.10}$$

by Young's inequality or Hardy-Littlewood-Sobolev's inequality (when $p = na/(2m-\gamma)$). This implies (i).

We now prove (ii). One easily sees that

$$1 + a < \frac{p_1 a}{p_1 - 1} < r(a+1). \tag{3.11}$$

In fact, noticing that

$$p_1 a > a p_1 - a + p_1 - 1 = (p_1 - 1)(1+a),$$
$$\frac{1}{a} > \frac{1}{ra} + \frac{1}{r(a+1)} \ge \frac{1}{p_1 a} + \frac{1}{r(1+a)}$$

by the condition $r > (2a+1)/(1+a)$, we get (3.11). Now we take $p = r(1+a) - \varepsilon$. In view of (3.11), we directly verify that

$$\max(r, a+1) < p < r(a+1), \tag{3.12}$$

$$r(a+1) - \varepsilon > \frac{p_1 a}{p_1 - 1}, \tag{3.13}$$

provided that $\varepsilon > 0$ is a suitable small constant. Hence (3.13) implies

$$\frac{a}{p} + \frac{1}{p_1} < 1 . \tag{3.14}$$

In this way we can choose a generalized admissble triplet $(p, q, r)$ satisfying (3.6) as

$$\frac{1}{q} = \frac{n}{2m}\left(\frac{1}{r} - \frac{1}{p}\right).$$

Using Hölder's inequality, we easily get

$$\begin{aligned} &\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{q_1}([0,T);L^{p_1})} \\ &\quad \le C\left\| \int_0^t |t-s|^{-\frac{\gamma}{2m} - \frac{n}{2m}\left(\frac{a}{p} + \frac{1}{p_1} - \frac{1}{p_1}\right)} \|f(x:s)\|_{\frac{p}{a+1}}^{\frac{a}{a+1}} \cdot \left\| |f|^{\frac{1}{a+1}} \right\|_{p_1} ds \right\|_{q_1} \\ &\quad \le CT^{1-\frac{\gamma}{2m} - \frac{na}{2mr}} \|f\|^{\frac{a}{a+1}}_{L^{\frac{q}{a+1}}([0,T);L^{\frac{p}{a+1}})} \cdot \| |f|^{\frac{1}{a+1}} \|_{L^{q_1}([0,T);L^{p_1})} , \end{aligned}$$

by Young's inequality or Hardy-Littlewood-Sobolev's inequality, and (3.14).

To prove (iii), let $p = \max(r, 1+a) + \varepsilon$. It is easy to see that for suitably small $\varepsilon > 0$, we choose a generalized admissible triplet $(p, q, r)$ by $1/q = (n/2m)(1/r - 1/p)$, which satisfies (3.6) and

$$\frac{a}{p} + \frac{1}{p_2} < 1 , \tag{3.15}$$

so we have

$$\begin{aligned} &\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{q_2}([0,T);L^{p_2})} \\ &\quad \le C\left\| \int_0^t |t-s|^{-\frac{\gamma}{2m} - \frac{n}{2m}\left(\frac{a}{p} + \frac{1}{p_2} - \frac{1}{p_2}\right)} \|f(x,s)\|_{\frac{p}{a+1}}^{\frac{a}{a+1}} \left\| |f|^{\frac{1}{a+1}} \right\|_{p_2} ds \right\|_{q_2} \\ &\quad \le CT^{1-\frac{\gamma}{2m} - \frac{na}{2mr}} \|f\|^{\frac{a}{a+1}}_{L^{\frac{q}{a+1}}([0,T);L^{\frac{p}{a+1}})} \left\| |f|^{\frac{1}{a+1}} \right\|_{L^{q_2}([0,T);L^{p_2})} , \end{aligned}$$

by Young's inequality or Hardy-Littlewood-Sobolev's inequality, and (3.15). The proof of Proposition 3.2 is complete.

PROPOSITION 3.3. (i) *Let* $0 \le \gamma \le 2m - 1$, $(p_1, q_1, r)$ *and* $(p_2, q_2, r)$ *be any two generalized admissible triplets satisfying* $p_1 \ge p_2$

$$q_1 > \frac{2m}{2m - \gamma}, \quad q_2 > \frac{2}{2m - \gamma} . \tag{3.16}$$

*Then we have*

$$\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{q(p_1,p_2)}(I;L^{p_1})} \le C\|f\|_{L^{q_\gamma}(I;L^r)} , \quad I = [0,\infty) \quad \text{or } I = [0,T) , \tag{3.17}$$

*where*

$$\begin{cases} \dfrac{1}{q(p_1,p_2)} = \dfrac{n}{2m}\left(\dfrac{1}{p_2} - \dfrac{1}{p_1}\right), \\ \dfrac{1}{q_\gamma} = 1 - \dfrac{\gamma}{2m} - \dfrac{1}{q_2} . \end{cases} \tag{3.18}$$

*In particular, when* $\gamma = 0$, *we have*

$$\|Gf(x,t)\|_{L^{q(p_1,p_2)}(I;L^{p_1})} \le C\|f\|_{L^{q_2'}(I;L^{r})}, \quad I=[0,\infty) \ \text{ or } \ I=[0,T), \tag{3.19}$$

*where* $q_2' = q_2/(q_2-1)$.

(ii) *Let* $0 \le \gamma \le 2m-1$, $(p, q_1, r_1)$ *and* $(p, q_2, r_2)$ *be any two generalized admissible triplets satisfying* $r_1 \ge r_2$ *and* (3.16). *Then we have*

$$\|A^{\frac{\gamma}{2m}}Gf(x,t)\|_{L^{q(r_1,r_2)}(I;L^{p})} \le C\|f\|_{L^{q_\gamma}(I;L^{r_2})}, \quad I=[0,\infty) \ \text{ or } \ I=[0,T), \tag{3.20}$$

*where*

$$\begin{cases} \dfrac{1}{q(r_1,r_2)} = \dfrac{n}{2m}\left(\dfrac{1}{r_2}-\dfrac{1}{r_1}\right), \\ \dfrac{1}{q_\gamma} = 1-\dfrac{\gamma}{2m}-\dfrac{1}{q_1}. \end{cases} \tag{3.21}$$

*In particular, when* $\gamma = 0$, *we have*

$$\|Gf(x,t)\|_{L^{q(r_1,r_2)}(I;L^{p})} \le C\|f\|_{L^{q_1'}(I;L^{r_2})}, \quad I=[0,\infty) \ \text{ or } \ I=[0,T), \tag{3.22}$$

*where* $q_1' = q_1/(q_1-1)$.

(iii) *Let* $0 \le \gamma \le 2m-1$, $(p_1, q_1, r)$ *and* $(r, q_2, p_2)$ *be any two generalized admissible triplets satisfying*

$$q_1 > \frac{2m}{2m-\gamma}, \quad q_2 = \frac{2m}{n}\left(\frac{1}{p_2}-\frac{1}{r}\right) > \frac{2m}{\gamma}, \tag{3.23}$$

$$p_1 < \begin{cases} \dfrac{np_2}{n-2mp_2}, & n > 2mp_2, \\ \infty, & n \le 2mp_2. \end{cases} \tag{3.24}$$

*Then we have*

$$\|A^{\frac{\gamma}{2m}}Gf(x,t)\|_{L^{q(p_1,p_2)}(I;L^{p_1})} \le C\|f\|_{L^{q_\gamma}(I;L^{r})}, \quad I=[0,\infty) \ \text{ or } \ I=[0,T), \tag{3.25}$$

*where*

$$\begin{cases} \dfrac{1}{q(p_1,p_2)} = \dfrac{n}{2m}\left(\dfrac{1}{p_2}-\dfrac{1}{p_1}\right), \\ \dfrac{1}{q_\gamma} = 1-\left(\dfrac{\gamma}{2m}-\dfrac{1}{q_2}\right). \end{cases} \tag{3.26}$$

PROOF. (i) One easily finds that

$$p_2 \le p_1 < \begin{cases} \dfrac{np_2}{n-2mp_2}, & n > 2mp_2, \\ \infty, & n \le 2mp_2, \end{cases}$$

so $(p_1, q(p_1,p_2), p_2)$ is a generalized admissible triplet. In view of (3.16) we have

$$0 < \frac{\gamma}{2m} + \frac{1}{q_j} < 1, \quad j = 1, 2. \tag{3.27}$$

Similar to the proof of proposition 3.2, we have

$$\|A^{\frac{\gamma}{2m}}Gf(x,t)\|_{L^{q(p_1,p_2)}(I;L^{p_1})} \le C\left\| \int_0^t |t-s|^{-\frac{\gamma}{2m}-\frac{n}{2m}\left(\frac{1}{r}-\frac{1}{p_1}\right)}\|f(x,s)\|_r ds\right\|_{q(p_1,p_2)}$$
$$\le \|f\|_{L^{q_\gamma}(I;L^r)}$$

by Hardy-Littlewood-Sobolev's inequality and (3.4), where we have used (3.27) and

$$\frac{1}{q(p_1,p_2)} = \frac{1}{q_\gamma} + \frac{\gamma}{2m} + \frac{1}{q_1} - 1. \tag{3.28}$$

(ii) It is easy to see that

$$r_2 \le r_1 < \begin{cases} \dfrac{nr_2}{n-2mr_2}, & n > 2mr_2, \\ \infty, & n \le 2mr_2, \end{cases}$$

by $r_1 \le p_1$ and the fact that $(p, q_2, r_2)$ is a generalized admissible triplet, so $q(r_1, r_2) > 1$. Noting that (3.16) implies (3.27), so we have

$$\|A^{\frac{\gamma}{2m}}Gf(x,t)\|_{L^{q(r_1,r_2)}(I;L^{p})} \le C\left\| \int_0^t |t-s|^{-\frac{\gamma}{2m}-\frac{n}{2m}\left(\frac{1}{r_2}-\frac{1}{p}\right)}\|f(x,s)\|_{r_2} ds\right\|_{q(r_1,r_2)}$$
$$\le \|f\|_{L^{q_\gamma}(I;L^{r_2})}$$

by Hardy-Littlewood-Sobolev's inequality and (3.4), where we have used

$$\frac{1}{q(r_1,r_2)} = \frac{1}{q_\gamma} + \frac{\gamma}{2m} + \frac{1}{q_2} - 1. \tag{3.29}$$

(iii) We directly verify

$$\|A^{\frac{\gamma}{2m}}Gf(x,t)\|_{L^{q(p_1,p_2)}(I;L^{p})} \le C\left\| \int_0^t |t-s|^{-\frac{\gamma}{2m}-\frac{n}{2m}\left(\frac{1}{r}-\frac{1}{p_1}\right)}\|f(x,s)\|_r ds\right\|_{q(p_1,p_2)}$$
$$\le \|f\|_{L^{q_\gamma}(I;L^r)},$$

by Hardy-Littlewood-Sobolev's inequality and (3.4), where we have used (3.23)–(3.25) and

$$\frac{1}{q(p_1,p_2)} = \frac{\gamma}{2m} + \frac{2m}{n}\left(\frac{1}{r} - \frac{1}{p_1}\right) + \frac{1}{q_\gamma} - 1. \tag{3.30}$$

Hence we complete the proof of Proposition 3.3.

PROPOSITION 3.4. *Let $r_1 = na_1/2m > 1$, $(p_1, q_1, r_1)$ be any generalized admissible triplet satisfying*

$$\max(r_1, a_1 + 1) < p_1 < r_1(1 + a_1). \tag{3.31}$$

*For any $\tilde{p}$ with $r_1 < \tilde{p} < r_1(a_1 + 1)$, let $(\tilde{p}, \tilde{q}, r_1)$ be a generalized admissible triplet which is determined by*

$$\frac{1}{\tilde{q}} = \frac{n}{2m}\left(\frac{1}{r_1} - \frac{1}{\tilde{p}}\right). \tag{3.32}$$

*Then we have*

$$\|A^{\frac{\gamma}{2m}}Gf(x,t)\|_{L^{\tilde q}(I;L^{\tilde p})}\le\left\||f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}}\right\|_{L^{q_1}(I;L^{p_1})}^{a_1(1-\frac{\gamma}{2m})}\left[\left\||f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}}\right\|_{L^{\tilde q}(I;L^{\tilde p})}+\left\||f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}}\right\|_{L^{q_1}(I;L^{p_1})}\right]. \tag{3.33}$$

*Moreover, for any* $r_2 = na_2/2m > 1$, $(p_2, q_2, r_2)$ *be any generalized admissible triplet satisfying*

$$\max(r_2, a_2+1) < p_2 < r_2(1+a_2), \tag{3.34}$$

$$\frac{r_2}{p_2} = \frac{r_1}{p_2}. \tag{3.35}$$

*Then, we have*

$$\|A^{\frac{\gamma}{2m}}Gf(x,t)\|_{L^{\tilde q}(I;L^{\tilde p})}\le\left\||f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}}\right\|_{L^{q_2}(I;L^{p_2})}^{a_2(1-\frac{\gamma}{2m})}\left[\left\||f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}}\right\|_{L^{\tilde q}(I;L^{\tilde p})}+\left\||f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}}\right\|_{L^{q_1}(I;L^{p_1})}\right], \tag{3.36}$$

*where* $g = |f|^{\frac{a_2(1-\frac{\gamma}{2m})+1}{a_1(1-\frac{\gamma}{2m})+1}}$.

PROOF. Take

$$\frac{1}{\sigma} = \frac{1}{\tilde p} + \frac{a_1\left(1-\frac{\gamma}{2m}\right)}{p_1}, \quad \text{when } \tilde p \ge p_1, \tag{3.37}$$

$$\frac{1}{\sigma} = \frac{\gamma}{2m}\frac{1}{\tilde p} + \left(1-\frac{\gamma}{2m}\right)\frac{1}{p_1} + \left(1-\frac{\gamma}{2m}\right)\frac{a_1}{p_1}, \quad \text{when } \tilde p < p_1. \tag{3.38}$$

It is easy to see that

$$\frac{1}{\sigma} \le \frac{1+a_1-a_1\frac{\gamma}{2m}}{p_1} < 1, \quad \text{when } \tilde p \ge p_1,$$

$$\frac{1}{\sigma} = \frac{\gamma}{2m}\frac{1}{\tilde p} + \left(1-\frac{\gamma}{2m}\right)\frac{1+a_1}{p_1} \le \frac{\gamma}{2m}\frac{1}{\tilde p} + \left(1-\frac{\gamma}{2m}\right) < 1, \quad \text{when } \tilde p < p_1,$$

so we have $\sigma > 1$. We now construct the following generalized admissible triplets

$$\begin{cases}(\tilde p, q(\tilde p,\sigma), \sigma),\\ (\sigma, q(\sigma, r_1), r_1), (\sigma > r_1) \quad \text{or} \quad (r_1, q(r_1,\sigma), \sigma), (r_1 \ge \sigma)\\ (\tilde p, \tilde q, r_1).\end{cases} \tag{3.39}$$

We claim $\tilde q > 1$ and

$$q(\tilde p, \sigma) > \frac{2m}{2m-\gamma}, \tag{3.40}$$

$$q(r_1, \sigma) > \frac{2m}{2m-\gamma}, \quad r_1 \ge \sigma, \tag{3.41}$$

$$q(\sigma, r_1) > \frac{2m}{\gamma}, \quad r_1 < \sigma. \tag{3.42}$$

Obviously, $\tilde{q} > 1$ by the definition of the generalized admissible triplet, so we need only to verify (3.40)–(3.42). One easily sees that

$$\begin{aligned}\frac{1}{q(\tilde{p}, \sigma)} &= \frac{n}{2m}\left(\frac{1}{\sigma} - \frac{1}{\tilde{p}}\right) = \frac{na_1}{2mp_1}\left(1 - \frac{\gamma}{2m}\right) \\ &= \frac{r_1}{p_1}\left(1 - \frac{\gamma}{2m}\right) < \frac{2m-\gamma}{2m}, \quad \text{when } \tilde{p} \geq p_1,\end{aligned}$$

$$\frac{1}{q(\tilde{p}, \sigma)} = \frac{n}{2m}\left(\frac{1+a_1}{p_1} - \frac{1}{\tilde{p}}\right)\left(1 - \frac{\gamma}{2m}\right) < 1 - \frac{\gamma}{2m}, \quad \text{when } \tilde{p} < p_1.$$

So we obtain (3.40). When $r_1 \geq \sigma$, noticing that $q(r_1, \sigma) \geq q(\tilde{p}, \sigma)$ since $\tilde{p} \geq r_1$, we get (3.41). At last, to prove (3.42), we obtain after a straightforward calculation that for $r_1 < \sigma$

$$\begin{aligned}\frac{1}{q(\sigma, r_1)} &= \frac{n}{2m}\left(\frac{1}{r_1} - \frac{1}{\sigma}\right) = \frac{n}{2m}\left(\frac{1}{r_1} - \frac{1}{\tilde{p}}\right) - \frac{r_1}{p_1}\left(1 - \frac{\gamma}{2m}\right) \\ &\leq \frac{1}{\tilde{q}} - \frac{1}{a_1+1} + \frac{r_1}{p_1}\frac{\gamma}{2m} < \frac{\gamma}{2m}, \quad \text{when } \tilde{p} \geq p_1,\end{aligned}$$

$$\begin{aligned}\frac{1}{q(\sigma, r_1)} &= \frac{\gamma}{2m}\frac{1}{\tilde{q}} + \left(1 - \frac{\gamma}{2m}\right)\frac{1}{q_1} - \frac{r_1}{p_1}\left(1 - \frac{\gamma}{2m}\right) \\ &= \frac{\gamma}{2m} - \frac{\gamma}{2m}\left(1 - \frac{1}{\tilde{q}}\right) + \left(1 - \frac{\gamma}{2m}\right)\left(\frac{1}{q_1} - \frac{r_1}{p_1}\right) \\ &\leq \frac{\gamma}{2m} - \frac{\gamma}{2m}\left(1 - \frac{1}{\tilde{q}}\right) + \left(1 - \frac{\gamma}{2m}\right)\left(\frac{1}{a_1+1} - \frac{r_1}{(a_1+1)r_1}\right) \\ &< \frac{\gamma}{2m}, \quad \text{when } \tilde{p} < p_1\end{aligned}$$

by (3.31), (3.37), (3.38) and $\tilde{q} > 1 + a_1$, thus we obtain (3.42). To obtain (3.33) and (3.36), we divide the proof into two cases.

Case I. When $\tilde{p} \geq p_1$, in view of Proposition 3.3, we have

$$\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \leq C\|f\|_{L^{q_\gamma}(I;L^{\sigma})}, \tag{3.43}$$

where

$$\begin{aligned}\frac{1}{q_\gamma} &= 1 - \frac{\gamma}{2m} - \frac{1}{q(r_1, \sigma)} \quad (\text{when } r_1 \geq \sigma) \\ &= 1 - \frac{\gamma}{2m} + \frac{n}{2m}\left(\frac{1}{r_1} - \frac{1}{\tilde{p}}\right) - \frac{r_1\left(1 - \frac{\gamma}{2m}\right)}{p_1} \\ &= \left(1 - \frac{\gamma}{2m}\right)\frac{a_1}{q(p_1, \sigma)} + \frac{1}{\tilde{q}},\end{aligned} \tag{3.44}$$

$$\frac{1}{q_\gamma} = 1 - \left(\frac{\gamma}{2m} - \frac{1}{q(\sigma, r_1)}\right) \quad (\text{when } \sigma > r_1)$$
$$= \left(1 - \frac{\gamma}{2m}\right)\frac{a_1}{q(p_1, \sigma)} + \frac{1}{\tilde{q}}, \tag{3.44'}$$

where we have used $r_1 = na_1/2m$. Noting that (3.37), (3.44) and

$$|f| = |f|^{\frac{a_1\left(1-\frac{\gamma}{2m}\right)}{1+a_1\left(1-\frac{\gamma}{2m}\right)}} |f|^{\frac{1}{1+a_1\left(1-\frac{\gamma}{2m}\right)}}, \tag{3.45}$$

we obtain

$$\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \le \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{q_1}(I;L^{p_1})}^{a_1\left(1-\frac{\gamma}{2m}\right)} \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{\tilde{q}}(I;L^{\tilde{p}})}$$

by Hölder's inequality. It is a special case of (3.33). Moreover, under the conditon (3.35), one easily sees

$$\frac{1}{\sigma} = \frac{1}{\tilde{p}} + \frac{a_2\left(1 - \frac{\gamma}{2m}\right)}{p_2}, \tag{3.46}$$

$$\frac{1}{q_\gamma} = 1 - \frac{\gamma}{2m} + \frac{n}{2m}\left(\frac{1}{r_1} - \frac{1}{\tilde{p}}\right) - \frac{r_2\left(1 - \frac{\gamma}{2m}\right)}{p_2}$$
$$= \left(1 - \frac{\gamma}{2m}\right)\frac{a_2}{q(p_2, \sigma)} + \frac{1}{\tilde{q}}. \tag{3.47}$$

Noting that (3.43) ($f$ is replaced by $g$), we have

$$\|A^{\frac{\gamma}{2m}} Gg(x,t)\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \le \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{q_2}(I;L^{p_2})}^{a_2\left(1-\frac{\gamma}{2m}\right)} \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{\tilde{q}}(I;L^{\tilde{p}})}$$

by Hölder's inequality, which is a special case of (3.36).

Case II. When $\tilde{p} < p_1$, in view of Proposition 3.3, we also have (3.43) and

$$\frac{1}{q_\gamma} = 1 - \frac{\gamma}{2m} - \frac{1}{q(r_1, \sigma)} \quad (\text{when } r_1 \ge \sigma)$$
$$= 1 - \frac{\gamma}{2m} - \frac{n}{2m}\left(\frac{\gamma}{2m}\frac{1}{\tilde{p}} + \left(1 - \frac{\gamma}{2m}\right)\frac{1}{p_1} + \left(1 - \frac{\gamma}{2m}\right)\frac{a_1}{p_1} - \frac{1}{r_1}\right)$$
$$= \left(1 - \frac{\gamma}{2m}\right)\frac{a_1}{q(p_1, \sigma)} + \frac{\gamma}{2m}\frac{1}{\tilde{q}} + \left(1 - \frac{\gamma}{2m}\right)\frac{1}{q_1}, \tag{3.48}$$

$$\frac{1}{q_\gamma} = 1 - \left(\frac{\gamma}{2m} - \frac{1}{q(\sigma, r_1)}\right) \quad (\text{when } \sigma > r_1)$$
$$= \left(1 - \frac{\gamma}{2m}\right)\frac{a_1}{q(p_1, \sigma)} + \frac{\gamma}{2m}\frac{1}{\tilde{q}} + \left(1 - \frac{\gamma}{2m}\right)\frac{1}{q_1}, \tag{3.48'}$$

where we have used $r_1 = na_1/2m$. Noticing that

$$|f| = |f|^{\frac{a_1\left(1-\frac{\gamma}{2m}\right)}{1+a_1\left(1-\frac{\gamma}{2m}\right)}} |f|^{\frac{\frac{\gamma}{2m}}{1+a_1\left(1-\frac{\gamma}{2m}\right)}} |f|^{\frac{1-\frac{\gamma}{2m}}{1+a_1\left(1-\frac{\gamma}{2m}\right)}},$$

and applying Hölder's inequality to the right side of (3.43), we easily obtain (3.33) by (3.38) and (3.48). Moreover, under the conditon (3.35), one easily infers

$$\frac{1}{\sigma} = \frac{\gamma}{2m}\frac{1}{\tilde{p}} + \left(1 - \frac{\gamma}{2m}\right)\frac{1}{p_1} + \left(1 - \frac{\gamma}{2m}\right)\frac{a_2}{p_2}, \tag{3.49}$$

$$\frac{1}{q_\gamma} = \left(1 - \frac{\gamma}{2m}\right)\frac{a_2}{q(p_2, \sigma)} + \frac{\gamma}{2m}\frac{1}{\tilde{q}} + \left(1 - \frac{\gamma}{2m}\right)\frac{1}{q_1}. \tag{3.50}$$

Noticing that

$$|g| = |f|^{\frac{a_2\left(1-\frac{\gamma}{2m}\right)}{1+a_1\left(1-\frac{\gamma}{2m}\right)}} |f|^{\frac{\frac{\gamma}{2m}}{1+a_1\left(1-\frac{\gamma}{2m}\right)}} |f|^{\frac{1-\frac{\gamma}{2m}}{1+a_1\left(1-\frac{\gamma}{2m}\right)}},$$

and (3.43), we easily obrtain (3.36) by Hölder's inequality.

PROPOSITION 3.5. *Let $r_1 = na_1/2m > 1$, $(\tilde{p}, \tilde{q}, r_1)$ be any generalized admissible triplet satisfying $\tilde{p} \geq r_1(1 + a_1)$. Then there exists at least a generalized admissible triplet $(p_1, q_1, r_1)$ satisfying* (3.31) *such that*

$$\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \leq \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{q_1}(I;L^{p_1})}^{a_1\left(1-\frac{\gamma}{2m}\right)} \left( \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{q_1}(I;L^{p_1})} + \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \right). \tag{3.51}$$

*Moreover, for any $r_2 = na_2/2m > 1$, let $(p_2, q_2, r_2)$ be any generalized admissible triplet satisfying* (3.34), (3.35), *then we have*

$$\|A^{\frac{\gamma}{2m}} Gg(x,t)\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \leq \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{q_2}(I;L^{p_2})}^{a_2\left(1-\frac{\gamma}{2m}\right)} \left( \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{q_1}(I;L^{p_1})} + \left\| |f|^{\frac{1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}} \right\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \right), \tag{3.52}$$

*where $g = |f|^{\frac{a_2\left(1-\frac{\gamma}{2m}\right)+1}{a_1\left(1-\frac{\gamma}{2m}\right)+1}}$.*

PROOF. Let $p_1 = r_1(a_1 + 1) - \varepsilon$. It is obvious that $p_1$ satisfies (3.31) when we choose $\varepsilon$ suitable small. Let $\sigma$ be defined by (3.38), then one easily sees that $\sigma > 1$ by

$$\frac{1}{\sigma} = \frac{\gamma}{2m}\frac{1}{\tilde{p}} + \left(1 - \frac{\gamma}{2m}\right)\frac{1 + a_1}{p_1} \leq \frac{\gamma}{2m}\frac{1}{\tilde{p}} + \left(1 - \frac{\gamma}{2m}\right) < 1.$$

We now verify the following triplets

$$\begin{cases} (\tilde{p}, q(\tilde{p}, \sigma), \sigma), \\ (\sigma, q(\sigma, r_1), r_1), (\sigma > r_1) \quad \text{or} \quad (r_1, q(r_1, \sigma), \sigma), (r_1 \ge \sigma) \\ (\tilde{p}, \tilde{q}, r_1). \end{cases}$$

satisfying $\tilde{q} > 1$ and (3.40)–(3.42). In fact, noticing that

$$\begin{cases} \dfrac{2m}{n} \ge \dfrac{1}{r_1} > \dfrac{1+a_1}{p_1} - \dfrac{1}{\tilde{p}}, & n \le 2mr_1, \\ \tilde{p} < \dfrac{nr_1}{n-2mr_1}, & n > 2mr_1, \end{cases}$$

where $\varepsilon > 0$ is suitably small. Hence we have

$$\frac{1}{q(\tilde{p}, \sigma)} = \frac{n}{2m}\left(\frac{1+a_1}{p_1} - \frac{1}{\tilde{p}}\right)\left(1 - \frac{\gamma}{2m}\right) < 1 - \frac{\gamma}{2m},$$

when $\varepsilon > 0$ is a suitably small constant. Therefore we obtain (3.40) and (3.41) by $r_1 \le \tilde{p}$. When $r_1 < \sigma$, we have

$$\begin{aligned} \frac{1}{q(\sigma, r_1)} &= \frac{\gamma}{2m}\frac{1}{\tilde{q}} + \left(1 - \frac{\gamma}{2m}\right)\frac{1}{q_1} - \frac{r_1}{p_1}\left(1 - \frac{\gamma}{2m}\right) \\ &= \frac{\gamma}{2m} - \frac{\gamma}{2m}\left(1 - \frac{1}{\tilde{q}}\right) + \left(1 - \frac{\gamma}{2m}\right)\left(\frac{1}{q_1} - \frac{r_1}{p_1}\right) \\ &\ge \frac{\gamma}{2m} - \frac{\gamma}{2m}\left(1 - \frac{1}{\tilde{q}}\right) + \left(1 - \frac{\gamma}{2m}\right)\left(\frac{1}{a_1+1} - \frac{r_1}{(a_1+1)r_1 - \varepsilon}\right) \\ &< \frac{\gamma}{2m}, \end{aligned}$$

so we have (3.42). Hence we obtain

$$\|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \le C\|f\|_{L^{q_\gamma}(I;L^{\sigma})}, \tag{3.43$'$}$$

by Proposition 3.3, where $1/q_\gamma$ is defined by (3.48). Applying Hölder's inequality to the right side of (3.43′), we easily obtain (3.51) by (3.38) and (3.48). In the same way as the proof of Proposition 3.4, we easily get (3.49). Thus we complete the proof of Proposition 3.5.

PROPOSITION 3.6. *Let $r_1 = na_1/2m > 1$, $(\tilde{p}, \tilde{q}, r_1)$ be any generalized admissible triplet satisfying $r_1 \le \tilde{p} \le \max(r_1, (1+a_1))$. Then there exists at least a generalized admissible triplet $(p_1, q_1, r_1)$ satisfying (3.31) such that*

$$\begin{aligned} \|A^{\frac{\gamma}{2m}} Gf(x,t)\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \le & \left\| |f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}} \right\|_{L^{q_1}(I;L^{p_1})}^{a_1(1-\frac{\gamma}{2m})} \times \left( \left\| |f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}} \right\|_{L^{q_1}(I;L^{p_1})} \right. \\ & \left. + \left\| |f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}} \right\|_{L^{\tilde{q}}(I;L^{\tilde{p}})} \right), \quad I = [0,\infty) \text{ or } [0,T). \end{aligned} \tag{3.53}$$

*Moreover, for any* $r_2 = na_2/2m > 1$, *let* $(p_2, q_2, r_2)$ *be any generalized admissible triplet satisfying* (3.34), (3.35), *then we have*

$$\|A^{\frac{\gamma}{2m}} Gg(x,t)\|_{L^{\tilde q}(I;L^{\tilde p})} \le \left\| |f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}} \right\|_{L^{q_2}(I;L^{p_2})}^{a_2(1-\frac{\gamma}{2m})} \times \left( \left\| |f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}} \right\|_{L^{q_1}(I;L^{p_1})} + \left\| |f|^{\frac{1}{a_1(1-\frac{\gamma}{2m})+1}} \right\|_{L^{\tilde q}(I;L^{\tilde p})} \right), \quad I = [0,\infty) \text{ or } [0,T), \tag{3.54}$$

*where*

$$g = |f|^{\frac{a_2(1-\frac{\gamma}{2m})+1}{a_1(1-\frac{\gamma}{2m})+1}}. \tag{3.55}$$

PROOF. Let $p_1 = \max(r_1, a_1+1)+\varepsilon$. It is obvious that $p_1$ satisfies (3.31) if we choose $\varepsilon > 0$ suitably small. Now we defined $\sigma$ by (3.50) and obtain that

$$\frac{1}{\sigma} = \frac{\gamma}{2m}\frac{1}{\tilde p} + \frac{1-\frac{\gamma}{2m}}{p_1/(1+a_1)} < 1, \tag{3.56}$$

so $\sigma > 1$. We now consider the generalized admissible triplets in (3.39). It is evident that $\tilde q > 1$. We can easily verify that

$$\begin{aligned}
\frac{1}{q(\tilde p,\sigma)} &= \frac{n}{2m}\left(\frac{1+a_1}{p_1} - \frac{1}{\tilde p}\right)\left(1-\frac{\gamma}{2m}\right) \\
&\le \begin{cases} \frac{n}{2m}\left(\frac{1+a_1}{\max(r_1,1+a_1)+\varepsilon} - \frac{1}{r_1}\right)\left(1-\frac{\gamma}{2m}\right), & \tilde p = r_1 \\ \frac{n}{2m}\left(\frac{1+a_1}{\max(r_1,1+a_1)+\varepsilon} - \frac{1}{1+a_1}\right)\left(1-\frac{\gamma}{2m}\right), & r_1 \le \tilde p \le 1+a_1, \end{cases} \\
&< \begin{cases} \frac{na_1}{2mr_1}\left(1-\frac{\gamma}{2m}\right), & \tilde p = r_1, \\ \frac{na_1}{2m(1+a_1)}\left(1-\frac{\gamma}{2m}\right), & r_1 \le \tilde p \le 1+a_1, \end{cases} \\
&< \left(1-\frac{\gamma}{2m}\right),
\end{aligned} \tag{3.57}$$

so we obtain (3.40) and (3.41) by $r_1 \le \tilde p$. When $\sigma > r_1$, it is easy to see that

$$\begin{aligned}
\frac{1}{q(\sigma,r_1)} &= \frac{\gamma}{2m}\frac{1}{\tilde q} + \left(1-\frac{\gamma}{2m}\right)\frac{1}{q_1} - \frac{r_1}{p_1}\left(1-\frac{\gamma}{2m}\right) \\
&= \frac{\gamma}{2m} - \frac{\gamma}{2m}\left(1-\frac{1}{\tilde q}\right) + \left(1-\frac{\gamma}{2m}\right)\left(\frac{1}{q_1} - \frac{r_1}{p_1}\right) \\
&< \frac{\gamma}{2m} - \frac{\gamma}{2m}\left(1-\frac{1}{\tilde q}\right) + \left(1-\frac{\gamma}{2m}\right)\left(\frac{1}{a_1+1} - \frac{r_1}{\max(r_1,a_1+1)+\varepsilon}\right) \\
&< \frac{\gamma}{2m},
\end{aligned} \tag{3.58}$$

for suitably small $\varepsilon > 0$. Hence we obtain (3.42). In the same way as leading to Proposition 3.4 and Proposition 3.5, we obtain Proposition 3.6 by Proposition 3.3.

REMARK 3.1. (i) One may find that the time space estimate of $e^{-At}\varphi$, see (3.5) in Lemma 3.1, is not vaild for a generalized admissible triplet $(p, q, r)$, see [6, 11] for the details. But the time-space estimates of the nonhomogeneous part of solutions for linear parabolic equations are vaild for a generalized admissible triplet $(p, q, r)$, see Proposition 3.2–3.6. This fact implies that there are better regularities for the nonhomogeneous part than those for free part, which are similar to the time-space estimates for solutions to dispersive wave equations [1,3]. For parabolic equations, regularity means that $L^p$ regularity, for dispersive wave equation it means the $C^\alpha$-regularity.

(ii) In the proof of Proposition 3.2–3.6, if generalized admissible triplets are replaced by admissible triplets, the results still remain valid. In fact, the main difference is that $p$ belongs to different intervals, see Definition 2.1 and 2.2. The process of the contruction of the generalized admissible triplets satisfying (3.6) from usual generalized admissible triplet is still valid for the case of admissible triplet.

(iii) From (i) it can be easily seen that we can not study the well-posedness in the $C$-spaces and $L^p$-spaces for generalized admissible triplet $(p, q, r)$. But for small global solutions in super-critial growth case, we can use generalized admissible triplets if we use

$$\|\varphi\|_{\bar{r}} + \|\varphi\|_{\tilde{r}} < \delta\,, \quad \forall\ \bar{r} < r < \tilde{r} < p\,, \tag{3.59}$$

to replace $\|v\|_r < \delta$. In fact, noting that

$$\|e^{-At}\varphi\|_p \le Ct^{-\frac{n}{2m}\left(\frac{1}{\bar{r}}-\frac{1}{p}\right)}\|\varphi\|_{\bar{r}}\,, \quad \|e^{-At}\varphi\|_p \le Ct^{-\frac{n}{2m}\left(\frac{1}{\tilde{r}}-\frac{1}{p}\right)}\|\varphi\|_{\tilde{r}}\,, \tag{3.60}$$

so we have

$$\|e^{-At}\varphi\|_p \le C\min(t^{-\frac{n}{2m}\left(\frac{1}{\bar{r}}-\frac{1}{p}\right)}, t^{-\frac{n}{2m}\left(\frac{1}{\tilde{r}}-\frac{1}{p}\right)})[\|\varphi\|_{\bar{r}} + \|\varphi\|_{\tilde{r}}]\,. \tag{3.61}$$

In view of Young's inequality we have

$$\|e^{At}\varphi\|_{L^q(I;L^p)} \le C[\|\varphi\|_{\bar{r}} + \|\varphi\|_{\tilde{r}}]\,. \tag{3.62}$$

It should be pointed out that the method of proof of small solution in this paper differs from that in other papers in twofolds: First, we do not use Picard's iteration method. Second, we establish Propositions 3.2–3.6 for the generalized admissible triplets, see Section 5.

## 4. The proof of Theorem 2.1 and Theorem 2.2.

Before giving the proof of Theorem 2.1 and Theorem 2.2, we first introduce some notations and derive some necessary nonlinear estimates. For $0 < T < \infty$, let $I = [0, T)$. For any admissible triplet $(p, q, r)$, we define

$$X_p(I) = \left\{u \in C_b(I; L^p) \mid \|u\|_{X_p(I)} = \sup_{t\in I} t^{\frac{1}{q}}\|u\|_p < \infty\right\}\,, \tag{4.1}$$

$$X_{p,q}(I) = L^q(I; L^p)\,. \tag{4.2}$$

In particular we denote $X_p = X_p([0, \infty))$, $X_{p,q} = X_{p,q}([0, \infty))$ when $T = \infty$.

We now prove some necessary nonlinear estimates of $\mathcal{J}u$ defined by (2.3) in $X_p(I)$ and $X_{p,q}(I)$.

LEMMA 4.1. *Let* $r_0 = \max_{|\beta|\le 2m-1}\{n\theta_\beta/(2m-|\beta|)\}$, $r \ge r_0$ *when* $r_0 > 1$ *or* $r > 1$ *when* $r_0 \le 1$, $(p,q,r)$ *be any admissible triplet satisfying* (2.9), $F(u, \partial u, \cdots, \partial^{2m-1}u)$ *satisfy* (1.5) *and* (1.6). *For any* $u, v \in X_p(I)$ *or* $X_{p,q}(I)$, *we have* $\mathcal{J}u$, $\mathcal{J}v \in X_p(I)$ *or* $X_{p,q}(I)$, *where* $\mathcal{J}$ *defined by* (2.3). *Moreover*

$$\|\mathcal{J}u\|_{X_p(I)} \le C \sum_{|\beta|\le 2m-1} \left[T^{1-\frac{|\beta|}{2m}}\|u\|_{X_p(I)} + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}\|u\|^{\theta_\beta+1}_{X_p(I)}\right], \tag{4.3}$$

$$\begin{aligned}\|\mathcal{J}u-\mathcal{J}v\|_{X_p(I)} \le & C \sum_{|\beta|\le 2m-1} \left[T^{1-\frac{|\beta|}{2m}} + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}(\|u\|^{\theta_\beta}_{X_p(I)} + \|v\|^{\theta_\beta}_{X_p(I)})\right] \\ & \times \|u-v\|_{X_p(I)}, \end{aligned}\tag{4.4}$$

$$\|\mathcal{J}u\|_{X_{p,q}(I)} \le C \sum_{|\beta|\le 2m-1} \left[T^{1-\frac{|\beta|}{2m}}\|u\|_{X_{p,q}(I)} + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}\|u\|^{\theta_\beta+1}_{X_{p,q}(I)}\right], \tag{4.5}$$

$$\begin{aligned}\|\mathcal{J}u-\mathcal{J}v\|_{X_{p,q}(I)} \le & C \sum_{|\beta|\le 2m-1} \left[T^{1-\frac{|\beta|}{2m}} + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}(\|u\|^{\theta_\beta}_{X_{p,q}(I)} + \|v\|^{\theta_\beta}_{X_{p,q}(I)})\right] \\ & \times \|u-v\|_{X_{p,q}(I)}. \end{aligned}\tag{4.6}$$

PROOF. Since (4.5) and (4.6) are immediate consequences of Proposition 3.2, it is sufficient to prove (4.3), (4.4). In view of $L^p - L^r$ estimates (3.4), (3.3) and definition of $X_p(T)$, we have

$$\begin{aligned}
\|\mathcal{J}u\|_{X_p(I)} \le & \sum_{|\beta|\le 2m-1} C_\beta \|A^{\frac{|\beta|}{2m}} G f_\beta(u)\|_{X_p(I)} \\
\le & C \sum_{|\beta|\le 2m-1} \left[\sup_{0\le t<T} |t|^{\frac{1}{q}} \int_0^t |t-s|^{-\frac{|\beta|}{2m}} \|u\|_p ds \right. \\
& \left. + \sup_{0\le t<T} |t|^{\frac{1}{q}} \int_0^t |t-s|^{-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mp}} \|u\|_p^{\theta_\beta+1} ds\right] \\
\le & C \sum_{|\beta|\le 2m-1} \left[|t|^{\frac{1}{q}} \int_0^t |t-s|^{-\frac{|\beta|}{2m}} s^{-\frac{1}{q}} ds \|u\|_{X_p(I)} \right. \\
& \left. + |t|^{\frac{1}{q}} \int_0^t |t-s|^{-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mp}} s^{-\frac{\theta_\beta+1}{q}} ds \|u\|^{\theta_\beta+1}_{X_p(I)}\right] \\
\le & C \sum_{|\beta|\le 2m-1} \left[|t|^{1-\frac{|\beta|}{2m}} \int_0^1 |1-s|^{-\frac{|\beta|}{2m}} s^{-\frac{1}{q}} ds \|u\|_{X_p(I)} \right. \\
& \left. + |t|^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}} \int_0^1 |1-s|^{-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mp}} s^{-\frac{\theta_\beta+1}{q}} ds \|u\|^{\theta_\beta+1}_{X_p(I)}\right],
\end{aligned}\tag{4.7}$$

Noting that $1/q < 1$ and

$$0 < \frac{|\beta|}{2m} + \frac{n\theta_\beta}{2mp} < 1, \quad 0 < \frac{\theta_\beta + 1}{q} < 1, \tag{4.8}$$

(4.7) implies (4.3). In the same way as in the derivation of (4.3), we easily obtain the estimate (4.4).

LEMMA 4.2. *Let* $r_0 = \max_{|\beta|\le 2m-1}\{n\theta_\beta/(2m - |\beta|)\}$, $r \ge r_0$ *when* $r_0 > 1$ *or* $r > 1$ *when* $r_0 \le 1$, $(p_1, q_1, r)$ *be any admissible triplet,* $F(u, \partial u, \cdots, \partial^{2m-1}u)$ *satisfy* (1.5) *and* (1.6), *we have the following results*:

(i) *If* $p_1 \ge r \max_{|\beta|\le 2m-1}(1 + \theta_\beta)$, *then there exists an admissible triplet* $(p, q, r)$ *satisfying* (2.9) *such that*

$$\|\mathcal{J}u\|_{X_{p_1}(I)} \le C \sum_{|\beta|\le 2m-1} \left[T^{1-\frac{|\beta|}{2m}} \|u\|_{X_{p_1}(I)} + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}} \|u\|^{\theta_\beta}_{X_p(I)} \|u\|_{X_{p_1}(I)}\right], \tag{4.9}$$

$$\begin{aligned}\|\mathcal{J}u - \mathcal{J}v\|_{X_{p_1}(I)} \le & C \sum_{|\beta|\le 2m-1} \left[T^{1-\frac{|\beta|}{2m}} + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}} (\|u\|^{\theta_\beta}_{X_p(I)} + \|v\|^{\theta_\beta}_{X_p(I)})\right] \\ & \times \|u - v\|_{X_{p_1}(I)},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\|\mathcal{J}u\|_{X_{p_1,q_1}(I)} \le & C \sum_{|\beta|\le 2m-1} \Big[T^{1-\frac{|\beta|}{2m}} \|u\|_{X_{p_1,q_1}(I)} \\ & + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}} \|u\|^{\theta_\beta}_{X_{p,q}(I)} \|u\|_{X_{p_1,q_1}(I)}\Big],\end{aligned} \tag{4.11}$$

$$\begin{aligned}\|\mathcal{J}u - \mathcal{J}v\|_{X_{p_1,q_1}(I)} \le & C \sum_{|\beta|\le 2m-1} \left[T^{1-\frac{|\beta|}{2m}} + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}} (\|u\|^{\theta_\beta}_{X_{p,q}(I)} + \|v\|^{\theta_\beta}_{X_{p,q}(I)})\right] \\ & \times \|u - v\|_{X_{p_1,q_1}(I)}.\end{aligned} \tag{4.12}$$

(ii) *If* $p_1 \le \max_{|\beta|\le 2m-1}(1 + \theta_\beta)$ *and* $r > \max_{|\beta|\le 2m-1}(1 + 2\theta_\beta)/(1 + \theta_\beta)$, *then there exists an admissible triplet* $(p, q, r)$ *satisfying* (2.9) *such that* (4.9) (4.12) *holds.*

PROOF. We shall prove (4.9) (4.12) using arguments similar to those used for Proposition 3.2, the key step is how to choose a admissible triplet appropriately. One easiy sees that (4.11) and (4.12) are immediate consequences of Proposition 3.2 and Young's inequality. Similar to the proof of Proposition 3.2, we take

$$p = \max\left(r, 1 + \max_{|\beta|\le 2m-1} \theta_\beta\right) + \varepsilon, \quad \text{when } p_1 \ge r\left(1 + \max_{|\beta|\le 2m-1} \theta_\beta\right). \tag{4.13}$$

Noting that $q \ge q_1$, it is easy to see that $(p, q, r)$ is an admissible triplet satisfying (2.9) and

$$\begin{cases} 0 \le \dfrac{|\beta|}{2m} + \dfrac{n\theta_\beta}{2mp} < 1, \\ 0 < \dfrac{\theta_\beta}{q} + \dfrac{1}{q_1} < 1. \end{cases} \tag{4.14}$$

In view of $L^p - L^r$ estimates (3.4), (3.3) and definition of $X_p(T)$, we have

$$
\begin{aligned}
\|\mathcal{J}u\|_{X_{p_1}(I)} \leq & \sum_{|\beta|\leq 2m-1} C_\beta \|A^{\frac{|\beta|}{2m}} G f_\beta(u)\|_{X_{p_1}(I)} \\
\leq & C \sum_{|\beta|\leq 2m-1} \sup_{0\leq t<T} \Big[ |t|^{\frac{1}{q_1}} \int_0^t |t-s|^{-\frac{|\beta|}{2m}} \|u\|_{p_1} ds \\
& + \sup_{0\leq t<T} |t|^{\frac{1}{q_1}} \int_0^t |t-s|^{-\frac{|\beta|}{2m} - \frac{n}{2m}\left(\frac{\theta_\beta}{p} + \frac{1}{p_1} - \frac{1}{p_1}\right)} \|u\|_p^{\theta_\beta} \|u\|_{p_1} ds \Big] \\
\leq & C \sum_{|\beta|\leq 2m-1} \Big[ |t|^{\frac{1}{q_1}} \int_0^t |t-s|^{-\frac{|\beta|}{2m}} s^{-\frac{1}{q_1}} ds \|u\|_{X_{p_1}(I)} \\
& + |t|^{\frac{1}{q_1}} \int_0^t |t-s|^{-\frac{|\beta|}{2m} - \frac{n\theta_\beta}{2mp}} s^{-\frac{\theta_\beta}{q} - \frac{1}{q_1}} ds \|u\|_{X_p(I)}^{\theta_\beta} \|u\|_{X_{p_1}(I)} \Big] \\
\leq & C \sum_{|\beta|\leq 2m-1} \Big[ |t|^{1-\frac{|\beta|}{2m}} \int_0^1 |1-s|^{\frac{|\beta|}{2m}} s^{-\frac{1}{q_1}} ds \|u\|_{X_p(I)} \\
& + |t|^{1-\frac{|\beta|}{2m} - \frac{n\theta_\beta}{2mr}} \int_0^1 |1-s|^{-\frac{|\beta|}{2m} - \frac{n\theta_\beta}{2mp}} s^{-\frac{\theta_\beta}{q} - \frac{1}{q_1}} ds \|u\|_{X_p(I)}^{\theta_\beta} \|u\|_{X_{p_1}(I)} \Big] \\
\leq & C \sum_{|\beta|\leq 2m-1} \left[ T^{1-\frac{|\beta|}{2m}} \|u\|_{X_{p_1}(I)} + T^{1-\frac{|\beta|}{2m} - \frac{n\theta_\beta}{2mr}} \|u\|_{X_p(I)}^{\theta_\beta} \|u\|_{X_{p_1}(I)} \right].
\end{aligned}
$$

As for the case (ii), we only take

$$
p = r\left(1 + \max_{|\beta|\leq 2m-1} \theta_\beta\right) - \varepsilon, \quad \text{when } r \leq p_1 \leq \left(1 + \max_{|\beta|\leq 2m-1} \theta_\beta\right), \tag{4.15}
$$

It can be easily seen that $(p, q, r)$ is an admissible triplet satisfying (2.9). In the same way as in the above proof and in the proof of Proposition 3.2 we obtain (4.9). In the same way as leading to (4.9), we easily obtain (4.10) in the cases (i), (ii).

THE PROOF OF THEOREM 2.1. We divide the proof into four steps.

Step 1. Let $(p, q, r)$ be any admissible triplet satisfying (2.9). It is easy to see that the right hand of the following integral equation

$$
\begin{aligned}
u &= e^{-tA}\varphi(x) + \int_0^t e^{-(t-s)A} F(u, \partial u, \cdots, \partial^{2m-1}u) ds \\
&= e^{-At}\varphi + \mathcal{J}u \triangleq \mathcal{T}u,
\end{aligned} \tag{4.16}
$$

defines a mapping $\mathcal{T}$ from $X_p(I)$ to itself by Lemma 4.1. Now we define the metric space as follows

$$X_p^s(I) = \left\{ u \in X_p(I) \,\middle|\, \|u\|_{X_p(I)} \le 2C\|\varphi\|_r, \sum_{|\beta|\le 2m-1} CT^{1-\frac{|\beta|}{2m}} < \frac{1}{4}, \right.$$

$$\left. \sum_{|\beta|\le 2m-1} 2CT^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}(2C\|\varphi\|_r)^{\theta_\beta} < \frac{1}{4} \right\}, \tag{4.17}$$

$$d(u,v) = \|u-v\|_{X_p(I)}. \tag{4.18}$$

Then the mapping $\mathcal{T}$ defined by (4.16) is a contraction mapping from $X_p^s(I)$ to itself. In fact, in view of Lemma 4.1 we have

$$\begin{aligned}\|\mathcal{T}u\|_{X_p(I)} \le& C\|\varphi\|_r + \sum_{|\beta|\le 2m-1} CT^{1-\frac{|\beta|}{2m}}(2C\|\varphi\|) \\ &+ \sum_{|\beta|\le 2m-1} CT^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}(2C\|\varphi\|_r)^{\theta_\beta}2C\|\varphi\|_r \\ &\le 2C\|\varphi\|_r,\end{aligned} \tag{4.19}$$

$$\begin{aligned}\|\mathcal{T}u-\mathcal{T}v\|_{X_p(I)} \le& \sum_{|\beta|\le 2m-1} CT^{1-\frac{|\beta|}{2m}}(2C\|\varphi\|)\|u-v\|_{X_p(I)} \\ &+ \sum_{|\beta|\le 2m-1} 2CT^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}(2C\|\varphi\|_r)^{\theta_\beta}\|u-v\|_{X_p(I)} \\ &\le \delta(T)\|u-v\|_{X_p(I)},\end{aligned} \tag{4.20}$$

where $u, v \in X_p^s(I)$ and

$$\delta(T) = \sum_{|\beta|\le 2m-1} CT^{1-\frac{|\beta|}{2m}} + \sum_{|\beta|\le 2m-1} 2CT^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}(2C\|\varphi\|_r^{\theta_\beta}) < 1. \tag{4.21}$$

So there exists a unique solution $u(t)$ of (4.16) on $[0, T)$ such that $u(t) \in X_p(I)$ by the Banach contraction mapping principle. Moreover, we also find that $T$ depends only on $\|\varphi\|_r$, which implies that we have the same existence inverval $[0, T)$ for any admissible triplet $(p_j, q_j, r)$. On the other hand, this fact implies that the iterative sequence

$$u_0 = e^{-At}\varphi, \quad u_{j+1} = e^{-tA}\varphi + \mathcal{J}u_j, \tag{4.22}$$

such that $\sum_{j=0}^\infty \|u_{j+1}-u_j\|_{X_p(I)}$ is a convergence series and $\|u_j\|_{X_p(I)} \le C(T)$. By Picard's methods we can extend the interval of existence up to the maximal interval $I = [0, T^*)$ such that

$$T^* = \infty, \quad \text{or} \quad T^* < \infty \quad \text{and} \quad \lim_{t\to T^*} \|u(t)\|_{X_p(I)} = \infty. \tag{4.23}$$

Step 2. For any admissible triplet $(p, q, r)$, whether $p \ge r(1+\max_{|\beta|\le 2m-1}\theta_\beta)$ or $p \le (1+\max_{|\beta|\le 2m-1}\theta_\beta)$, there is at least an admissible triplet $(\tilde{p}, \tilde{q}, r)$ satisfying (2.9) by Lemma 4.2. According to the above proof and the statement in Step 1, one can easily see that $u \in X_{\tilde{p}}^s(I)$ with $I = [0, T)$, and the iterative sequence $\{u_j\}$ given by (4.22) such that

$\sum_{j=0}^{\infty} \|u_{j+1} - u_j\|_{X_{\tilde{p}}(I)}$ is convergent and $\|u_j\|_{X_{\tilde{p}}(I)} \leq C(T)$. In view of Lemma 4.2 we have

$$\|u_{j+1} - u_j\|_{X_p(I)} \leq C \sum_{|\beta| \leq 2m-1} \Big[ T^{1-\frac{|\beta|}{2m}} + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}} (\|u_j\|_{X_{\tilde{p}}(I)}^{\theta_\beta} + \|u_{j-1}\|_{X_{\tilde{p}}(I)}^{\theta_\beta}) \Big] \|u_j - u_{j-1}\|_{X_p(I)}, \tag{4.24}$$

so we conclude that $\sum_{j=0}^{\infty} \|u_{j+1} - u_j\|_{X_p(I)}$ is convergent and $\|u_j\|_{X_p(I)} \leq C(T)$. Therefore, there exists a solution $u(t) \in L^\infty((0,T); L^p)$ satisfying the integral equation (4.16). Let $u(t)$, $v(t) \in L^\infty((0,T); L^p)$ satisfy (4.16), then we have

$$\|u(t) - v(t)\|_{X_p(I)} \leq C \sum_{|\beta| \leq 2m-1} \Big[ T^{1-\frac{|\beta|}{2m}} + T^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}} (\|u\|_{X_{\tilde{p}}(I)}^{\theta_\beta} + \|v\|_{X_{\tilde{p}}(I)}^{\theta_\beta}) \Big] \|u - v\|_{X_p(I)}.$$

Notice that $u(t)$, $v(t) \in X_{\tilde{p}}^s(I)$ and $\delta(T) < 1$, one easily sees that $u(t) = v(t)$.

Step 3. Let $(p, q, r)$ be any admissible triplet satisfying (2.9). If $p > r$, then we have

$$t^{\frac{1}{q}} \|e^{tA}\varphi\|_r \to 0, \quad t \to 0, \tag{4.25}$$

(see Giga's [3]). In this case we can use the space

$$\tilde{X}_p(I) = \Big\{ u : u \in C_b(I; L^p),\ \|u\|_{\tilde{X}_p(I)} = \sup_{t\in I} t^{\frac{1}{q}} \|u\|_p < \infty, \quad \text{and} \quad \lim_{t\to 0} t^{\frac{1}{q}} \|u\|_r = 0. \Big\}$$

to replace $X_p(I)$ in the Step 1, in this way we conclude that the solution to (4.16) satisfies (2.10). We now consider the regularity in the form integrability by Ginibre-Velo's mothed. In other words, we want to prove the following claim:

Let $u$ be solution to (4.16) with $u(t) \in C([(t_0, T)]; L^{p_1})$ for any admissible triplet $(p_1, q_1, r)$ satisfying (2.9). Then, for any $p_1 \leq p \leq \infty$, we have

$$u(t) \in ([t_0, T); L^p). \tag{4.26}$$

For this purpose, we prove inductively that for any $\varepsilon > 0$, $u(t) \in C([t_0 + j\varepsilon, T), L^{p_j})$ for a increasing sequence of exponent $\{p_j\}$, $1 \leq j \leq k$ reaching infinity in a finite number of steps. At the $j$-th step, we estimate $u$ from the integral equation with initial data $u(t_0 + j\varepsilon)$ as follows

$$\begin{aligned} \|u; L^\infty([t_0 + (j+1)\varepsilon, T); L^{p_j+1})\| &= C\varepsilon^{-\frac{n}{2m}\left(\frac{1}{p_j} - \frac{1}{p_{j+1}}\right)} \|u(t_0 + t\varepsilon)\|_{p_j} \\ &+ C \sup_{t_0+(j+1)\varepsilon \leq t < T} \int_{t_0+(j+1)\varepsilon}^{t} \Big[ \sum_{|\beta|\leq 2m-1} |t-s|^{-\frac{|\beta|}{2m} - \frac{n}{2m}\left(\frac{1}{p_j} - \frac{1}{p_{j+1}}\right)} \|u(s)\|_{p_j} \\ &+ \sum_{|\beta|\leq 2m-1} |t-s|^{-\frac{|\beta|}{2m} - \frac{n}{2m}\left(\frac{\theta_\beta+1}{p_j} - \frac{1}{p_{j+1}}\right)} \|u(s)\|_{p_j}^{1+\theta_\beta} \Big] ds, \end{aligned} \tag{4.27}$$

where $((1 + \max_{|\beta|\leq 2m-1} \theta_\beta)/p_j - 1/p_{j+1}) < (2m)/n(1 - |\beta|/2m)$. Now we choose

$$\frac{1}{p_{j+1}} = \max\left( \frac{1}{p_j}\left(1 + \max_{|\beta|\leq 2m-1} \theta_\beta\right) - \frac{2m}{n}\left(1 - \frac{|\beta|}{2m}\right) + \delta, 0 \right), \tag{4.28}$$

so that

$$\frac{1}{p_j}-\frac{1}{p_{j+1}}\geq\frac{2m}{n}\left(1-\frac{|\beta|}{2m}\right)-\frac{1}{p_j}\left(\max_{|\beta|\leq 2m-1}\theta_\beta\right)-\delta$$
$$\geq\frac{2m}{n}\left(1-\frac{|\beta|}{2m}\right)-\frac{1}{p_1}\left(\max_{|\beta|\leq 2m-1}\theta_\beta\right)-\delta>0 \tag{4.29}$$

by $p_1=r>\left(n\max_{|\beta|\leq 2m-1}\theta_\beta\right)/(2m-|\beta|)$, where $\delta>0$ is a sufficiently small constant. Hence, $L^\infty$ can be reached in a finite number of steps. As an immediate result, we have

$$u(t)\in C((0,T^*);L^r\cap L^\infty)\,. \tag{4.30}$$

Step 4. If we take $T>0$ sufficiently small, there always exists an $\varepsilon_0>0$ such that

$$\varepsilon_0\leq\max_{|\beta|\leq 2m-1}CT^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}(2C\|\varphi\|_r)^{\theta_\beta})\leq\frac{1}{4}\,. \tag{4.31}$$

Thus we conclude that

$$T\geq\min_{|\beta|\leq 2m-1}\left(C\|\varphi\|_r^{-\theta_\beta/\left(1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}\right)}\right)\,. \tag{4.32}$$

Let $[0,T^*)$ be the maximal interval such that $u$ solves (4.16), then we have

$$\lim_{t\to T^*}\|u(t)\|_p=\infty\,,\qquad r\leq p\leq\infty\,,\qquad p>r_0\,. \tag{4.33}$$

Otherwise, one easily sees that

$$\|u(T^*)\|_p<\infty\,. \tag{4.34}$$

We take $\eta>0$ sufficiently small such that $(p+\eta,q(p+\eta,p),p)$ is an admissible triplet. Similar to the proof in the first steps, using regularity we solve the integral equation

$$u=e^{-(t-T^*)A}u(T^*)+\int_{T^*}^t e^{-(t-s)A}F(u,\partial u,\cdots,\partial^{2m-1}u)ds \tag{4.35}$$

in the following metric space

$$\begin{aligned}X^s_{p+\eta}([T^*,T))=\Bigg\{u\,\Big|\,&u\in X_{p+\eta}(T^*;T),\ \|u(t)\|_{X_{p+\eta}([T^*,T))}\leq 2C\|u(T^*)\|_p\\ &\sum_{|\beta|\leq 2m-1}C(T-T^*)^{1-\frac{|\beta|}{2m}}<\frac{1}{4},\\ &\sum_{|\beta|\leq 2m-1}2C(T-T^*)^{1-\frac{|\beta|}{2m}-\frac{n\theta_\beta}{2mr}}(2C\|u(T^*)\|_p)^{\theta_\beta})<\frac{1}{4}\Bigg\}\end{aligned} \tag{4.36}$$

where

$$\begin{aligned}X_{p+\eta}([T^*,T))=\Bigg\{u\,\Big|\,&u\in C_b([T^*,T);L^{p+\eta}),\ \|u\|_{X_{p+\eta}(T^*,T)}\\ &=\sup_{T^*\leq t<T}(t-T^*)^{\frac{1}{q(p+\eta,p)}}\|u\|_{p+\eta}<\infty\Bigg\},\end{aligned} \tag{4.37}$$

This contradicts with the fact that $[0, T^*)$ is the maximal interval. Meanwhile, for $s < T^*$ we have $\|u(s)\|_r < \infty$, so if we take $s$ sufficiently close to $T^*$ and $u(t)$ solves

$$u = e^{-(t-s)A}u(s) + \int_s^t e^{-(t-\tau)A}F(u, \partial u, \cdots, \partial^{2m-1}u)d\tau \tag{4.38}$$

in space $X_p([s, T)) \cap X_{\tilde p}([s, T))$ by Step 3, where $\tilde p \geq p > r$. We easily see, analogously to (4.32), that

$$\|u(s)\|_p \geq \min_{|\beta| \leq 2m-1} \left( \frac{C}{(T^* - s)^{\frac{1}{b_\beta} - \frac{n}{2mp}}} \right), \tag{4.39}$$

with the constant $C$ being independent of $T^*$ and $s$. This completes the proof of Theorem 2.1.

PROOF OF THEOREM 2.2. In the exactly same way as in the proof of Theorem 2.1 we can obtain Theorem 2.2 except (2.12) by space-time estimates and Lemma 4.1 and Lemma 4.2. It remains to prove (2.12). In fact, noticing that

$$\left(\frac{1}{b_\beta} - \frac{n}{2p}\right) = \left(\frac{2m - |\beta|}{2m\theta_\beta} - \frac{n}{2mp}\right) = \frac{n}{2m}\left(\frac{2m - |\beta|}{n\theta_\beta} - \frac{1}{p}\right) > \frac{1}{q}, \tag{4.40}$$

and (4.39), we obtain (2.12).

## 5. The proof of Theorem 2.3.

We denote $L^q(I; L^p)$ by $X_{p,q}(I)$, in particular, denote $X_{p,q}$ by $X_{p,q}(0, \infty)$. As an immediate result of Proposition 3.4 3.6, we have

LEMMA 5.1. *Let* $\bar b = \min_{|\beta| \leq 2m-1} b_\beta$, $\tilde b = \max_{|\beta| \leq 2m-1} b_\beta$, $2m/n < \bar b \leq \tilde b < \infty$, $\bar r = n\bar b/2m$, $\tilde r = n\tilde b/2m$.

$$|f'_\beta(u)| \leq C\left(|u|^{\bar b\left(1 - \frac{|\beta|}{2m}\right)} + |u|^{\tilde b\left(1 - \frac{|\beta|}{2m}\right)}\right) \quad |\beta| \leq 2m - 1. \tag{5.1}$$

*Let* $(\bar p, \bar q, \bar r)$, $(\tilde p, \tilde q, \tilde r)$ *be any generalized admissible triplets satisfying* (2.15) *and* (2.16), *then*

$$\begin{aligned} &\|A^{\frac{|\beta|}{2m}} G(f_\beta(u) - f_\beta(v))\|_{X_{\tilde p,\tilde q}(I)} \\ &\leq C\left(\|u\|_{X_{\bar p,\bar q}(I)}^{\bar b\left(1-\frac{|\beta|}{2m}\right)} + \|u\|_{X_{\tilde p,\tilde q}(I)}^{\tilde b\left(1-\frac{|\beta|}{2m}\right)} + \|v\|_{X_{\bar p,\bar q}(I)}^{\bar b\left(1-\frac{|\beta|}{2m}\right)} + \|v\|_{X_{\tilde p,\tilde q}(I)}^{\tilde b\left(1-\frac{|\beta|}{2m}\right)}\right) \|u - v\|_{X_{\tilde p,\tilde q}(I)}, \\ &\qquad I = [0, \infty) \quad or \quad I = [0, T). \end{aligned} \tag{5.2}$$

$$\begin{aligned} &\|A^{\frac{|\beta|}{2m}} G(f_\beta(u) - f_\beta(v))\|_{X_{\bar p,\bar q}(I)} \\ &\leq C\left(\|u\|_{X_{\bar p,\bar q}(I)}^{\bar b\left(1-\frac{|\beta|}{2m}\right)} + \|u\|_{X_{\tilde p,\tilde q}(I)}^{\tilde b\left(1-\frac{|\beta|}{2m}\right)} + \|v\|_{X_{\bar p,\bar q}(I)}^{\bar b\left(1-\frac{|\beta|}{2m}\right)} + \|v\|_{X_{\tilde p,\tilde q}(I)}^{\tilde b\left(1-\frac{|\beta|}{2m}\right)}\right) \|u - v\|_{X_{\bar p,\bar q}(I)}, \\ &\qquad I = [0, \infty) \quad or \quad I = [0, T). \end{aligned} \tag{5.3}$$

*Moreover, for any generalized admissible triplet* $(\hat p, \hat q, \tilde r)$ *or* $(\hat p, \hat q, \bar r)$ *satisfying*

$$\tilde r < \hat p < \tilde r(1 + \tilde b), \quad or \quad \bar r < \hat p < \bar r(1 + \bar b), \tag{5.4}$$

*then*

$$\|A^{\frac{|\beta|}{2m}}G(f_\beta(u)-f_\beta(v))\|_{X_{\hat{p},\hat{q}}(I)}$$
$$\leq C\left(\|u\|_{X_{\tilde{p},\tilde{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|u\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\tilde{p},\tilde{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}\right)$$
$$\times[\|u-v\|_{X_{\hat{p},\hat{q}}}+\|u-v\|_{X_{\tilde{p},\tilde{q}}}],\quad I=[0,\infty)\quad or\quad I=[0,T)\,. \tag{5.5}$$

*or*

$$\|A^{\frac{|\beta|}{2m}}G(f_\beta(u)-f_\beta(v))\|_{X_{\hat{p},\hat{q}}(I)}$$
$$\leq C\left(\|u\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|u\|_{X_{\bar{p},\bar{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\bar{p},\bar{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}\right)$$
$$\times[\|u-v\|_{X_{\hat{p},\hat{q}}(I)}+\|u-v\|_{X_{\bar{p},\bar{q}}}],\quad I=[0,\infty)\quad or\quad I=[0,T)\,. \tag{5.6}$$

LEMMA 5.2. *Let* $\bar{b}=\min_{|\beta|\leq 2m-1}b_\beta$, $\tilde{b}=\max_{|\beta|\leq 2m-1}b_\beta$, $2m/n<\bar{b}\leq\tilde{b}<\infty$, $f_\beta$ *satisfies* (5.1). *Let* $(\hat{p},\hat{q},\tilde{r})$ *or* $(\hat{p},\hat{q},\bar{r})$ *be any generalized admissible triplet with*

$$\hat{p}\geq\tilde{r}(\tilde{b}+1)\,,\quad or\quad \hat{p}\geq\bar{r}(\bar{b}+1)\,. \tag{5.7}$$

*Then there exist two generalized admissible triplets* $(\tilde{p},\tilde{q},\tilde{r})$ *and* $(\bar{p},\bar{q},\bar{r})$ *with* (2.15) *and* (2.16) *such that*

$$\|A^{\frac{|\beta|}{2m}}G(f_\beta(u)-f_\beta(v))\|_{X_{\hat{p},\hat{q}}(I)}$$
$$\leq C\left(\|u\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|u\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}\right)$$
$$\times(\|u-v\|_{X_{\tilde{p},\tilde{q}}(I)}+\|u-v\|_{X_{\hat{p},\hat{q}}(I)})\,,\quad I=[0,\infty)\ or\ [0,T)\,. \tag{5.8}$$

*or*

$$\|A^{\frac{|\beta|}{2m}}G(f_\beta(u)-f_\beta(v))\|_{X_{\hat{p},\hat{q}}(I)}$$
$$\leq C\left(\|u\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|u\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}\right)$$
$$\times(\|u-v\|_{X_{\bar{p},\bar{q}}(I)}+\|u-v\|_{X_{\hat{p},\hat{q}}(I)})\,,\quad I=[0,\infty)\ or\ [0,T)\,. \tag{5.8'}$$

LEMMA 5.3. *Let* $\bar{b}=\min_{|\beta|\leq 2m-1}b_\beta$, $\tilde{b}=\max_{|\beta|\leq 2m-1}b_\beta$, $2m/n<\bar{b}\leq\tilde{b}<\infty$, $f_\beta$ *satisfy* (5.1). *Let* $(\hat{p},\hat{q},\tilde{r})$ *or* $(\hat{p},\hat{q},\bar{r})$ *be any generalized admissible triplet with*

$$\hat{p}\leq(\bar{b}+1)\,,\quad or\quad \hat{p}\leq(\tilde{b}+1)\,, \tag{5.9}$$

*Then there exist two admissible triplet* $(\tilde{p},\tilde{q},\tilde{r})$ *or* $(\bar{p},\bar{q},\bar{r})$ *with* (2.15) *and* (2.16) *such that*

$$\|A^{\frac{|\beta|}{2m}}G(f_\beta(u)-f_\beta(v))\|_{X_{\hat{p},\hat{q}}(I)}$$
$$\leq C\left(\|u\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|u\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}\right)$$
$$\times(\|u-v\|_{X_{\bar{p},\bar{q}}(I)}+\|u-v\|_{X_{\hat{p},\hat{q}}(I)})\,,\quad I=[0,\infty)\ or\ I=[0,T)\,. \tag{5.10}$$

$$\|A^{\frac{|\beta|}{2m}}G(f_\beta(u)-f_\beta(v))\|_{X_{\hat{p},\hat{q}}(I)}$$
$$\le C\left(\|u\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\|u\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\tilde{p},\tilde{q}}(I)}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\|v\|_{X_{\bar{p},\bar{q}}(I)}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}\right)$$
$$\times\left(\|u-v\|_{X_{\tilde{p},\tilde{q}}(I)}+\|u-v\|_{X_{\hat{p},\hat{q}}(I)}\right),\quad I=[0,\infty)\ or\ I=[0,T). \tag{5.10'}$$

REMARK 5.1. Since Remark 3.1 (iii), Lemma 5.1 are still vaild if generalized admissible triplets are replaced by admissible triplets.

THE PROOF OF THEOREM 2.3. We divide the proof into four steps.

Step 1. Let $\tilde{r}=n\tilde{b}/2m$, $\bar{r}=n\bar{b}/2m$, $(\tilde{p},\tilde{q},\tilde{r})$ and $(\bar{p},\bar{q},\bar{r})$ be any admissible triplet with (2.15) and (2.16). We construct the work-spaces as follows

$$Y=\{u:u(t)\in X_{\tilde{p},\tilde{q}}\cap X_{\bar{p},\bar{q}},\ \|u\|_{X_{\tilde{p},\tilde{q}}}\le\delta,\ \|u|_{X_{\bar{p},\bar{q}}}\le\delta\} \tag{5.11}$$

$$\rho(u,v)\overset{\Delta}{=}\|u-v\|_{X_{\tilde{p},\tilde{q}}}+\|u-v\|_{X_{\bar{p},\bar{q}}}\,. \tag{5.12}$$

Obviously, $(Y,\rho)$ is a complete metric space. We claim that the mapping $\mathcal{T}$ defined by the right hand of integral equation (4.16)

$$\mathcal{T}:u(t)\to e^{-At}\varphi+\mathcal{J}u\,, \tag{5.13}$$

is a contraction mapping from $(Y,\rho)$ to itself for sufficiently small $\delta>0$. In fact, we have

$$\|\mathcal{T}u\|_Y\le\|e^{-At}\varphi\|_{X_{\tilde{p},\tilde{q}}}+\|e^{-At}\varphi\|_{X_{\bar{p},\bar{q}}}+\sum_{|\beta|\le 2m+1}C\big[\|A^{\frac{|\beta|}{2m}}Gf_\beta\|_{X_{\tilde{p},\tilde{q}}}+\|A^{\frac{|\beta|}{2m}}Gf_\beta\|_{X_{\bar{p},\bar{q}}}\big]$$
$$\le C\|\varphi\|_{\tilde{r}}+C\|\varphi\|_{\bar{r}}+\sum_{|\beta|\le 2m+1}C\left[\|u\|_{X_{\tilde{p},\tilde{q}}}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)+1}+\|u\|_{X_{\bar{p},\bar{q}}}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)+1}\right]$$
$$\le C\|\varphi\|_{\tilde{r}}+C\|\varphi\|_{\bar{r}}+C\sum_{|\beta|\le 2m+1}\left[\delta^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)+1}+\delta^{\bar{b}\left(1-\frac{|\beta|}{2m}+1\right)}\right], \tag{5.14}$$

$$\|\mathcal{T}u-\mathcal{T}v\|_Y\le C\sum_{|\beta|\le 2m+1}\left[\delta^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\delta^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}\right]\|u-v\|_Y\,, \tag{5.15}$$

that is

$$\rho(\mathcal{T}u,\mathcal{T}v)\le C\sum_{|\beta|\le 2m+1}\left[\delta^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\delta^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}\right]\rho(u,v)\,. \tag{5.16}$$

We take $\delta>0$ satisfy

$$C\sum_{|\beta|\le 2m+1}\left[\delta^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}+\delta^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}\right]<\frac{1}{2}\,, \tag{5.17}$$

and let

$$\|\varphi\|_{\tilde{r}}+\|\varphi\|_{\bar{r}}\le\delta_1=\frac{\delta}{2C}\,, \tag{5.18}$$

then $\mathcal{T}$ is a contraction mapping from $(Y,\rho)$ to itself by (5.15)–(5.18). In particular, $\delta$ only depends on $\|\varphi\|_{\tilde{r}}$ and $\|\varphi\|_{\bar{r}}$, so $\delta$ is dependent of $\tilde{r}$ and $\bar{r}$, but independent of $\tilde{p}$ and $\bar{p}$. Thus, if $(\tilde{p}, \tilde{q}, \tilde{r})$ and $(\bar{p}, \bar{q}, \bar{r})$ are replaced by any admissible triplets $(\tilde{p}_j, \tilde{q}_j, \tilde{r})$ and $(\bar{p}_j, \bar{q}_j, \bar{r})$ satisfying (2.15) and (2.16), we have the same result as (i) for the same $\delta$.

Step 2. Let $(\hat{p}, \hat{q}, \hat{r})$ be any admissible triplet such that

$$\frac{\tilde{r}}{\tilde{p}} = \frac{\bar{r}}{\bar{p}} = \frac{\hat{r}}{\hat{p}}, \quad \hat{r} \in [\bar{r}, \tilde{r}], \tag{5.19}$$

then Remark 2.3 implies

$$(1 + \hat{b}) < \hat{p} < \hat{r}(1 + \hat{b}), \quad \hat{b} = \frac{2m\hat{r}}{n}. \tag{5.20}$$

Hence we have

$$\|u\|_{X_{\hat{p},\hat{q}}} \leq \|u\|_{X_{\tilde{p},\tilde{q}}}^{\frac{\hat{r}}{\tilde{r}}} \|u\|_{X_{\bar{p},\bar{q}}}^{1-\frac{\hat{r}}{\tilde{r}}} < \infty$$

by the interpolation theorem.

Step 3. Let $(p, q, \tilde{r})$ and $(p, q, \bar{r})$ be any admissible triplet. If

$$\tilde{r} < p < \tilde{r}(1 + \tilde{b}), \quad \text{or} \quad \bar{r} < p < \bar{r}(1 + \bar{b}), \tag{5.21}$$

then we have

$$\begin{aligned} \|u\|_{X_{p,q}} &\leq C\|\varphi\|_{\tilde{r}} + C\|\varphi\|_{\bar{r}} \\ &\quad + \sum_{|\beta| \leq 2m+1} C\left[\|u\|_{X_{\tilde{p},\tilde{q}}}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)} + \|u\|_{X_{\bar{p},\bar{q}}}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}\right](\|u\|_{X_{p,q}} + \delta) \\ &\leq C\|\varphi\|_{\tilde{r}} + C\|\varphi\|_{\bar{r}} \\ &\quad + C\sum_{|\beta| \leq 2m+1} \left[\delta^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)} + \delta^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}\right](\|u\|_{X_{p,q}} + \delta), \end{aligned} \tag{5.22}$$

by Proposition 3.4. Hence we obtain $u \in X_{p,q}$ by (5.17). When $p = \tilde{r}$ or $p \geq \tilde{r}(1 + \tilde{b})$, we easily find by Proposition 3.5 and 3.6 that the admissible triplet $(\tilde{p}_1, \tilde{q}_1, \tilde{r})$ and $(\bar{p}_1, \bar{q}_1, \bar{r})$ satisy (2.15), (2.16) and

$$\tilde{r} < \tilde{p}_1 < \tilde{r}(1 + \tilde{b}), \quad \text{or} \quad \bar{r} < \bar{p}_1 < \bar{r}(1 + \bar{b}).$$

Moreover, we have

$$\begin{aligned} \|u\|_{X_{p,q}} &\leq C\|\varphi\|_{\tilde{r}} + \sum_{|\beta| \leq 2m+1} C\left[\|u\|_{X_{\tilde{p}_1,\tilde{q}_1}}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)} + \|u\|_{X_{\bar{p}_1,\bar{q}_1}}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}\right] \\ &\quad \times (\|u\|_{X_{p,q}} + \|u\|_{X_{\tilde{p}_1,\tilde{q}_1}}) \\ &\leq \|\varphi\|_{\tilde{r}} + C\sum_{|\beta| \leq 2m+1} \left[\delta^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)} + \delta^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}\right](\|u\|_{X_{p,q}} + \delta), \end{aligned}$$

or

$$\|u\|_{X_{p,q}} \le C\|\varphi\|_{\bar{r}} + \sum_{|\beta|\le 2m+1} C\left[\|u\|_{X_{\bar{p}_1,\bar{q}_1}}^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)} + \|u\|_{X_{\tilde{p}_1,\tilde{q}_1}}^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}\right] \times (\|u\|_{X_{p,q}} + \|u\|_{X_{\bar{p}_1,\bar{q}_1}})$$

$$\le C\|\varphi\|_{\bar{r}} + C\sum_{|\beta|\le 2m+1}\left[\delta^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)} + \delta^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)}\right](\|u\|_{X_{p,q}} + \delta),$$

by Step 1, which implies $u \in X_{p,q}$.

Step 4. Let $(\hat{p}, \hat{q}, \hat{r})$ be any generalized admissible triplet with $\bar{r} \le \hat{r} \le \tilde{r}$. Similar to the proof of Step 3 and taking into account Remark 3.1, we have

$$\|e^{-At}\varphi\|_{X_{\hat{p},\hat{q}}} \le C\|\varphi\|_{\tilde{r}} + C\|\varphi\|_{\bar{r}}.$$

Thus we have

$$\|u\|_{X_{\hat{p},\hat{q}}} \le C\|\varphi\|_{\tilde{r}} + C\|\varphi\|_{\bar{r}} + C\sum_{|\beta|\le 2m+1}\left[\delta^{\tilde{b}\left(1-\frac{|\beta|}{2m}\right)} + \delta^{\bar{b}\left(1-\frac{|\beta|}{2m}\right)}\right](\|u\|_{X_{\hat{p},\hat{q}}} + \delta),$$

this implies $u \in X_{\hat{p},\hat{q}}$. Therefore we complete the proof of Theorem 2.3.

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