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The Life Span of Blow-up Solutions for a Weakly Coupled System of Reaction-Diffusion Equations

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Abstract. We consider the weakly coupled system of reaction-diffusion equations

$$u_t = \Delta u + |x|^{\sigma_1} v^p, \quad v_t = \Delta v + |x|^{\sigma_2} u^q,$$
$$u(x,0) = \lambda^\mu \varphi(x), \quad v(x,0) = \lambda^\nu \psi(x)$$

where $x \in \mathbf{R}^N$, $t > 0$, $p, q > 1$ and $0 \le \sigma_1 < N(p-1)$, $0 \le \sigma_2 < N(q-1)$. The existence of solutions, blow-up conditions, and global solutions of the above equations are studied by Mochizuki-Huang. In this paper, we consider the estimate of maximal existence time of blow-up solutions in $I^{\delta_1} \times I^{\delta_2}$ as λ goes to 0 or ∞.

1. Introduction.

We consider nonnegative solutions of the initial value problem for a weakly coupled system

$$\begin{cases} u_t = \Delta u + |x|^{\sigma_1} v^p & (x \in \mathbf{R}^N, t > 0), \\ v_t = \Delta v + |x|^{\sigma_2} u^q & (x \in \mathbf{R}^N, t > 0), \\ u(x,0) = \lambda^\mu \varphi(x) & (x \in \mathbf{R}^N), \\ v(x,0) = \lambda^\nu \psi(x) & (x \in \mathbf{R}^N), \end{cases} \tag{1}$$

where $N \ge 1$, $p, q > 1$, $0 \le \sigma_1 < N(p-1)$, $0 \le \sigma_2 < N(q-1)$, $\lambda > 0$ is a parameter, μ, ν are positive constants. Since the nonlinearities, $|x|^{\sigma_1} v^p$, $|x|^{\sigma_2} u^q$, are locally Hölder continuous in x and locally Lipschitz in u, v, it follows from standard results that any solution $u(x,t), v(x,t) \ge 0$ of the equation (1) is in fact classical; that is, $u, v \in C^{2,1}(\mathbf{R}^N \times (0,T)) \cap C(\mathbf{R}^N \times [0,T))$.

Throughout the rest of this paper we shall use the following notations. We put

$$\alpha = \frac{2(p+1)}{pq-1}, \quad \beta = \frac{2(q+1)}{pq-1}, \quad \delta_1 = \frac{\sigma_2 p + \sigma_1}{pq-1}, \quad \delta_2 = \frac{\sigma_1 q + \sigma_2}{pq-1}.$$

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We set BC to be the space of all bounded continuous functions in $\mathbf{R}^N$ and for $a \geq 0$,

$$I^a \equiv \left\{\xi \in BC; \xi(x) \geq 0, \limsup_{|x|\to\infty} |x|^a \xi(x) < \infty\right\},$$

$$I_a \equiv \left\{\xi \in BC; \xi(x) \geq 0, \liminf_{|x|\to\infty} |x|^a \xi(x) > 0\right\}.$$

It is obvious that

$$\|\xi\|_{\infty,a} \equiv \sup_{x\in\mathbf{R}^N} \langle x\rangle^a |\xi(x)| < \infty$$

holds if $\xi \in I^a$, where $\langle x\rangle = \sqrt{1+|x|^2}$. The letter C denotes a positive generic constant which may vary from line to line. We shall use the notation $S(t)\xi$ to represent the solution of the heat equation with initial value $\xi(x)$:

$$S(t)\xi(x) \equiv (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \exp\left(-\frac{|x-y|^2}{4t}\right) \xi(y)dy.$$

In the following we require

$$(\varphi, \psi) \in I^{\delta_1} \times I^{\delta_2}. \tag{2}$$

Then as is proved by K. Mochizuki and Q. Huang in [7], problem (1) has a unique, nonnegative solution $(u(\cdot, t), v(\cdot, t))$ which satisfies

$$\sup_{t\in[0,T)} \{\|u(t)\|_{\infty,\delta_1} + \|v(t)\|_{\infty,\delta_2}\} < \infty \quad \text{for } 0 < \exists T \leq \infty$$

when (2) holds. We let $T_\lambda^* > 0$ be the maximal existence time:

$$T_\lambda^* \equiv \sup\left\{T > 0;\ \sup_{t\in[0,T)} \{\|u(t)\|_{\infty,\delta_1} + \|v(t)\|_{\infty,\delta_2}\} < \infty\right\}.$$

If $T_\lambda^* = \infty$, the solutions are global. The global existence and nonexistence are studied by Escobedo-Herrero [1] and Mochizuki [6] in the case $\sigma_1 = \sigma_2 = 0$, and are generalized in [7] to the problem (1). The following two results are proved in [7].

1. if $\max\{\alpha+\delta_1, \beta+\delta_2\} \geq N$, then $T_\lambda^* < \infty$ for every nontrivial solution $(u(t), v(t))$ of (1);

2. if $\max\{\alpha+\delta_1, \beta+\delta_2\} < N$ and $\varphi \in I_a$ with $a < \alpha+\delta_1$ (or $\psi \in I_b$ with $b < \beta+\delta_2$), then $T_\lambda^* < \infty$ for every nontrivial solution $(u(t), v(t))$ of (1).

In this paper, we shall consider a precise estimate of T_λ^* as λ goes to 0 or ∞. This problem is studied in Huang-Mochizuki-Mukai [4] and Mochizuki [6] in the special case $\sigma_1 = \sigma_2 = 0$. We shall extend the results to the case $0 \leq \sigma_1 < N(p-1)$ and $0 \leq \sigma_2 < N(q-1)$.

THEOREM 1. *Suppose that $\delta_1 \leq a < \alpha+\delta_1$ and $a < N$ (or $\delta_2 \leq b < \beta+\delta_2$ and $b < N$). Let $\varphi \in I_a$ (or $\psi \in I_b$). Then there exist $\lambda_1 > 0$ and $C > 0$ such that*

$$T_\lambda^* \leq C\lambda^{-\frac{2\mu}{\alpha+\delta_1-a}} \ (or \leq C\lambda^{-\frac{2\nu}{\beta+\delta_2-b}}) \quad for\ \lambda < \lambda_1.$$

THEOREM 2. *Suppose that* $\delta_1 \le a < \alpha + \delta_1$, $\delta_2 \le b < \beta + \delta_2$, $a, b < N$ *and*

$$0 \le pb - a - \sigma_1 < 2 \quad or \quad 0 \le qa - b - \sigma_2 < 2 . \tag{3}$$

Let μ, ν *be chosen to satisfy*

$$\frac{\mu}{\nu} = \frac{\alpha + \delta_1 - a}{\beta + \delta_2 - b} , \tag{4}$$

and let $(\varphi, \psi) \in (I^a \cap I_a) \times (I^b \cap I_b)$. *Then we have*

$$T_\lambda^* \sim \lambda^{-\frac{2\mu}{\alpha+\delta_1-a}} = \lambda^{-\frac{2\nu}{\beta+\delta_2-b}} \quad as \ \lambda \to 0 .$$

REMARK 1. Let s satisfies

$$0 \le s < \min\left\{1, \frac{N-\delta_1}{\alpha}, \frac{N-\delta_2}{\beta}\right\} ,$$

and put $a = s\alpha + \delta_1$, $b = s\beta + \delta_2$. Then a, b satisfies (3).

REMARK 2. If we put $\sigma_1 = \sigma_2 = 0$ in Theorems 1 and 2, these results are the same one as Theorem 4.4 (i) and (ii) in [6] respectively.

These theorems are the upper and lower bound estimate of T_λ^* as $\lambda \to 0$, and we shall prove them in Sections 2 and 3, respectively. The methods of proof are quite similar to [6]. But, about the upper bound, the definition of $F_{k,\varepsilon}(t)$ and $G_{k,\varepsilon}(t)$ is slightly different from [6] to satisfy $F_{k,\varepsilon}(t) \le C\|u_k(t)\|_{\infty,\delta_1}$ and $G_{k,\varepsilon}(t) \le C\|v_k(t)\|_{\infty,\delta_2}$, and about the lower bound, we will add Lemma 3.3 to prove Lemma 3.4 (iii) in this paper.

It is also an interesting problem to obtain the estimates of T_λ^* as $\lambda \to \infty$. Some results of this problem are proved in the last Section 4.

2. Proof of Theorem 1.

$(u(t), v(t))$ is a solution of (1). We put

$$u_k(x,t) = k^{\alpha+\delta_1} u(kx, k^2 t) , \quad v_k(x,t) = k^{\beta+\delta_2} v(kx, k^2 t)$$

for $k > 0$. As is easily seen $(u_k(t), v_k(t))$ solves the system

$$\begin{cases} u_{kt} = \Delta u_k + |x|^{\sigma_1} v_k^p , \\ v_{kt} = \Delta v_k + |x|^{\sigma_2} u_k^q , \\ u_k(x,0) = k^{\alpha+\delta_1} \lambda^\mu \varphi(kx) , \\ v_k(x,0) = k^{\beta+\delta_2} \lambda^\nu \psi(kx) . \end{cases} \tag{5}$$

Let $\tilde{T}_k^*$ be the life span of $(u_k(t), v_k(t))$. Then obviously

$$T_\lambda^* = k^2 \tilde{T}_k^* . \tag{6}$$

We define

$$F_{k,\varepsilon}(t) = \varepsilon^{\frac{N-\delta_1}{2}} \int_{\mathbf{R}^N} u_k(x,t) e^{-\varepsilon|x|^2} dx , \quad G_{k,\varepsilon}(t) = \varepsilon^{\frac{N-\delta_2}{2}} \int_{\mathbf{R}^N} v_k(x,t) e^{-\varepsilon|x|^2} dx ,$$

then it follows that

$$\begin{aligned} F_{k,\varepsilon}(t) &\leq \|u_k(t)\|_{\infty,\delta_1} \int_{\mathbf{R}^N} |y|^{-\delta_1} e^{-|y|^2} dy \\ &\leq C\|u_k(t)\|_{\infty,\delta_1}, \\ G_{k,\varepsilon}(t) &\leq C\|v_k(t)\|_{\infty,\delta_2}, \end{aligned}$$

where $C > 0$ is independent of $\varepsilon > 0$. Since

$$\Delta e^{-\varepsilon|x|^2} = (-2N\varepsilon + 4\varepsilon^2|x|^2)e^{-\varepsilon|x|^2} \geq -2N\varepsilon e^{-\varepsilon|x|^2}$$

and, by Hölder's inequality,

$$\begin{aligned} \varepsilon^{\frac{N-\delta_2}{2}} \int_{\mathbf{R}^N} v_k(x,t) e^{-\varepsilon|x|^2} dx &= \int_{\mathbf{R}^N} \varepsilon^{\frac{N-\delta_1}{2p}} |x|^{\frac{\sigma_1}{p}} v_k(x,t) \cdot \varepsilon^{\frac{N(p-1)-\sigma_1}{2p}} |x|^{-\frac{\sigma_1}{p}} e^{-\varepsilon|x|^2} dx \\ &\leq \left(\varepsilon^{\frac{N-\delta_1}{2}} \int_{\mathbf{R}^N} |x|^{\sigma_1} v_k(x,t)^p e^{-\varepsilon|x|^2} dx\right)^{\frac{1}{p}} \\ &\quad \times \left(\varepsilon^{\frac{N}{2}-\frac{\sigma_1}{2(p-1)}} \int_{\mathbf{R}^N} |x|^{-\frac{\sigma_1}{p-1}} e^{-\varepsilon|x|^2} dx\right)^{\frac{p-1}{p}}, \end{aligned}$$

we have

$$\begin{aligned} &\frac{d}{dt}\varepsilon^{\frac{N-\delta_1}{2}} \int_{\mathbf{R}^N} u_k(x,t) e^{-\varepsilon|x|^2} dx \\ &= \varepsilon^{\frac{N-\delta_1}{2}} \int_{\mathbf{R}^N} (\Delta u_k(x,t) + |x|^{\sigma_1} v_k(x,t)^p) e^{-\varepsilon|x|^2} dx \\ &\geq -2N\varepsilon \cdot \varepsilon^{\frac{N-\delta_1}{2}} \int_{\mathbf{R}^N} u_k(x,t) e^{-\varepsilon|x|^2} dx \\ &\quad + \left(\int_{\mathbf{R}^N} |y|^{-\frac{\sigma_1}{p-1}} e^{-|y|^2} dy\right)^{-p+1} \left(\varepsilon^{\frac{N-\delta_2}{2}} \int_{\mathbf{R}^N} v_k(x,t) e^{-\varepsilon|x|^2} dx\right)^p, \end{aligned}$$

so the following inequalities hold:

$$\begin{cases} F'_{k,\varepsilon}(t) \geq -2N\varepsilon F_{k,\varepsilon}(t) + \bar{C}_p G_{k,\varepsilon}(t)^p & (t > 0) \\ G'_{k,\varepsilon}(t) \geq -2N\varepsilon G_{k,\varepsilon}(t) + \bar{C}_q F_{k,\varepsilon}(t)^q & (t > 0), \end{cases} \tag{7}$$

where

$$\bar{C}_p = \left(\int_{\mathbf{R}^N} |y|^{-\frac{\sigma_1}{p-1}} e^{-|y|^2} dy\right)^{-p+1}, \quad \bar{C}_q = \left(\int_{\mathbf{R}^N} |y|^{-\frac{\sigma_2}{q-1}} e^{-|y|^2} dy\right)^{-q+1}.$$

Let us consider the system of ordinary differential equations

$$\begin{cases} f'_\varepsilon(t) = -2N\varepsilon f_\varepsilon(t) + \bar{C}_p g_\varepsilon(t)^p & (t > 0) \\ g'_\varepsilon(t) = -2N\varepsilon g_\varepsilon(t) + \bar{C}_p f_\varepsilon(t)^q & (t > 0) \\ f_\varepsilon(0) = F_{k,\varepsilon}(0), \quad g_\varepsilon(0) = G_{k,\varepsilon}(0). \end{cases} \tag{8}$$

Then $(F_{k.\varepsilon}(t), G_{k,\varepsilon}(t))$ is a supersolution of (8). By the scaling

$$f(t) = (2N\varepsilon)^{-\frac{\alpha}{2}} \bar{C}_p^{\frac{1}{pq-1}} \bar{C}_q^{\frac{p}{pq-1}} f_\varepsilon(t/2N\varepsilon)$$

$$g(t) = (2N\varepsilon)^{-\frac{\beta}{2}} \bar{C}_p^{\frac{q}{pq-1}} \bar{C}_q^{\frac{1}{pq-1}} g_\varepsilon(t/2N\varepsilon),$$

we obtain the simpler system of equations

$$\begin{cases} f'(t) = -f(t) + g(t)^p & (t > 0) \\ g'(t) = -g(t) + f(t)^q & (t > 0). \end{cases} \tag{9}$$

LEMMA 2.1. *Let $(f(t), g(t))$ be the solution to* (9) *with the initial data*

$$f(0) > 1, \quad g(0) = 0.$$

If $f(0)$ is sufficiently large, then $(f(t), g(t))$ blows up in finite time. Moreover, the life span T_0 of $(f(t), g(t))$ is estimated from above like

$$T_0 \le t_0 + \int_{f(t_0)g(t_0)}^{\infty} \{C(p, q)\xi^{\frac{(p+1)(q+1)}{p+q+2}} - 2\xi\}^{-1} d\xi, \tag{10}$$

where

$$C(p, q) = \left(\frac{p+q+2}{p+1}\right)^{\frac{p+1}{p+q+2}} \left(\frac{p+q+2}{q+1}\right)^{\frac{q+1}{p+q+2}}$$

and $0 < t_0 < T_0$ is chosen to satisfy $\{f(t_0)g(t_0)\}^{(pq-1)/(p+q+2)} > 2$.

PROOF. See e.g., K. Mochizuki [6]. □

As is shown is above lemma, there exist $A_1 > 0$ and $B_1 > 0$ such that if

$$f(0) > A_1 \quad or \quad g(0) > B_1,$$

then $(f(t), g(t))$ blows up in finite time. As a result of these arguments and a comparison principle, we have the following about $(F_{k,\varepsilon}(t), G_{k,\varepsilon}(t))$.

LEMMA 2.2. *Let $(F_{k,\varepsilon}(t), G_{k,\varepsilon}(t))$ satisfy differential inequalities* (7). *If*

$$F_{k.\varepsilon}(0) > A\varepsilon^{\frac{\alpha}{2}}, \quad or \quad G_{k,\varepsilon}(0) > B\varepsilon^{\frac{\beta}{2}} \tag{11}$$

for some $\varepsilon > 0$, then $(F_{k,\varepsilon}(t), G_{k,\varepsilon}(t))$ blows up in finite time, where

$$A = (2N)^{-\frac{\alpha}{2}} \bar{C}_p^{\frac{1}{pq-1}} \bar{C}_q^{\frac{p}{pq-1}} A_1, \quad B = (2N)^{-\frac{\beta}{2}} \bar{C}_p^{\frac{q}{pq-1}} \bar{C}_q^{\frac{1}{pq-1}} B_1.$$

Moreover, its life span is estimated from above by $(2N\varepsilon)^{-1}T_0$. Thus, we obtain

$$\tilde{T}_k^* \le (2N\varepsilon)^{-1} T_0. \tag{12}$$

Note that there is only one equilibria of system (8) in $\mathbf{R}_+^2$, say

$$P = ((2N\varepsilon)^{\frac{\alpha}{2}}, (2N\varepsilon)^{\frac{\beta}{2}}).$$

As is easily seen, P is a saddle point. One of the separatrix starts from 0 and runs to ∞. Another one intersects f-axis and g-axis at $A\varepsilon^{\alpha/2}$ and $B\varepsilon^{\beta/2}$, respectively. Moreover, every solution $(f_\varepsilon(t), g_\varepsilon(t))$ of (8) with the initial value of $(f_\varepsilon(0), g_\varepsilon(0))$ lying above this separatrix runs into

$$Q = \{(f, g) \in \mathbf{R}_+^2; (\bar{C}_p^{-1} 2N\varepsilon f)^{\frac{1}{p}} < g < \bar{C}_q (2N\varepsilon)^{-1} f^q\},$$

and then blows up in finite time. As for these argument, see e.g., Galktionov-Kurdyumov-Samarskii [2], [3] or Qi-Levine [8].

LEMMA 2.3. *Suppose $\varphi \in I_a$ for some $a < N$ and $\delta_1 \le a < \alpha + \delta_1$. Put*

$$\lambda^\mu = k^{-\alpha-\delta_1+a} \tag{13}$$

in (7). *Then there exists a constant $K_1 > 0$ independent of $\varepsilon > 0$ such that*

$$\liminf_{\lambda \to 0} F_{k,\varepsilon}(0) \ge K_1 \varepsilon^{\frac{a-\delta_1}{2}}.$$

PROOF. By (13),

$$F_{k,\varepsilon}(0) = k^a \varepsilon^{\frac{N-\delta_1}{2}} \int_{\mathbf{R}^N} \varphi(kx) e^{-\varepsilon |x|^2} dx.$$

Putting $y = \varepsilon^{1/2} x$, it holds

$$F_{k,\varepsilon}(0) = k^a \varepsilon^{-\frac{\delta_1}{2}} \int_{\mathbf{R}^N} \varphi(\varepsilon^{-\frac{1}{2}} ky) e^{-|y|^2} dy.$$

For an arbitraly $M > 0$ such that $\liminf_{k\to\infty} |x|^a \varphi(x) > M$, there exists a $k_0 > 0$ such that

$$\varepsilon^{-\frac{a}{2}} k^a |y|^a \varphi(\varepsilon^{-\frac{1}{2}} ky) > M \quad \text{for } k > k_0.$$

Thus, we obtain

$$\liminf_{k\to\infty} F_{k,\varepsilon}(0) \ge M \varepsilon^{\frac{a-\delta_1}{2}} \int_{\mathbf{R}^N} |y|^{-a} e^{-|y|^2} dy \ge K_1 \varepsilon^{\frac{a-\delta_1}{2}}.$$

□

PROOF OF THEOREM 1. We only prove the first inequality. Let k be chosen as in (13). Then by assumption we see $k \to \infty$ as $\lambda \to 0$. Thus, by Lemma 2.3, if we choose $\varepsilon > 0$ to satisfy $\varepsilon^{(\alpha+\delta_1-a)/2} < K_1 A^{-1}$, there exists $\lambda_1 > 0$ such that $F_{k,\varepsilon}(0) > A\varepsilon^{\alpha/2}$ for $\lambda < \lambda_1$. Thus, we can apply Lemma 2.2 and (6) to conclude the result. □

3. Proof of Theorem 2.

We set for $\gamma > 0$

$$\eta_\gamma(x, t) = S(t) \langle x \rangle^{-\gamma}. \tag{14}$$

LEMMA 3.1. *We have*

$$\eta_\gamma(x, t) \ge C \min\{\langle x \rangle^{-\gamma}, (1+t)^{-\frac{\gamma}{2}}\}.$$

PROOF. Assume first $t < 1$. If $|z| < 1/\sqrt{2}$, then

$$1 + |x - z|^2 \le 1 + 2(|x|^2 + |z|^2) < 2(1 + |x|^2),$$

so we have

$$\begin{aligned}\eta_\gamma(x,t) &\geq (4\pi t)^{-\frac{N}{2}} \int_{|z|<1/\sqrt{2}} \langle x-z\rangle^{-\gamma} e^{-\frac{|z|^2}{4t}} dz \\ &\geq 2^{-\frac{\gamma}{2}} (4\pi t)^{-\frac{N}{2}} \langle x\rangle^{-\gamma} \int_{|z|<1/\sqrt{2}} e^{-\frac{|z|^2}{4t}} dz \\ &= 2^{-\frac{\gamma}{2}} \pi^{-\frac{N}{2}} \langle x\rangle^{-\gamma} \int_{|y|<(8t)^{-1/2}} e^{-|y|^2} dy \\ &\geq C\langle x\rangle^{-\gamma} \quad \text{for } t<1\,.\end{aligned}$$

Next, let $t \geq 1$. Since

$$1+|x-\sqrt{t}z|^2 = 1+t|xt^{-\frac{1}{2}}-z|^2 \leq t(1+|xt^{-\frac{1}{2}}-z|^2)\,,$$

we have

$$\begin{aligned}\eta_\gamma(x,t) &= (4\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \langle x-\sqrt{t}z\rangle^{-\gamma} e^{-\frac{|z|^2}{4}} dz \\ &\geq (4\pi)^{-\frac{N}{2}} t^{-\frac{\gamma}{2}} \int_{\mathbf{R}^N} \langle xt^{-\frac{1}{2}}-z\rangle^{-\gamma} e^{-\frac{|z|^2}{4}} dz\,.\end{aligned}$$

If $|x|t^{-1/2} < 1/\sqrt{2}$, this shows

$$\eta_\gamma(x,t) \geq 2^{-\frac{\gamma}{2}} (4\pi)^{-\frac{N}{2}} t^{-\frac{\gamma}{2}} \int_{\mathbf{R}^N} \langle z\rangle^{-\gamma} e^{-\frac{|z|^2}{4}} dz \geq Ct^{-\frac{\gamma}{2}}\,.$$

On the other hand, if $\xi = |x|t^{-1/2} \geq 1/\sqrt{2}$, we have

$$\begin{aligned}|x|^\gamma \eta_\gamma(x,t) &\geq (4\pi)^{-\frac{N}{2}} |\xi|^\gamma \int_{\mathbf{R}^N} \langle \xi-z\rangle^{-\gamma} e^{-\frac{|z|^2}{4}} dz \\ &\leq 2^{-\frac{\gamma}{2}} (4\pi)^{-\frac{N}{2}} |\xi|^\gamma \langle\xi\rangle^{-\gamma} \int_{|z|<1/\sqrt{2}} e^{-\frac{|z|^2}{4}} dz \\ &\to 2^{-\frac{\gamma}{2}} (4\pi)^{-\frac{N}{2}} \int_{|z|<1/\sqrt{2}} e^{-\frac{|z|^2}{4}} dz > 0 \quad \text{as } \xi\to\infty\,.\end{aligned}$$

Summarizing these results, we obtain the inequality in the lemma. □

LEMMA 3.2. *Let $0<\gamma<N$ and $0\leq\delta\leq\gamma$. Then we have for any $t>0$,*

$$\|\eta_\gamma(\cdot,t)\|_{\infty,\delta} \leq C(1+t)^{\frac{-\gamma+\delta}{2}}\,.$$

PROOF.

$$\begin{aligned}
|x|^\delta \eta_\gamma(x,t) &= |x|^\delta \left\{ \int_{|y|<|x|/\sqrt{2}} + \int_{|y|>|x|/\sqrt{2}} \right\} (4\pi t)^{-\frac{N}{2}} \langle x-y\rangle^{-\gamma} e^{-\frac{|y|^2}{4t}} dy \\
&\equiv I + II \\
I &\le 2^{\frac{\delta}{2}} (4\pi t)^{-\frac{N}{2}} \int_{|y|<|x|/\sqrt{2}} \langle x-y\rangle^{\delta-\gamma} e^{-\frac{|y|^2}{4t}} dy \\
&\le 2^{\frac{\delta}{2}} S(t)\langle x\rangle^{\delta-\gamma} \\
II &\le 2^{\frac{N}{2}} (8\pi t)^{-\frac{N}{2}} |x|^\delta e^{-\frac{|x|^2}{16t}} \int_{|y|>|x|/\sqrt{2}} \langle x-y\rangle^{-\gamma} e^{-\frac{|y|^2}{8t}} dy \\
&\le c t^{\frac{\delta}{2}} S(2t)\langle x\rangle^{-\gamma},
\end{aligned}$$

where we use the fact

$$2\langle x-y\rangle^2 \ge 2|x-y|^2 \ge 2(|x|^2-|y|^2) \ge |x|^2 \quad \text{for } |y| < |x|/\sqrt{2},$$

$$\sup_{(x,t)\in\mathbf{R}^N\times(0,\infty)} \{|x|^\delta t^{-\frac{\delta}{2}} e^{-\frac{|x|^2}{16t}}\} < \infty.$$

Since $\gamma < N$,

$$\begin{aligned}
t^{\frac{\gamma}{2}} S(t)\langle x\rangle^{-\gamma} &= (4\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} t^{\frac{\gamma}{2}} \langle x-\sqrt{t}z\rangle^{-\gamma} e^{-\frac{|z|^2}{4}} dz \\
&\le (4\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} |xt^{-\frac{1}{2}} - z|^{-\gamma} e^{-\frac{|z|^2}{4}} dz \\
&\to (4\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} |z|^{-\gamma} e^{-\frac{|z|^2}{4}} dz < \infty \quad \text{as } t\to\infty.
\end{aligned}$$

Thus, we obtain (cf., [5, Lemma 2.12])

$$\|S(t)\langle x\rangle^{-\gamma}\|_\infty \le C(1+t)^{-\frac{\gamma}{2}}.$$

By this inequality, we have $II \le C(1+t)^{(-\gamma+\delta)/2}$ and, since

$$\begin{aligned}
S(t)\langle x\rangle^{\delta-\gamma} &\le (1+t)^{\frac{\delta}{2}} (4\pi t)^{-\frac{N}{2}} \int_{|y|<\sqrt{t}} \langle y\rangle^{-\gamma} e^{-\frac{|x-y|^2}{4t}} dy \\
&\quad + (1+t)^{\frac{\delta-\gamma}{2}} (4\pi t)^{-\frac{N}{2}} \int_{|y|>\sqrt{t}} e^{-\frac{|x-y|^2}{4t}} dy \\
&\le (1+t)^{\frac{\delta}{2}} S(t)\langle x\rangle^{-\gamma} + (1+t)^{\frac{\delta-\gamma}{2}},
\end{aligned}$$

we also have $I \le C(1+t)^{(-\gamma+\delta)/2}$. □

LEMMA 3.3. *We have in* $\mathbf{R}^N \times (0,\infty)$,

$$\begin{aligned}
\langle x\rangle^{\sigma_1} \eta_b(x,t)^p &\le C(1+t)^{\frac{\sigma_1+a-pb}{2}} \eta_a(x,t), \\
\langle x\rangle^{\sigma_2} \eta_a(x,t)^q &\le C(1+t)^{\frac{\sigma_2+b-qa}{2}} \eta_b(x,t).
\end{aligned} \tag{15}$$

PROOF. We only consider the case $b \geq a$. A similar argument can be applied also to the case $b \leq a$. We have by Lemma 3.1,

$$\begin{aligned}\langle x\rangle^{\sigma_1}\eta_b(x,t)^p &= \langle x\rangle^{\sigma_1}\eta_b(x,t)^p\eta_a(x,t)^{-1}\eta_a(x,t)\\ &\leq C\langle x\rangle^{\sigma_1}\max\{\langle x\rangle^a, (1+t)^{\frac{a}{2}}\}\eta_b(x,t)^p\eta_a(x,t)\,.\end{aligned}$$

Suppose $t \leq |x|^2$. Since $a+\sigma_1 \leq pb$ by (3), we have by Lemma 3.2

$$\begin{aligned}\langle x\rangle^{\sigma_1}\eta_b(x,t)^p &\leq C\{\langle x\rangle^{\frac{a+\sigma_1}{p}}\eta_b(x,t)\}^p\eta_a(x,t)\\ &\leq C(1+t)^{\frac{-pb+a+\sigma_1}{2}}\eta_a(x,t)\,.\end{aligned}$$

Suppose $t \geq |x|^2$. Then we have by Lemma 3.2

$$\begin{aligned}\langle x\rangle^{\sigma_1}\eta_b(x,t)^p &\leq C(1+t)^{\frac{a}{2}}\{\langle x\rangle^{\frac{\sigma_1}{p}}\eta_b(x,t)\}^p\eta_a(x,t)\\ &\leq C(1+t)^{\frac{a}{2}}(1+t)^{\frac{-pb+\sigma_1}{2}}\eta_a(x,t)\,.\end{aligned}$$

Next, by Jensen's inequality, we have

$$\begin{aligned}\langle x\rangle^{\sigma_2}\eta_a(x,t)^q &= \langle x\rangle^{\sigma_2}\eta_a(x,t)^{q-\frac{b}{a}}\eta_a(x,t)^{\frac{b}{a}}\\ &\leq \langle x\rangle^{\sigma_2}\eta_a(x,t)^{q-\frac{b}{a}}\eta_b(x,t)\\ &\leq \langle x\rangle^{\sigma_2}\max\{\langle x\rangle^b, (1+t)^{\frac{b}{2}}\}\eta_a(x,t)^q\eta_b(x,t)\\ &\leq C(1+t)^{\frac{-qa+b+\sigma_2}{2}}\eta_b(x,t)\,,\end{aligned}$$

since $b \geq a$ and $b+\sigma_2 \leq qa$. □

We put

$$W_1(x,t) = \lambda^\mu\|\varphi\|_{\infty,a}\eta_a(x,t)\,,\quad W_2(x,t) = \lambda^\nu\|\psi\|_{\infty,b}\eta_b(x,t)\,, \tag{16}$$

where $\varphi \in I^a$, $\psi \in I^b$. As is easily verified from Lemmas 3.2 and 3.3, we have the following

LEMMA 3.4. (i) *$W_j(x,t) > 0$ $(j=1,2)$ and $|x|^a W_1(x,t)$, $|x|^b W_2(x,t)$ are bounded in $\mathbf{R}^N \times [0,\infty)$.*

(ii) *There exists a constant $C > 0$ such that for any $t \geq 0$,*

$$\|W_1(\cdot,t)\|_{\infty,\delta_1} \leq C(1+t)^{\frac{-a+\delta_1}{2}}\,,\quad \|W_2(\cdot,t)\|_{\infty,\delta_2} \leq C(1+t)^{\frac{-b+\delta_2}{2}}\,.$$

(iii) *There exists a constant $C_1 > 0$ such that for any $t \geq 0$,*

$$\begin{aligned}\|W_2(\cdot,t)^p/W_1(\cdot,t)\|_{\infty,\sigma_1} &\leq C_1\lambda^{p\nu-\mu}(1+t)^{\frac{a+\sigma_1-pb}{2}}\,,\\ \|W_1(\cdot,t)^q/W_2(\cdot,t)\|_{\infty,\sigma_2} &\leq C_1\lambda^{q\mu-\nu}(1+t)^{\frac{b+\sigma_2-qa}{2}}\,.\end{aligned}$$

Now, let $(\alpha(t),\beta(t))$ be the solution of

$$\begin{cases}\alpha' = \|W_2(\cdot,t)^p/W_1(\cdot,t)\|_{\infty,\sigma_1}\beta^p & (t>0)\,,\\ \beta' = \|W_1(\cdot,t)^q/W_2(\cdot,t)\|_{\infty,\sigma_2}\alpha^q & (t>0)\,,\\ \alpha(0)=\beta(0)=1\,,\end{cases} \tag{17}$$

and let us define $(\bar{u}(x,t), \bar{v}(x,t))$ as follows:

$$\bar{u}(x,t) = \alpha(t) W_1(x,t), \quad \bar{v}(x,t) = \beta(t) W_2(x,t). \tag{18}$$

LEMMA 3.5. (i) *Let $(x(t), y(t))$ be the solution to*

$$\begin{cases} x' = h(t) y^p, \quad y' = h(t) x^q \quad (t > 0), \\ x(0) = y(0) = 1, \end{cases} \tag{19}$$

where $h(t) = C_1 \max\{\lambda^{p\nu-\mu}(1+t)^{\frac{a+\sigma_1-pb}{2}}, \lambda^{q\mu-\nu}(1+t)^{\frac{b+\sigma_2-qa}{2}}\}$. Then $(\alpha(t), \beta(t))$ is a subsolution of (19).

(ii) *Suppose that $\varphi \in I^a$ and $\psi \in I^b$. Then $(\bar{u}(x,t), \bar{v}(x,t))$ gives a supersolution of* (1).

PROOF. (i) is obvious from Lemma 3.4 (iii). (ii) We have

$$\begin{aligned} \bar{u}_t &= \alpha'(t) W_1(x,t) + \alpha(t) W_{1t}(x,t) \\ &= \| W_2^p / W_1 \|_{\infty,\sigma_1} \beta^p W_1 + \alpha \Delta W_1 \\ &\geq \langle x \rangle^{\sigma_1} W_2^p \beta^p + \alpha \Delta W_1 \geq |x|^{\sigma_1} \bar{v}^p + \Delta \bar{u}. \end{aligned}$$

Similarly, we have $\bar{v}_t \geq |x|^{\sigma_2} \bar{u}^q + \Delta \bar{v}$. Moreover, as is easily verified from (16) and $\alpha(0) = \beta(0) = 1$, we have

$$\bar{u}(x,0) \geq \lambda^{\mu} \varphi(x), \quad \bar{v}(x,0) \geq \lambda^{\nu} \psi(x).$$

These results show the assertion. □

PROOF OF THEOREM 2. It follows from Lemma 3.5 and comparison principle that

$$\begin{gathered} u(x,t) \leq \bar{u}(x,t), \quad v(x,t) \leq \bar{v}(x,t), \\ \alpha(t) \leq x(t), \quad \beta(t) \leq y(t). \end{gathered}$$

Then we see from (18) that T_λ^* is not less than the life span of $(x(t), y(t))$.

From equation (19) it follows that

$$\int_0^t x^q x' dt = \int_0^t y^p y' dt.$$

Suppose $p \geq q$ (another case can be proved in the same way). Then we have

$$\frac{x(t)^{q+1}}{q+1} \geq \frac{y(t)^{p+1}}{p+1}, \quad y(t) \leq \left(\frac{p+1}{q+1}\right)^{\frac{1}{p+1}} x(t)^{\frac{q+1}{p+1}}.$$

Substitute this in the first equation of (19), we have

$$x(t)^{-\frac{p(q+1)}{p+1}} x' \leq \left(\frac{p+1}{q+1}\right)^{\frac{p}{p+1}} h(t).$$

Integrating this again from 0 to t, we obtain

$$-\frac{\beta}{2}(x(t)^{-\frac{2}{\beta}} - 1) \leq C_1(p,q) \int_0^t h(t) dt,$$

$$x(t) \le \left\{1 - \frac{2}{\beta} C_1(p,q) \int_0^t h(t)dt\right\}^{-\frac{\beta}{2}} .$$

From the equation (3), $a + \sigma_1 - pb \le 0$, $b + \sigma_2 - qa \le 0$, thus

$$x(t) \le \left\{1 - C \int_0^t \max\{\lambda^{p\nu-\mu} t^{\frac{a+\sigma_1-pb}{2}}, \lambda^{q\mu-\nu} t^{\frac{b+\sigma_2-qa}{2}}\}dt\right\}^{-\frac{\beta}{2}} .$$

From the equation (3),

$$\frac{a+\sigma_1-pb}{2} > -1 , \quad \frac{b+\sigma_2-qa}{2} > -1 ,$$

so the integrand is integrable from 0 to t, thus,

$$x(t) \le \{1 - C \max\{\lambda^{p\nu-\mu} t^{\frac{2+a+\sigma_1-pb}{2}}, \lambda^{q\mu-\nu} t^{\frac{2+b+\sigma_2-qa}{2}}\}\}^{-\frac{\beta}{2}} ,$$

where $C_1(p,q), C > 0$ are independent of λ and t. This implies that $x(t)$ remains finite at least for t less than

$$C \min\{\lambda^{-\frac{2(p\nu-\mu)}{2+a+\sigma_1-pb}}, \lambda^{-\frac{2(q\mu-\nu)}{2+b+\sigma_2-qa}}\} .$$

Integrating the second equation of (19) shows that $y(t)$ is finite in the same interval. Thus, we obtain

$$T_\lambda^* > C \min\{\lambda^{-\frac{2(p\nu-\mu)}{2+a+\sigma_1-pb}}, \lambda^{-\frac{2(q\mu-\nu)}{2+b+\sigma_2-qa}}\} \quad \text{for } \forall\lambda > 0 . \tag{20}$$

Remember here that we have assumed (4). Then since

$$p\beta - \alpha = q\alpha - \beta = 2 , \quad p\delta_2 - \delta_1 = \sigma_1 , \quad q\delta_1 - \delta_2 = \sigma_2 ,$$

it follows that

$$\begin{aligned} \frac{\mu}{\alpha+\delta_1-a} &= \frac{p\nu-\mu}{2+a+\sigma_1-pb} \\ = \frac{\nu}{\beta+\delta_2-b} &= \frac{q\mu-\nu}{2+b+\sigma_2-qa} . \end{aligned}$$

Thus, we can combine (20) and Theorem 1 to conclude assertion of Theorem 2. □

4. Some results for the case $\lambda \to \infty$.

About the estimate of T_λ^* as λ goes to ∞, the following two theorems are obtained.

THEOREM 3. *Suppose that* $\varphi \in I^{\delta_1}$ *and* $\varphi(0) > 0$ (or $\psi \in I^{\delta_2}$ *and* $\psi(0) > 0$). *Then there exist* $\lambda_0 > 0$ *and* $C > 0$ *such that*

$$T_\lambda^* \le C\lambda^{-\frac{2\mu}{\alpha+\delta_1}} \ (or \le C\lambda^{-\frac{2\nu}{\beta+\delta_2}}) \quad for\ \lambda > \lambda_0 .$$

PROOF. We only prove the first inequality. Put $\lambda^\mu = k^{-\alpha-\delta_1}$ in (7). Then

$$\begin{aligned} \lim_{\lambda\to\infty} F_{k,\varepsilon}(0) &= \lim_{k\to 0} \varepsilon^{-\frac{\delta_1}{2}} \int_{\mathbf{R}^N} \varphi(\varepsilon^{-\frac{1}{2}} ky) e^{-|y|^2} dy \\ &= \pi^{\frac{N}{2}} \varphi(0) \varepsilon^{-\frac{\delta_1}{2}} . \end{aligned}$$

So if we choose $\varepsilon > 0$ to satisfy $\varepsilon^{(\alpha+\delta_1)/2} < \pi^{N/2}\varphi(0)A^{-1}$, there exists $\lambda_0 > 0$ such that $F_{k,\varepsilon}(0) > A\varepsilon^{\alpha/2}$ for $\lambda > \lambda_0$. Thus, we can apply Lemma 2.2 and (6) to conclude the result. □

THEOREM 4. *Suppose that* $(\varphi, \psi) \in I^{\delta_1} \times I^{\delta_2}$ *and let* μ, ν *be chosen to satisfy* $\mu/\nu = \alpha/\beta$. *Then we have*

$$T_\lambda^* \geq C\lambda^{-\frac{2\mu}{\alpha}} = C\lambda^{-\frac{2\nu}{\beta}}.$$

PROOF. Put $a = \delta_1$, $b = \delta_2$ in the proof of Theorem 2. Then we have the inequality. □

REMARK 3. The order of λ in upper bound estimate is different from in lower bound estimate.

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