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On a Diophantine Equation Concerning Eisenstein Numbers

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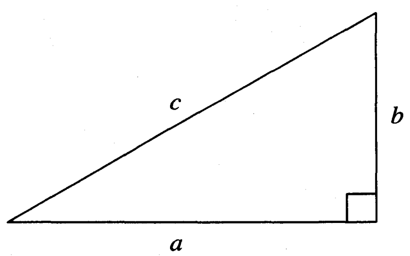
1. Introduction

In 1956, Jeśmanowicz [J] conjectured that if a, b, c are *Pythagorean numbers*, i.e., positive integers satisfying $a^2 + b^2 = c^2$, then the Diophantine equation

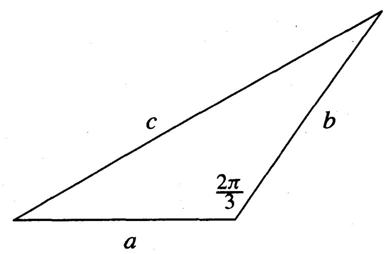
$$a^x + b^y = c^z$$

has only the positive integral solution $(x, y, z) = (2, 2, 2)$. It has been verified that this conjecture holds for many Pythagorean numbers (cf. Sierpiński [S1], [S2], [TA1], [TA2], [Ta1], [Ta2], [GL] and [Le]). This conjecture, however, is still open.

If a, b, c are positive integers satisfying $a^2 + ab + b^2 = c^2$, we call a, b, c *Eisenstein numbers*. Eisenstein numbers have some properties similar to those of Pythagorean numbers. As shown in Lemma 1 below, Eisenstein numbers $a,\ b,\ c$ can be expressed in terms of positive integers $u,\ v$ by factoring $a^2 + ab + b^2 = c^2$ in $\mathbf{Q}(\omega)$, where $\omega = e^{2\pi i/3} = (-1 + \sqrt{-3})/2$. It is worth noting that, geometrically, Pythagorean numbers a, b, c are the sides of a right triangle, and that Eisenstein numbers a, b, c are the sides of a triangle with an interior angle $2\pi/3$. See the figures below.



Pythagorean numbers a, b, c.



Eisenstein numbers a, b, c.

As an analogue to Jeśmanowicz' conjecture, we propose the following (cf. Terai [Te1], [Te2]):

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CONJECTURE. *If a, b, c are fixed positive integers satisfying $a^2+ab+b^2=c^2$ with $(a, b)=1$, then the Diophantine equation*

$$a^{2x}+a^x b^y+b^{2y}=c^z \tag{1}$$

has only the positive integral solution $(x, y, z)=(1, 1, 2)$.

In Sections 3, 4, we show that when a or b is a power of a prime, the Conjecture above holds under some conditions. The proof is based on the results concerning the Diophantine equations of second degree established by using properties of $\mathbf{Q}(\sqrt{-3})$. In Section 5, we also deduce that for Eisenstein numbers a, b, c with $a=p^e q^f$ or $b=p^e q^f$, an upper bound of y or x of equation (1) is derived by applying a result due to Bugeaud [B], which is proved by means of estimates for linear forms in two logarithms.

In Section 6, we verify that the Conjecture holds for all Eisenstein numbers a, b, c with $3 \le a, b \le 100$ and $(a, b)=1$.

2. Lemmas.

LEMMA 1. *Eisenstein numbers a, b, c with $(a, b)=1$ and $a-b \equiv 1 \pmod 3$ are given as follows*:

$$a=u^2-v^2, \quad b=v(2u+v), \quad c=u^2+uv+v^2, \tag{2}$$

where u, v are positive integers such that $(u, v)=1$, $u>v$ and $u \not\equiv v \pmod 3$.

PROOF. We have $c^2=(a-b\omega)(a-b\omega^2)$. Note that $c \not\equiv 0 \pmod 3$, since $(2a+b)^2+3b^2=4c^2$ and $(a, b)=1$. We claim that $\alpha=a-b\omega$ and $\bar{\alpha}=a-b\omega^2$ are relatively prime in $\mathbf{Z}[\omega]$. Indeed, let π be a prime in $\mathbf{Z}[\omega]$ such that $\pi \mid \alpha$ and $\pi \mid \bar{\alpha}$. Then $\pi \mid \bar{\alpha}-\alpha=b\omega(1-\omega)$, which implies that $\pi \mid 1-\omega$, since $(a, b)=1$. In view of $3=-\omega^2(1-\omega)^2$, we see that $c \equiv 0 \pmod 3$, which is a contradiction. Hence there are rational integers u, v such that

$$a-b\omega=\varepsilon\,(u-v\omega)^2,$$

where $\varepsilon=\pm 1, \pm\omega, \pm\omega^2$, and $c=u^2+uv+v^2$. We may suppose that $\varepsilon=\pm 1$, because $\omega=\omega^4$. Then $a-b\omega=\pm\{(u^2-v^2)-(2uv+v^2)\omega\}$. Therefore it is easy to see that

$$a=u^2-v^2, \quad b=v(2u+v), \quad c=u^2+uv+v^2,$$

where u, v are positive integers such that $(u, v)=1$, $u>v$ and $u \not\equiv v \pmod 3$.

We note that

$$a=u^2-v^2, \quad b=v(2u+v) \Leftrightarrow a-b \equiv 1 \pmod 3.$$

Indeed, if $a=u^2-v^2$, $b=v(2u+v)$, then $a-b=u^2-v^2-v(2u+v)=(u-v)^2-3v^2 \equiv 1 \pmod 3$, since $u \not\equiv v \pmod 3$. □

REMARK. In the table below, we give all Eisenstein numbers a, b, c with $(a, b)=1$, $a-b \equiv 1 \pmod 3$ and $3 \le a, b \le 100$.

TABLE.

a	b	c	a	b	c
3	5	7	35	13	43
5	16	19	40	51	79
7	33	37	45	32	67
8	7	13	55	57	97
9	56	61	63	17	73
11	85	91	65	88	133
16	39	49	77	40	103
24	11	31	80	19	91
24	95	109	91	69	139

LEMMA 2. (1) (Nagell [N1]). *The Diophantine equation*

$$x^2 + x + 1 = y^n$$

has only the positive integral solution $(x, y, n) = (18, 7, 3)$ *with* $n \geq 2$.
(2) (Nagell [N2]). *The Diophantine equation*

$$x^2 + 3 = y^n$$

has only the positive integral solution $(x, y, n) = (1, 2, 2)$ *with* $n \geq 2$.
(3) (Nagell [N3]). *The Diophantine equation*

$$3x^2 + 1 = y^n$$

has no positive integral solutions x, y, n *with* $n \geq 3$.
(4) (Ljunggren [Lj]). *The Diophantine equation*

$$3x^2 + 1 = 4y^n$$

has no positive integral solutions x, y, n *with* $y > 1$ *and* $n \geq 3$.

For any prime number p, we denote by v_p the standard p-adic valuation over $\mathbf{Q}_p$, normalized by $v_p(p) = 1$.

LEMMA 3 (Bugeaud [B]). *Let* p *be a prime number. Let* $a = a_1/a_2$ *and* $b = b_1/b_2$ *be two irreducible rational numbers satisfying* $v_p(a) = v_p(b) = 0$ *and put* $A = \max\{a_1, a_2, b_1, b_2, 3\}$. *If the diophantine equation*

$$p^m = ax^n + by^n$$

has positive integral solutions x, y, n *with* $\gcd(x, y) = 1$ *and* $n \geq 2$, *then we have*

$$n \leq 34000\, p \log p \log A\,.$$

3. The case $a = p^t$ or $b = p^t$ (p : odd prime).

To begin with, we consider the Conjecture when $b(= v(2u + v))$ is a power of an odd prime. Then Eisenstein numbers a, b, c can be expressed as follows:

$$a = m^2 - 1\,, \quad b = 2m + 1\,, \quad c = m^2 + m + 1\,, \tag{3}$$

where m is a positive integer with $m \not\equiv 1 \pmod 3$. (Put $u = m,\ v = 1$ in (2).)

We first prove the following:

PROPOSITION 1. *Let m be a positive integer with $m \not\equiv 1 \pmod 3$. Let a, b, c be fixed positive integers satisfying* (3). *If equation* (1) *has positive integral solutions x, y, z, then z is even.*

PROOF. Let $m = 2$. Then we have $(a, b, c) = (3, 5, 7)$. It follows from (1) that z is even. Hence we may suppose that $m \geq 3$. Then we see that x is odd. Indeed, taking (1) modulo m yields $1 + (-1)^x \cdot 1 + 1 \equiv 1 \pmod m$, so $(-1)^x \equiv -1 \pmod m$. Since $m \geq 3$, x must be odd.

When m is even, let $2^s \parallel m$, where s is a positive integer. Then it follows from (3) that

$$m \equiv 2^s \pmod{2^{s+1}}\,, \quad a \equiv -1 \pmod{2^{s+1}}\,, \quad b \equiv 1 \pmod{2^{s+1}}\,, \quad c \equiv 2^s + 1 \pmod{2^{s+1}}\,.$$

Since x is odd, equation (1) implies that

$$1 + (-1) \cdot 1 + 1 \equiv (2^s + 1)^z \pmod{2^{s+1}}\,, \quad \text{so} \quad 1 \equiv 1 + z2^s \pmod{2^{s+1}}\,.$$

Hence z is even.

When m is odd, let $2^s \parallel m + 1$, where s is a positive integer. Then it follows from (3) that

$$m \equiv 2^s - 1 \pmod{2^{s+1}}\,, \ a \equiv 0 \pmod{2^{s+1}}\,, \ b \equiv -1 \pmod{2^{s+1}}\,, \ c \equiv 2^s + 1 \pmod{2^{s+1}}\,.$$

Equation (1) implies that $1 \equiv 1 + z2^s \pmod{2^{s+1}}$. Hence z is even. □

THEOREM 1. *Let m be a positive integer with $m \not\equiv 1 \pmod 3$. Let a, b, c be fixed positive integers satisfying* (3). *Suppose that b is a power of an odd prime. Then equation* (1) *has only the positive integral solution $(x, y, z) = (1, 1, 2)$.*

PROOF. Let (x, y, z) be a solution of (1). Then it follows from Proposition 1 that z is even, say $z = 2Z$. By Lemma 1, we have

$$a^x = U^2 - V^2\,, \quad b^y = V(2U + V)\,, \quad c^Z = U^2 + UV + V^2\,, \qquad (E_1)$$

or

$$a^x = V(2U + V)\,, \quad b^y = U^2 - V^2\,, \quad c^Z = U^2 + UV + V^2\,, \qquad (E_2)$$

where $U,\ V$ are positive integers such that $(U, V) = 1$, $U > V$ and $U \not\equiv V \pmod 3$.

First consider (E_1). Since b is a power of an odd prime and $(U, V) = 1$, we have $2U + V = b^y$ and $V = 1$, so

$$U^2 + U + 1 = c^Z\,.$$

Now Lemma 2, (1) implies that $Z = 1$. Then $c = m^2 + m + 1 = U^2 + U + 1$ and so $U = m$. Therefore we obtain $x = 1,\ y = 1,\ z = 2$.

Next consider (E_2). Since b is an odd prime power and $(U, V) = 1$, we have $U + V = b^y$ and $U - V = 1$, so

$$3(2V + 1)^2 + 1 = 4c^Z\,.$$

Now Lemma 2, (4) implies that $Z = 1,\ 2$. We show that the cases $Z = 1,\ 2$ do not occur.

If $Z = 1$, then $c = m^2 + m + 1 = 3V^2 + 3V + 1$ and so $V < m$. Hence $b^y = U^2 - V^2 = 2V + 1 < 2m + 1 = b$, which is impossible.

If $Z = 2$, then it follows from Lemma 1 that

$$U, \quad V = r^2 - s^2, \quad s(2r + s); \quad c = r^2 + rs + s^2,$$

where r, s are positive integers such that $(r, s) = 1$, $r > s$ and $r \not\equiv s \pmod 3$. Thus we have

$$U + V = r(2s + r) = b^y.$$

Since b is a power of an odd prime and $(r,\ s) = 1$, we have $2s + r = b^y$ and $r = 1$, which is impossible. □

We next consider the Conjecture when $a(= u^2 - v^2)$ is a power of an odd prime. Then Eisenstein numbers $a,\ b,\ c$ can be expressed as follows:

$$a = 2m + 1, \quad b = 3m^2 + 2m, \quad c = 3m^2 + 3m + 1, \tag{4}$$

where m is a positive integer. (Put $u = m + 1,\ v = m$ in (2).)

THEOREM 2. *Let m be a positive integer. Let a, b, c be fixed positive integers satisfying* (4). *Suppose that a is a power of an odd prime. Then equation* (1) *has only the positive integral solution* $(x, y, z) = (1, 1, 2)$.

PROOF. The proof of Theorem 2 is similar to that of Theorem 1. □

4. The case $a = 2^t$ or $b = 2^t$.

To begin with, we consider the Conjecture when $b(= v(2u + v)) = 2^t$. Then Eisenstein numbers $a,\ b,\ c$ can be expressed as follows:

$$a = 2^{2t-4} - 2^{t-1} - 3, \quad b = 2^t, \quad c = 2^{2t-4} + 3, \tag{5}$$

where t is a positive integer with $t \geq 4$. (Put $u = 2^{t-2} - 1,\ v = 2$ in (2).)

We first show the following:

PROPOSITION 2. *Let t be a positive integer with $t \geq 4$. Let a, b, c be fixed positive integers satisfying* (5). *If equation* (1) *has positive integral solutions $x,\ y,\ z$, then z is even.*

PROOF. Since $t \geq 4$, we have $a \equiv 1 \pmod 4$, $b \equiv 0 \pmod 4$, $c \equiv -1 \pmod 4$. Taking (1) modulo 4 yields $1 \equiv (-1)^z \pmod 4$, so z is even. □

THEOREM 3. *Let t be a positive integer with $t \geq 4$. Let a, b, c be fixed positive integers satisfying* (5). *Then equation* (1) *has only the positive integral solution* $(x, y, z) = (1, 1, 2)$.

PROOF. Let (x, y, z) be a solution of (1). Then it follows from Proposition 2 that z is even, say $z = 2Z$. By Lemma 1, we have two cases (E_1), (E_2) as in the proof of Theorem 1.

First consider (E_1). Then from (5), we have $V = 2$ and $2U + V = 2^{ty-1}$, so

$$(U+1)^2 + 3 = c^Z .$$

Now Lemma 2, (2) implies that $Z = 1$. Then from (1), we have $x = y = 1$.

Next consider (E_2). Then from (5), we have $U + V = 2^{ty-1}$ and $U - V = 2$, so

$$3(V+1)^2 + 1 = c^Z .$$

Now Lemma 2, (3) implies that $Z = 1,\ 2$. We show that the cases $Z = 1,\ 2$ do not occur.

If $Z = 1$, then we have $c \equiv -1 \pmod 4$ from (5). On the other hand, since $U \equiv 1 \pmod 4$ and $V \equiv -1 \pmod 4$, we have $c = c^Z \equiv 1 \pmod 4$ from (E_2). This is impossible.

If $Z = 2$, then it follows from Lemma 1 that

$$U + V = r(2s + r)\,, \quad c = r^2 + rs + s^2\,,$$

where $r,\ s$ are positive integers such that $(r,\ s) = 1$, $r > s$ and $r \not\equiv s \pmod 3$. Since $r(2s + r) = 2^{ty-1}$, we have $r = 2,\ 2s + r = 2^{ty-2}$ and so $s = 1,\ t = 4$. But this is impossible, since $c = 2^{2t-4} + 3 = r^2 + rs + s^2$. □

We next consider the Conjecture when $a (= u^2 - v^2) = 2^t$. Then Eisenstein numbers $a,\ b,\ c$ can be expressed as follows:

$$a = 2^t\,, \quad b = 3 \cdot 2^{2t-4} - 2^{t-1} - 1\,, \quad c = 3 \cdot 2^{2t-4} + 1\,, \tag{6}$$

where t is a positive integer with $t \geq 3$. (Put $u + v = 2^{t-1},\ u - v = 2$ in (2).)

THEOREM 4. *Let t be a positive integer such that $t \geq 3$ and $t \not\equiv 3 \pmod 4$. Let a, b, c be fixed positive integers satisfying* (6). *Then equation* (1) *has only the positive integral solution* $(x, y, z) = (1, 1, 2)$.

PROOF. Taking (1) modulo 5 implies that z is even. The proof of Theorem 4 is similar to that of Theorem 3. □

5. The case $a = p^e q^f$ or $b = p^e q^f$.

In this section, we consider the Conjecture when $a = p^e q^f$ or $b = p^e q^f$. Then we apply the theory of linear forms in two logarithms to derive an upper bound of y or x of equation (1).

We first prove the following:

PROPOSITION 3. (i) *Let $a,\ b,\ c$ be fixed positive integers satisfying* (2). *Suppose that $a = p^e q^f$, where $e,\ f$ are positive integers, and $p,\ q$ are odd primes such that $p^e > q^f$ and $q \equiv 5,\ 7 \pmod{12}$. If equation* (1) *has positive integral solutions $x,\ y,\ z$, then z is even.*

(ii) *Let $a,\ b,\ c$ be fixed positive integers satisfying* (2). *Suppose that $b = p^e q^f$, where $e,\ f$ are positive integers, and $p,\ q$ are odd primes such that $p^e > q^f$ and $p \equiv 5,\ 7 \pmod{12}$. If equation* (1) *has positive integral solutions x, y, z, then z is even.*

PROOF. (i) Since $a = p^e q^f = u^2 - v^2$ in (2), we have $u - v = 1$ or $u - v = q^f$. If $u - v = 1$, then z is even, as in Proposition 1.

If $u - v = q^f$, then $u \equiv v \pmod q$ and so $\left(\frac{c}{q}\right) = \left(\frac{3}{q}\right) = -1$ from (2). Now equation (1) implies that z is even.

(ii) Since $b = p^e q^f = v(2u + v)$ in (2), we have $v = 1$ or $v = q^f$. If $v = 1$, then z is even, as in Proposition 1.

If $v = q^f$, then $u = (p^e - q^f)/2$ and so $\left(\frac{c}{p}\right) = \left(\frac{3}{p}\right) = -1$ from (2). Now equation (1) implies that z is even. □

We use Lemma 3 to show the following:

THEOREM 5. *Put* $Q = \max\{p, q\}$.

(i) *Let a, b, c be fixed positive integers as in Proposition 3,* (i). *Suppose that b is even. Then the solution y of equation* (1) *satisfies*

$$y \leq 47135\, Q \log Q\,.$$

(ii) *Let a, b, c be fixed positive integers as in Proposition* 3, (i). *Suppose that b is odd. Then the solution y of equation* (1) *satisfies*

$$y \leq 23568\, Q \log Q\,.$$

(iii) *Let a, b, c be fixed positive integers as in Proposition* 3, (ii). *Suppose that a is even. Then the solution x of equation* (1) *satisfies*

$$x \leq 47135\, Q \log Q\,.$$

(iv) *Let a, b, c be fixed positive integers as in Proposition* 3, (ii). *Suppose that a is odd. Then the solution x of equation* (1) *satisfies*

$$x \leq 23568\, Q \log Q\,.$$

PROOF. (i) Let (x, y, z) be a solution of (1). Then it follows from Proposition 3 that z is even, say $z = 2Z$. By Lemma 1, we have two cases (E_1), (E_2) as in the proof of Theorem 1.

First consider (E_1). Then since $a = p^e q^f$ and $p^e > q^f$, we have $U - V = 1$ or q^{fx}.

If $U - V = 1$, then we obtain $Z = 1, 2$, as in Theorem 1. Since $c < b^2$ from Lemma 1, we have

$$b^{2y} < a^{2x} + a^x b^y + b^{2y} = c^{2Z} \leq c^4 < b^8\,,$$

so $y < 4$.

If $U - V = q^{fx}$, then we have $U + V = p^{ex}$. The assumption that b is even implies that

$$V = 2b_1^y,\ 2U + V = 2^{sy-1} b_2^y \quad \text{or} \quad V = 2^{sy-1} b_1^y,\ 2U + V = 2b_2^y\,,$$

where $b = 2^s b_0$ with b_0 odd and $b_0 = b_1 b_2$ with $(b_1, b_2) = 1$. Let $(X, Y) = (b_1, b_2)$ or (b_2, b_1). Then $2X^y + 2^{sy-1}Y^y = 2(U + V) = 2 \cdot p^{ex}$ and so

$$X^y + \frac{1}{4}(2^s Y)^y = p^{ex}.$$

Hence it follows from Lemma 3 that

$$y \leq 34000\, p \log p \log 4 \leq 47135\, p \log p\,.$$

Next consider (E_2). Then since $a = p^e q^f$ and $p^e > q^f$, we have $V = 1$ or q^{fx}.

If $V = 1$, then we obtain $(x, y, z) = (1, 1, 2)$, as in Theorem 2.

Let $V = q^{fx}$. The assumption that b is even implies that

$$U - V = 2b_1^y\,, \quad U + V = 2^{sy-1}b_2^y \quad \text{or} \quad U - V = 2^{sy-1}b_1^y\,, \quad U + V = 2b_2^y\,,$$

where $b = 2^s b_0$ with b_0 odd and $b_0 = b_1 b_2$ with $(b_1, b_2) = 1$. Let $(X, Y) = (b_1, b_2)$ or (b_2, b_1). Then $2X^y - 2^{sy-1}Y^y = \pm 2V = \pm 2 \cdot q^{fx}$ and so

$$\pm X^y \mp \frac{1}{4}(2^s Y)^y = q^{fx}\,.$$

Hence it follows from Lemma 3 that

$$y \leq 34000\, q \log q \log 4 \leq 47135\, q \log q\,.$$

(ii), (iii), (iv) Similarly, we obtain the desired assertions. □

6. Examples.

Using the preceding Theorems, we verify that when a and b satisfy $a^2 + ab + b^2 = c^2$, $3 \leq a,\ b \leq 100$ and $(a, b) = 1$, the Conjecture holds. In fact, we show the following (cf. Table in Section 2):

THEOREM 6. *Let a, b, c be fixed positive integers satisfying*

$$a^2 + ab + b^2 = c^2\,, \quad 3 \leq a, b \leq 100 \quad \text{and} \quad (a, b) = 1\,.$$

Then equation (1) *has only the positive integral solution* $(x, y, z) = (1, 1, 2)$.

PROOF. We may suppose that $a - b \equiv 1 \pmod 3$. We have to consider the cases $(a, b) = (24, 95), (40, 51), (55, 57), (65, 88), (77, 40), (91, 69)$, since all the other cases are covered by Theorems 1, 2, 3, 4.

• (i): $(a, b) = (24, 95)$. Proposition 3 implies that z is even, say $z = 2Z$.

Case (E_1): $24^x = U^2 - V^2$, $95^y = V(2U + V)$, $109^Z = U^2 + UV + V^2$.

If $V = 1$, then we have $Z = x = y = 1$ as in the proof of Theorem 1. But this is impossible. Thus $V = 5^y$ and $2U + V = 19^y$, which imply that y is odd and $U + V \equiv 0 \pmod 4$. Since $U \not\equiv V \pmod 3$, we have $U - V = 2$ and $U + V = 2^{3x-1}3^x$. Hence as in the proof of Theorem 3, we obtain $Z = 1$ and so $U = 7$, $V = 5$, $x = y = 1$.

Case (E_2): $24^x = V(2U + V)$, $95^y = U^2 - V^2$, $109^Z = U^2 + UV + V^2$.

If $U - V = 1$, then we have $Z = x = y = 1$ or $Z = 2$ as in the proof of Theorem 1. But this is impossible. Thus $U - V = 5^y$ and $U + V = 19^y$, which imply that y is even, $V \equiv 0 \pmod 3$ and $V \equiv 0 \pmod 4$. Hence we obtain have $V = 2^{3x-1}3^x$ and $2U + V = 2$. But this is impossible.

• (ii): $(a, b) = (40, 51)$. Proposition 3 implies that z is even, say $z = 2Z$.

Case (E_1): $40^x = U^2 - V^2, \quad 51^y = V(2U + V), \quad 79^Z = U^2 + UV + V^2$.

If $V = 1$, then we have $Z = x = y = 1$ as before. But this is impossible. Thus $V = 3^y$ and $2U+V = 17^y$, which imply that y is odd and $U-V \equiv 0 \pmod 4$. Hence $U-V = 2^{3x-1}$ and $U + V = 2 \cdot 5^x$. Therefore we obtain $x = 1$ and so $U = 7, \ V = 3, \ y = Z = 1$.

Case (E_2): $40^x = V(2U + V), \quad 51^y = U^2 - V^2, \quad 79^Z = U^2 + UV + V^2$.

If $U - V = 1$, then we have $Z = x = y = 1$ or $Z = 2$ as before. But this is impossible. Thus $U - V = 3^y$, which is also impossible, since $U \not\equiv V \pmod 3$.

• (iii): $(a, b) = (55, 57)$. Proposition 3 implies that z is even, say $z = 2Z$.

Case (E_1): $55^x = U^2 - V^2, \quad 57^y = V(2U + V), \quad 97^Z = U^2 + UV + V^2$.

If $U - V = 1$ or $V = 1$, then we have $Z = x = y = 1$ or $Z = 2$ as before. But this is impossible. Thus we have $U - V = 5^x$, $U + V = 11^x$, $V = 3^y$ and $2U + V = 19^y$. Eliminating U and V yields

$$11^x - 5^x = 2 \cdot 3^y .$$

Taking the equation modulo 4 implies that x is odd. Suppose that $x > 1$. Then $x \equiv 0 \pmod 3$. Indeed, since x is odd, we have

$$3^{y-1} = 11^{x-1} + 11^{x-2} \cdot 5 + \cdots + 11 \cdot 5^{x-2} + 5^{x-1} \equiv 2^{x-1} + 2^{x-1} + \cdots + 2^{x-1} \equiv x \pmod 3 .$$

Hence $2 \cdot 3^y$ is divisible by $11^3 - 5^3 = 2 \cdot 3^2 \cdot 67$, which is a contradiction. Therefore we obtain $x = 1$ and so $U = 8, \ V = 3, \ y = Z = 1$.

Case (E_2): $55^x = V(2U + V), \quad 57^y = U^2 - V^2, \quad 97^Z = U^2 + UV + V^2$.

If $U - V = 1$ or $V = 1$, then we have $Z = x = y = 1$ or $Z = 2$ as before. But this is impossible. Thus we have $U - V = 3^y$, $U + V = 19^y$, $V = 5^x$ and $2U + V = 11^x$. Eliminating U and V yields

$$19^y + 3^y = 11^x - 5^x .$$

Taking the equation modulo 3 leads to a contradiction.

• (iv): $(a, b) = (65, 88)$. Proposition 3 implies that z is even, say $z = 2Z$.

Case (E_1): $65^x = U^2 - V^2, \quad 88^y = V(2U + V), \quad 133^Z = U^2 + UV + V^2$.

If $U - V = 1$, then we have $Z = x = y = 1$ or $Z = 2$ as before. But this is impossible. Thus $U - V = 5^x$ and $U + V = 13^x$, which imply that $V \equiv 0 \pmod 4$. Hence we have $V = 2^{3y-1}$ and $2U + V = 2 \cdot 11^y$. Eliminating U and V yields

$$2^{3y-2} + 11^y = 13^x .$$

Since $(-1)^{3y-2} + (-1)^y \equiv 1 \pmod 3$, y is odd. Then $2^{3y-2} \equiv 2 \pmod 4$. Therefore we obtain $y = 1$ and so $V = 4, \ U = 9, \ x = Z = 1$.

Case (E_2): $65^x = V(2U + V), \quad 88^y = U^2 - V^2, \quad 133^Z = U^2 + UV + V^2$.

If $V = 1$, then we have $Z = x = y = 1$ as before. But this is impossible. Thus $V = 5^x$ and $2U + V = 13^x$. Taking the equations above modulo 4 leads to a contradiction, since U is odd.

• (v): $(a, b) = (77, 40)$. Proposition 3 implies that z is even, say $z = 2Z$.

Case (E_1): $77^x = U^2 - V^2, \quad 40^y = V(2U + V), \quad 103^Z = U^2 + UV + V^2$.

If $U - V = 1$, then we have $Z = x = y = 1$ or $Z = 2$ as before. But this is impossible. Thus $U - V = 7^x$ and $U + V = 11^x$, which imply that x is odd and $V \equiv 2 \pmod 4$. Since x is odd, we have $V = 2$ and $2U + V = 2^{3y-1}5^y$. Hence as before, we obtain $Z = 1$ and so $U = 9, \ x = y = 1$.

Case (E_2): $77^x = V(2U + V), \quad 40^y = U^2 - V^2, \quad 103^Z = U^2 + UV + V^2$.

If $V = 1$, then we have $Z = x = y = 1$ as before. But this is impossible. Thus $V = 7^x$ and $2U + V = 11^x$. Hence we have $0 \equiv 2(U + V) = 11^x + 7^x \equiv 2 \pmod 4$, which is also impossible.

• (vi): $(a, b) = (91, 69)$. Proposition 3 implies that z is even, say $z = 2Z$.

Case (E_1): $91^x = U^2 - V^2, \quad 69^y = V(2U + V), \quad 139^Z = U^2 + UV + V^2$.

If $U - V = 1$ or $V = 1$, then we have $Z = x = y = 1$ or $Z = 2$ as before. But this is impossible. Thus we have $V = 3^y$, $2U + V = 23^y$, $U - V = 7^x$ and $U + V = 13^x$. Eliminating U and V yields

$$13^x - 7^x = 2 \cdot 3^y .$$

Taking the equation modulo 4 implies that x is odd. Suppose that $x > 1$. Then $x \equiv 0 \pmod 3$. Indeed, since x is odd, we have

$$3^{y-1} = 13^{x-1} + 13^{x-2} \cdot 7 + \cdots + 13 \cdot 7^{x-2} + 7^{x-1} \equiv x \pmod 3 .$$

Hence $2 \cdot 3^y$ is divisible by $13^3 - 7^3 = 2 \cdot 3^2 \cdot 103$, which is a contradiction. Therefore we obtain $x = 1$ and so $U = 10, \ V = 3, \ y = Z = 1$.

Case (E_2): $91^x = V(2U + V), \quad 69^y = U^2 - V^2, \quad 139^Z = U^2 + UV + V^2$.

If $U - V = 1$ or $V = 1$, then we have $Z = x = y = 1$ or $Z = 2$ as before. But this is impossible. Thus we have $U - V = 3^y$, $U + V = 23^y$, $V = 7^x$ and $2U + V = 13^x$. Taking the equations above modulo 3 leads to a contradiction. □

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