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Analytic Singularities of Solutions to Certain Nonlinear Ordinary Differential Equations Associated with p-Laplacian

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Abstract. Analytic singularities of local solutions to the nonlinear ordinary differential equation $(|u_x|^{p-2}u_x)_x + |u|^{q-2}u = 0$ are obtained through Briot-Bouquet type nonlinear analytic differential equations with regular singularity.

1. Introduction

In connection with the determination of the best possible constant for Sobolev - Poincaré inequalities, the following one dimensional nonlinear Dirichlet problem (1) and (2) associated with the so-called p-Laplace operator has been studied by M. Ôtani ([3], [4]) and T. Idogawa and M. Ôtani ([1]) and by others (e.g. P. Lindqvist [5]):

$$(|u_x|^{p-2}u_x)_x + |u|^{q-2}u = 0 \tag{1}$$

on (a, b) and

$$u(a) = u(b) = 0 \tag{2}$$

where $1 < p < \infty$.

The existence of a unique positive solution in (a, b), determination of the set of the nontrivial solutions and classical differentiability of solutions are established in [3] and [4], when $u \in W_0^{1,p}(a, b)$ satisfies (1) in the distribution sense.

We consider in this paper local solutions. Let I be a subinterval contained in $[a, b]$. A real-valued function u is said to be a local solution to (1) on I, if $u \in W^{1,p}(I)$ and u satisfies (1) in distribution sense. The objective of this paper is to give local analytic singularity of solutions on I to (1), making use of Briot-Bouquet type nonlinear differential equations with regular singularity. Our analytic expression provides convergent expansions when x tends to a point x_0 where $u(x_0) = 0$ or $u_x(x_0) = 0$. They also reproduce easily the known differentiability and analyticity obtained in [3], [4] and [1].

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CASE 1. Analytic Expression of a local solution $u(x)$ on I near a point $\sigma \in I$ where $u(\sigma) = 0$ and $u_x(\sigma) = A \neq 0$. We can assume $A > 0$ without loss of generality, since $-u$ is also a solution when u is a solution.

THEOREM 1.1. *For any p and q satisfying $1 < p, q < \infty$, there exists a unique analytic function $F(\xi)$ near the origin such that we have near $x = \sigma$*

$$u(x) = (x - \sigma)F(|x - \sigma|^q) . \tag{3}$$

$F(\xi)$ is a unique holomorphic solution to

$$(p-1)[F(\xi) + q\xi F'(\xi)]^{p-2}[q(q+1)F'(\xi) + q^2\xi F''(\xi)] + (F(\xi))^{q-1} = 0 \tag{4}$$

with $F(0) = A$ and $F'(0) = \frac{-A^{q-p+1}}{q(q+1)(p-1)}$.

Consequently, $u(x)$ has a convergent expansion near $x = \sigma$:

$$u(x) = A(x - \sigma) - \frac{A^{q-p+1}}{q(q+1)(p-1)}(x - \sigma)|x - \sigma|^q + \cdots . \tag{5}$$

CASE 2. Analytic expression of a local solution $u(x)$ on I near $\tau \in I$ where $u(\tau) = A \neq 0$ and $u_x(\tau) = 0$. We can assume $A > 0$ without loss of generality as before.

THEOREM 1.2. *For any p and q satisfying $1 < p, q < \infty$, there exists a unique analytic function $G(\xi)$ near the origin such that we have near $x = \tau$*

$$u(x) = G\left(|x - \tau|^{\frac{p}{p-1}}\right). \tag{6}$$

$G(\xi)$ is a unique holomorphic solution to a nonlinear equation:

$$\left(\frac{p}{p-1}\right)^{p-1}(-G'(\xi))^{p-2}[G'(\xi) + p\xi G''(\xi)] + (G(\xi))^{q-1} = 0 \tag{7}$$

with $G(0) = A$ and $G'(0) = -\frac{p-1}{p}A^{\frac{q-1}{p-1}}$.

Consequently, *we have a convergent expansion near $x = \tau$*:

$$u(x) = A - \frac{p-1}{p}A^{\frac{q-1}{p-1}}|x - \tau|^{\frac{p}{p-1}} + \cdots . \tag{8}$$

In the extreme case $p = q = 2$, the equation (1) reduces to

$$u_{xx} + u = 0 . \tag{9}$$

We explain our heuristic procedure by this simplest case.

Assume $\sigma = 0$ for simplicity. The solution $u(x)$ with $u(0) = 0$ and $u_x(0) = A$ has a convergent power series expansion $u(x) = A\{\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\}$, which is equal to $A \sin x$. If we put $xF(x^2) = A \sin x$, then $F(\xi)$ is a well defined analytic function and satisfies a linear equation with regular singularity

$$4\xi F''(\xi) + 6F'(\xi) + F(\xi) = 0$$

with

$$F(0) = A\,, \quad F'(0) = -\frac{A}{6}\,.$$

Next, we suppose $u(0) = A$ and $u_x(0) = 0$, assuming $\tau = 0$. The solution $u(x)$ has a convergent expansion $u(x) = A\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\}$, which is equal to $A\cos x$. If we put $G(x^2) = A\cos x$, $G(\xi)$ is a well defined analytic function and satisfies a linear equation with regular singularity

$$4\xi G''(\xi) + 2G'(\xi) + G(\xi) = 0$$

with

$$G(0) = A\,, \quad G'(0) = -\frac{A}{2}\,.$$

Thus appear analytic differential equations with regular singularity. For general p and q, the nonlinear equations with regular singularity (4) and (7) describe the solution near $x_0 = \sigma$ or τ.

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2. Local uniqueness

We need local uniqueness of solutions to the Cauchy problem to (1). This will be proved through localizing the energy equality obtained in [4] for global problems. Let I be a subinterval and $[c, d] = \bar{I}$, its closure in $[a, b]$.

PROPOSITION 2.1. *Every nonzero local solution u on I has $C^1(\bar{I})$ regularity and satisfies for some positive constant C the energy equality*

$$(p-1)|u_x(x)|^p/p + |u(x)|^q/q = C \tag{10}$$

for all $x \in \bar{I}$.

PROOF. At first, we recall an expression (11) below, which was obtained in [4] and is a key equality in our discussion. Since $W^{1,p}(I)$ is embedded in $C(\bar{I})$, u is continuous on $\bar{I}$. Therefore, $\langle(|u_x|^{p-2}u_x)_x + |u|^{q-2}u, \varphi\rangle = \langle|u_x|^{p-2}u_x, -\varphi_x\rangle + \langle-\int_{x_0}^{\bullet}|u(\sigma)|^{q-2}u(\sigma)d\sigma, \varphi_x(\cdot)\rangle = 0$ for arbitrarily fixed $x_0 \in [c, d]$ and for every $\varphi \in C_0^\infty(c, d)$. Hence, there exists a constant K such that $|u_x(x)|^{p-2}u_x(x) = -\int_{x_0}^{x}|u(\sigma)|^{q-2}u(\sigma)d\sigma + K$. It follows that u is in $C^1([c, d])$ and

$$|u_x(x)|^{p-2}u_x(x) = -\int_{x_0}^{x}|u(\sigma)|^{q-2}u(\sigma)d\sigma + |u_x(x_0)|^{p-2}u_x(x_0)\,. \tag{11}$$

We see that u is in $C^2([c,d])$ where u_x does not vanish.

If $u(x)$ does not identically vanish on $[c,d]$, $|u(x)|$ is positive for a certain open subinterval (c_0, d_0) in I. We assume $u(x)$ is positive on (c_0, d_0). Then, $u_x(x)$ is strictly decreasing in (c_0, d_0). Since $u_x(x)$ vanishes at most once at $x = x_1$ in (c_0, d_0), $u(x)$ is of C^2 in (c_0, x_1) and (x_1, d_0). Multiplying the equation (1) by u_x, we have $((p-1)|u_x(x)|^p/p + |u(x)|^q/q)_x = 0$ for all $x \in (c_0, d_0) \setminus \{x_1\}$. Hence, by continuity, we have a positive constant C such that $(p-1)|u_x(x)|^p/p + |u(x)|^q/q = C$ for all $x \in [c_0, d_0]$. In case $u(x)$ is negative, we have similarly (10) on $[c_0, d_0]$.

Next, starting with (c_0, d_0) where $u(x)$ is positive, we enlarge the subinterval (c_0, d_0), as long as u is positive. If $(c_0, d_0) = (c, d)$, we have (10) on $[c, d]$ and the proof is complete. We assume c_0 be the first zero point of u in continuation to the negative direction. Note that the right derivative $u_x(c_0)$ is positive. $u(x)$ is negative in a left neighborhood of c_0, since $u_x(c_0) > 0$. While u is negative, we have the energy equality as in the positive case. We have the same conclusion, starting with (c_0, d_0) where $u(x)$ is negative.

At last, repeating this process, we arrive at c after at most finite zero points. In fact, if there exists accumulation of zero points at $\xi(\geq c)$, we have $u(\xi) = 0$ and $|u_x(\xi)| > 0$ by continuity and the energy equality (10). This contradicts to accumulation of zero points. Thus, the energy equality (10) holds in $[c, d_0]$. Similar argument to the right direction leads us the energy equality (10) on $[c, d]$. □

PROPOSITION 2.2 (Local uniqueness). *Let x_0 be an arbitrary point in I. Local solutions on I are uniquely determined by initial data $u(x_0)$ and $u_x(x_0)$.*

PROOF. Take any two solutions $u_1(x)$ and $u_2(x)$ on I such that $u_1(x_0) = u_2(x_0)$ and $u_{1,x}(x_0) = u_{2,x}(x_0)$. Define a subset of I by $U = \{x \in I;\ u_1(x) = u_2(x),\ \text{and}\ u_{1,x}(x) = u_{2,x}(x)\}$. We will show that U is a closed and open set in I. Then, $U = I$ and the desired conclusion is proved.

Since U is clearly closed, we will show that U is open. Take any point $x_1 \in U$. We consider the three cases.

(a): if $u_i(x_1)u_{i,x}(x_1) \neq 0$, x_1 is an interior point of U by the usual uniqueness theorem for the Cauchy problems of explicit analytic differential equations.

(b): if $u_i(x_1) = u_{i,x}(x_1) = 0$, $u_i(x)$'s identically vanish on I in virtue of the energy equality (10). Hence, $U = I$.

(c): if either $u_i(x_1)$ or $u_{i,x}(x_1)$ is 0 and if the other is not zero, we will show as follows that x_1 is an interior point of U.

If $u_{i,x}(x_1) = 0$, then $|u_{i,x}(x)|^{p-2}u_{i,x}(x) = -\int_{x_1}^{x} |u_i(\sigma)|^{q-2}u_i(\sigma)d\sigma$ by (11). If $u_i(x_1) = 0$, then $u_i(x) = \int_{x_1}^{x} u_{i,x}(\sigma)d\sigma$. We see through these equalities that there exists ε_0 in the case (c) such that $u_1(x)u_2(x)$ and $u_{1,x}(x)u_{2,x}(x)$ are positive on a deleted neighborhood D of x_1, that is, $D = (x_1 - \varepsilon_0, x_1) \cup (x_1, x_1 + \varepsilon_0)$. Especially, if $u_1(x_2) = u_2(x_2)$ for some $x_2 \in D$, then $x_2 \in U$ by (10). Moreover, x_2 is an interior point of U by the case (a).

Assume that x_1 is not an interior point of U. Then, there exsits $\xi \in D$ such that $\xi \notin U$. Therefore, $u_1(\xi) \neq u_2(\xi)$. We assume that $\xi \in (x_1, x_1 + \varepsilon_0)$.

We claim that $u_1(x) \neq u_2(x)$ on $(x_1, \xi]$. Put $Y = \{x \in (x_1, \xi);\ u_1(x) = u_2(x)\}$. Assume that $Y \neq \phi$. Since Y is open and closed in (x_1, ξ), $Y = (x_1, \xi)$. Hence, $u_1(\xi) = u_2(\xi)$. This contradicts to the definition of ξ. Hence, $Y = \phi$. We can then assume that $u_1(x) < u_2(x)$ on $x \in (x_1, \xi]$. We have from (11) $|u_{1,x}(x)|^{p-2}u_{1,x}(x) - |u_{2,x}(x)|^{p-2}u_{2,x}(x) = -\int_{x_1}^{x}\{|u_1(\sigma)|^{q-2}u_1(\sigma) - |u_2(\sigma)|^{q-2}u_2(\sigma)\}d\sigma$. Therefore, $u_{1,x}(x) > u_{2,x}(x)$ on $(x_1, \xi]$. On the other hand, $u_1(x) - u_2(x) = \int_{x_1}^{x}\{u_{1,x}(\sigma) - u_{2,x}(\sigma)\}d\sigma$. Hence, $u_1(x) > u_2(x)$, which contradicts to the above inequality.

Similarly, we have contradiction, when $\xi \in (x_1 - \varepsilon_0, x_1)$. Therefore, x_1 is an interior point of U. Since x_1 is arbitrary, U is an open set. The proof of the proposition is complete. □

3. Analytic singularities

We shall now describe local analytic singularities of the solution $u(x)$ to (1). If $u(x_0) \neq 0$ and $u_x(x_0) \neq 0$, the solution u is real analytic at $x = x_0$ through Cauchy's theorem and the local uniqueness in the previous section. Therefore, we restrict ourselves near a point $x_0 = \sigma$ where $u(\sigma) = 0$ and $u_x(\sigma) = A \neq 0$ and at a point $x_0 = \tau$ where $u(\tau) = A \neq 0$ and $u_x(\tau) = 0$.

We quote a classical Briot-Bouquet type theorem on the unique existence of analytic solution to a system of analytic nonlinear ordinary differential equations with singularity of regular type (e.g. [1] p.261, Prop.1.1.1).

THEOREM 3.1. *Consider a system of equations*

$$\xi y_i'(\xi) = U_i(\xi,\ y_1,\ y_2) \quad (i = 1,\ 2)\,, \tag{12}$$

where the $U_i(\xi,\ y_1,\ y_2)$ *are analytic at* $\xi = 0$, $y_1 = 0$, $y_2 = 0$ *and satisfy*

$$U_i(0,\ 0,\ 0) = 0 \quad (i = 1,\ 2)\,.$$

If none of the eigenvalues of the matrix $\{\partial U_i/\partial y_j;\ i,\ j = 1,\ 2\}$ *at* $(0, 0, 0)$ *is a positive integer, then the equation* (12) *has a unique analytic solution at* $\xi = 0$ *satisfying* $y_i(0) = 0$, $i = 1,\ 2$.

CASE 1. Analytic Expression of a local solution $u(x)$ on I near a point σ where $u(\sigma) = 0$ and $u_x(\sigma) = A \neq 0$. We assume $A > 0$ without loss of generality, since $-u$ is also a solution when u is a solution.

THEOREM 3.2. *For any* p *and* q *satisfying* $1 < p, q < \infty$, *there exists a unique analytic function* $F(\xi)$ *in a neigborhood of the origin such that we have near* $x = \sigma$

$$u(x) = (x - \sigma)F(|x - \sigma|^q)\,. \tag{13}$$

$F(\xi)$ is a holomorphic solution to

$$(p-1)[F(\xi)+q\xi F'(\xi)]^{p-2}[q(q+1)F'(\xi)+q^2\xi F''(\xi)]+(F(\xi))^{q-1}=0 \tag{14}$$

with

$$F(0)=A \text{ and } F'(0)=B\,, \tag{15}$$

where $B=\frac{-A^{q-p+1}}{q(q+1)(p-1)}$.

Consequently, $u(x)$ has an expansion near $x=\sigma$:

$$u(x)=(x-\sigma)\{A+B|x-\sigma|^q+C|x-\sigma|^{2q}+\cdots\}\,, \tag{16}$$

and $C=\frac{1+3q-p-pq}{2(q+1)q^2(2q+1)(p-1)^2}A^{2q-2p+1}$.

PROOF. At first, we prove unique existence of the solution $F(\xi)$ to (14) with (15). We reduce equation (14) by change of variable

$$F(\xi)=A+B\xi+y(\xi)=A-\frac{A^{q-p+1}}{q(q+1)(p-1)}\xi+y(\xi)$$

into

$$\xi y''(\xi)=-\frac{q+1}{q}y'(\xi)+\frac{A^{q-p+1}}{q^2(p-1)} - \frac{\left(A-\frac{A^{q-p+1}}{q(q+1)(p-1)}\xi+y(\xi)\right)^{q-1}}{q^2(p-1)\left[A-\frac{A^{q-p+1}}{q(p-1)}\xi+y(\xi)+q\xi y'(\xi)\right]^{p-2}} \tag{17}$$

with

$$y(0)=y'(0)=0\,. \tag{18}$$

Introducing $y_1=y(\xi)$ and $y_2=y'(\xi)$ will convert (17) into the following system of first order equations:

$$\xi y_1'(\xi)=U_1(\xi,y_1,y_2)=\xi y_2, \tag{19}$$

$$\xi y_2'(\xi)=U_2(\xi,y_1,y_2)=-\frac{(q+1)}{q}y_2+\frac{A^{q-p+1}}{q^2(p-1)} - \frac{\left(A-\frac{A^{q-p+1}}{q(q+1)(p-1)}\xi+y_1\right)^{q-1}}{q^2(p-1)\left[A-\frac{A^{q-p+1}}{q(p-1)}\xi+y_1+q\xi y_2\right]^{p-2}} \tag{20}$$

with

$$y_1(0) = y_2(0) = 0\,. \tag{21}$$

We have clearly $U_1(0,0,0) = U_2(0,0,0) = 0$. Since we have also

$$\begin{pmatrix} \dfrac{\partial U_1}{\partial y_1}(0,0,0)\,, & \dfrac{\partial U_1}{\partial y_2}(0,0,0) \\ \dfrac{\partial U_2}{\partial y_1}(0,0,0)\,, & \dfrac{\partial U_2}{\partial y_2}(0,0,0) \end{pmatrix} = \begin{pmatrix} 0 & , & 0 \\ \dfrac{(p-q-1)A^{q-p}}{q^2(p-1)} & , & -\dfrac{q+1}{q} \end{pmatrix},$$

we have nonpositive eigenvalues 0 and $-(q+1)/q$. By Theorem 3.1, we have a unique analytic solution $y(\xi)$ to (17) with (18). This gives an analytic solution $F(\xi) = A + B\xi + y(\xi)$ to (14) with (15).

Next, $(x-\sigma)F(|x-\sigma|^q)$ is a C^2 function near σ. It satisfies (1) with the prescribed Cauchy data. By Proposition 2.2, it is equal to the unique solution $u(x)$ with the same Cauchy data.

Putting

$$y_1 \sim C\xi^2 + o(\xi^2)\,, \quad y_2 \sim 2C\xi + o(\xi)\,, \tag{22}$$

we substitute them into (20). We have

$$C = \frac{1+3q-p-pq}{2(q+1)q^2(2q+1)(p-1)^2} A^{2q-2p+1}\,. \tag{23}$$

□

COROLLARY 3.1 ([1], [4]). (i) *When q is an even integer more than* 1*, the solution $u(x)$ is real analytic near σ.*

(ii) *When q is not an even integer, the solution $u(x)$ is of class $C^{<q>}$ at σ, where $< q >$ is the least integer greater than or equal to q.*

CASE 2. Analytic expression of a local solution $u(x)$ on I near a point τ where $u(\tau) = A$ and $u_x(\tau) = 0$. As in the case 1, we can assume without loss of generality that $A > 0$ by symmetry of the equation.

THEOREM 3.3. *For any p and q satisfying $1 < p, q < \infty$, there exists a unique analytic function $G(\xi)$ in a neigborhood of the origin such that we have near $x = \tau$*

$$u(x) = G(|x-\tau|^{\frac{p}{p-1}})\,, \tag{24}$$

where $G(\xi)$ is a holomorphic solution to the nonlinear equation:

$$\left(\frac{p}{p-1}\right)^{p-1} (-G'(\xi))^{p-2}\left[G'(\xi) + p\xi G''(\xi)\right] + (G(\xi))^{q-1} = 0 \tag{25}$$

with

$$G(0) = A \quad \textit{and} \quad G'(0) = B\,, \tag{26}$$

where $B = -\frac{p-1}{p} A^{\frac{q-1}{p-1}}$.

Consequently, we have a convergent expansion near $x = \tau$:

$$u(x) = A + B|x-\tau|^{\frac{p}{p-1}} + C|x-\tau|^{\frac{2p}{p-1}} + \cdots , \tag{27}$$

where $C = \frac{q-1}{2(2p-1)} \left(\frac{p-1}{p}\right)^2 A^{1+\frac{2(q-p)}{p-1}}$.

PROOF. We show unique existence of the solution $G(\xi)$. Setting

$$G(\xi) = A - \frac{p-1}{p} A^{\frac{q-1}{p-1}} \xi + z(\xi),$$

we obtain an equation for $z(\xi)$:

$$\xi z'' = \frac{p-1}{p^2} A^{\frac{q-1}{p-1}} - \frac{1}{p} z' - \frac{1}{p} \left(\frac{p-1}{p}\right)^{p-1} \frac{(A - \frac{p-1}{p} A^{\frac{q-1}{p-1}} \xi + z)^{q-1}}{(\frac{p-1}{p} A^{\frac{q-1}{p-1}} - z')^{p-2}} \tag{28}$$

with

$$z(0) = 0 \quad \text{and} \quad z'(0) = 0 . \tag{29}$$

If we let $z_1 = z(\xi)$ and $z_2 = z'(\xi)$, the corresponding system of first order equations is

$$\xi z_1'(\xi) = V_1(\xi, z_1, z_2) = \xi z_2, \tag{30}$$

$$\begin{aligned} \xi z_2'(\xi) = V_2(\xi, z_1, z_2) = {} & \frac{p-1}{p^2} A^{\frac{q-1}{p-1}} - \frac{z_2}{p} \\ & - \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \frac{\left(A - \frac{p-1}{p} A^{\frac{q-1}{p-1}} \xi + z_1\right)^{q-1}}{\left(\frac{p-1}{p} A^{\frac{q-1}{p-1}} - z_2\right)^{p-2}} \end{aligned} \tag{31}$$

with

$$z_1(0) = z_2(0) = 0 . \tag{32}$$

Note that $V_1(0,0,0) = V_2(0,0,0) = 0$. Since we have also

$$\begin{pmatrix} \frac{\partial V_1}{\partial z_1}(0,0,0) & \frac{\partial V_1}{\partial z_2}(0,0,0) \\ \frac{\partial V_2}{\partial z_1}(0,0,0) & \frac{\partial V_2}{\partial z_2}(0,0,0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{(p-1)(q-1)}{p^2} A^{\frac{q-p}{p-1}} & -\frac{p-1}{p} \end{pmatrix},$$

we have nonpositive eigenvalues 0 and $-(p-1)/p$. By Theorem 3.1, we have a unique analytic solution $z(\xi)$ to (28) with (29). This gives an analytic solution $G(\xi) = A + B\xi + z(\xi)$ to (25) with (26).

Next, we show $v(x) = G(|x-\tau|^{\frac{p}{p-1}})$ is a solution near τ. By construction, $v(x)$ is a real analytic solution to (1) in $(\tau, \tau+\varepsilon)$ and in $(\tau-\varepsilon, \tau)$ for sufficiently small positive ε, where $v(x)$ and $v_x(x)$ have constant signature. We notice that $v_x(x)$ is continuous and $v_{xx}(x)$ is integrable on $(\tau-\varepsilon, \tau+\varepsilon)$, since $p-1$ is positive. Hence, $v(x)$ is a local solution on $(\tau-\varepsilon, \tau+\varepsilon)$ with the prescribed Cauchy data. By Proposition 2.2, $v(x) = u(x)$. Using the equation (30) and (31), C is determined as before. □

COROLLARY 3.2 ([1], [4]). (i) *If $p/(p-1)$ is an even integer, i.e. $p = (2m+2)/(2m+1)$ $(m = 0, 1, 2, \cdots)$, $u(x)$ is real analytic at τ.*

(ii) *If $p/(p-1)$ is not an even integer, the solution $u(x)$ is of class $C^{\langle\frac{2-p}{p-1}\rangle+1}$ at τ, where $\langle r\rangle$ is the least integer greater than or equal to r. Especially, when $1 < p \le 2$, $u(x)$ is of class C^2 at τ. When $2 < p$, $u(x)$ is not of class C^2 at τ.*

Derivation of equations for F and G with the prescribed Cauchy data is given in the appendix A and B as below.

A. Asymptotic expansion at σ

We compute assuming $\sigma = 0$. Since $u(0^+) = 0$ and $u_x(0^+) = A > 0$, we assume a differentiable asymptotic expansion of the form

$$u(x) \sim Ax + Bx^\beta + o(x^\beta) \quad \text{as } x \to 0^+ \tag{33}$$

where $1 < \beta$.

Since $|u_x| = u_x$ and $|u| = u$, (1) becomes

$$(p-1)(u_x)^{p-2}u_{xx} + u^{q-1} = 0\,. \tag{34}$$

If we differentiate (33) and substitute this in (34), we get

$$(p-1)(A + B\beta x^{\beta-1} + o(x^{\beta-1}))^{p-2}(B\beta(\beta-1)x^{\beta-2} + o(x^{\beta-2})) + (Ax + Bx^\beta + o(x^\beta))^{q-1} \sim 0\,.$$

Expanding the left hand side, we have

$$(p-1)(A^{p-2} + (p-2)A^{p-3}B\beta x^{\beta-1} + o(x^{\beta-1}))(B\beta(\beta-1)x^{\beta-2} + o(x^{\beta-2})) + x^{q-1}(A^{q-1} + (q-1)A^{q-2}Bx^{\beta-1} + o(x^{\beta-1})) \sim 0\,.$$

Therefore, we have

$$(p-1)A^{p-2}B\beta(\beta-1)x^{\beta-2} + o(x^{\beta-2}) + A^{q-1}x^{q-1} + o(x^{q-1}) \sim 0$$

We get the following:

1. $\beta - 2 = q - 1$ and hence $\beta = q + 1$,
2. $(p-1)A^{p-2}B\beta(\beta-1) + A^{q-1} = 0$ and hence $B = \frac{-A^{q-p+1}}{q(q+1)(p-1)}$. Next, we assume

$$u(x) \sim (-A)(-x) + B'(-x)^{\beta} + o(x^{\beta}) \quad \text{as } x \to 0^- .$$

Since $|u_x| = u_x$ and $|u| = -u$, (1) becomes

$$(p-1)(u_x)^{p-2}u_{xx} - (-u)^{q-1} = 0 . \tag{35}$$

We have $\beta = q + 1$ and $B' = -B$ as above.

We postulate a solution of the form $u(x) = xF(|x|^q)$ with $F(0) = A$ and $F'(0) = B$. When $x > 0$, substituting $u(x) = xF(x^q)$ into the equation (34) we get

$$(p-1)(F(\xi) + q\xi F'(\xi))^{p-2}(q(q+1)x^{q-1}F'(\xi) \\ + q^2x^{2q-1}F''(\xi)) + x^{q-1}(F(\xi))^{q-1} = 0$$

where $\xi = x^q$. Dividing both sides by x^{q-1}, we have

$$(p-1)[F(\xi) + q\xi F'(\xi)]^{p-2}[q(q+1)F'(\xi) + q^2\xi F''(\xi)] + (F(\xi))^{q-1} = 0 , \tag{36}$$

for $x > 0$.

When $x < 0$, substituting $u(x) = xF((-x)^q)$ into (35), we get

$$(p-1)[F(\xi) + q(-x)^qF'(\xi)]^{p-2}[-q(q+1)(-x)^{q-1}F'(\xi) \\ - q^2(-x)^{2q-1}F''(\xi)] - (-x)^{q-1}(F(\xi))^{q-1} = 0 ,$$

where $\xi = (-x)^q$. Simplifying this, we obtain the same equation for F as (36).

B. Asymptotic expansion near $x = \tau$

We compute assuming $\tau = 0$. Since $u(0^-) = A > 0$ and $u_x(0^-) = 0$, we assume this time that we have a differentiable asymptotic expansion

$$u(x) \sim A + B(-x)^{\beta} + o((-x)^{\beta}) \quad \text{as } x \to 0^- , \tag{37}$$

where $1 < \beta$. Since $|u| = u$ and $|u_x| = u_x$ by (11), (1) becomes (34).

If we differentiate (37) and substitute this in (34), we get

$$(p-1)(-B\beta(-x)^{\beta-1} - o((-x)^{\beta-1}))^{p-2}(B\beta(\beta-1)(-x)^{\beta-2} \\ + o((-x)^{\beta-2})) + (A + B(-x)^{\beta} + o((-x)^{\beta}))^{q-1} \sim 0 .$$

Expanding the left hand side, we have

$$(p-1)(-B\beta)^{p-2}(-x)^{(\beta-1)(p-2)}[1+o(1)][B\beta(\beta-1)(-x)^{\beta-2}$$
$$+o((-x)^{\beta-2})]+A^{q-1}\left[1+(q-1)\frac{B}{A}(-x)^{\beta}+o((-x)^{\beta})\right]\sim 0\,.$$

Therefore, we have

$$-(p-1)(-B\beta)^{p-1}(\beta-1)(-x)^{\beta(p-1)-p}+o((-x)^{\beta(p-1)-p})$$
$$+A^{q-1}+(q-1)A^{q-2}B(-x)^{\beta}+o((-x)^{\beta})\sim 0\,.$$

We have necessarily:

1. $\beta(p-1)-p=0$ and hence $\beta=\frac{p}{p-1}$,
2. $-(p-1)(-B\beta)^{p-1}(\beta-1)+A^{q-1}=0$ and hence $B=-\frac{p-1}{p}A^{\frac{q-1}{p-1}}$.

Next, we assume

$$u(x)\sim A+B'x^{\beta}+o(x^{\beta})\quad \text{as } x\to 0^{+}\,.$$

Since $|u_x|=-u_x$ and $|u|=u$, (1) becomes

$$(p-1)(-u_x)^{p-2}u_{xx}+u^{q-1}=0\,. \tag{38}$$

We have $\beta=p/(p-1)$ and $B'=B$ as above. Based on this trial computation, we seek for a solution of the form

$$u(x)=G(|x|^{\frac{p}{p-1}})$$

with $G(0)=A$ and $G'(0)=B$.

When $x<0$, we get from (34)

$$(p-1)\left[\frac{-p}{p-1}(-x)^{\frac{1}{p-1}}G'(\xi)\right]^{p-2}\left[\frac{p}{(p-1)^2}(-x)^{\frac{2-p}{p-1}}G'(\xi)+\left(\frac{p}{p-1}\right)^2(-x)^{\frac{2}{p-1}}G''(\xi)\right]$$
$$+(G(\xi))^{q-1}=0\,,$$

where $\xi=(-x)^{\frac{p}{p-1}}$. Simplifying this, we obtain

$$\left(\frac{p}{p-1}\right)^{p-1}(-G'(\xi))^{p-2}\left[G'(\xi)+p\xi G''(\xi)\right]+(G(\xi))^{q-1}=0\,. \tag{39}$$

When $x>0$, substituting $u(x)=G(x^{\frac{p}{p-1}})$ into (38), we get

$$(p-1)\left[\frac{-p}{p-1}x^{\frac{1}{p-1}}G'(\xi)\right]^{p-2}\left[\frac{p}{(p-1)^2}x^{\frac{2-p}{p-1}}G'(\xi)+\left(\frac{p}{p-1}\right)^2x^{\frac{2}{p-1}}G''(\xi)\right]$$
$$+(G(\xi))^{q-1}=0\,,$$

where $\xi=x^{\frac{p}{p-1}}$.

Simplifying this, we obtain the same equation for G as (39).

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