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On the Descriptions of $\mathbf{Z}/p^2\mathbf{Z}$-Torsors by the Kummer-Artin-Schreier-Witt Theory

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Introduction

The Kummer-Artin-Schreier-Witt theory is the unified theory of the Kummer theory and the Artin-Schreier-Witt theory. We denote by p a prime number and ζ_n a primitive p^n-th root of unity such that ${\zeta_n}^p = \zeta_{n-1}$. Let $A = \mathbf{Z}_{(p)}[\zeta_n]$. The Kummer-Artin-Schreier-Witt sequence

$$0 \longrightarrow (\mathbf{Z}/p^n\mathbf{Z})_A \xrightarrow{i_n} \mathcal{W}_n \xrightarrow{\Psi^n} \mathcal{V}_n \longrightarrow 0$$

has the Artin-Schreier-Witt sequence as the special fiber and the Kummer type sequence as the generic fiber, where $\mathcal{W}_n$ and $\mathcal{V}_n$ are group schemes related to deformations of the additive group scheme to the multiplicative group scheme (cf. Section 2). This sequence is a key of the Kummer-Artin-Schreier-Witt theory. The case $n = 1$ of this theory (the Kummer-Artin-Schreier theory) was presented by Waterhouse [10] and Sekiguchi-Oort-Suwa [3] independently. In the general case, this theory was formulated by Sekiguchi-Suwa [5], [8] and [7].

Let X be a scheme, G a flat group scheme locally of finite type over X and X' a scheme over X such that G acts on X'. The scheme X' is a G-torsor over X if X' is locally isomorphic to G for the flat topology on X. In particular, if G is a finite group scheme, a G-torsor is a Galois G-extension. Now let $\mathrm{PHS}(G/X)$ be the set of all isomorphism classes of G-torsors over X. If G is a commutative affine group scheme over X, then $\mathrm{PHS}(G/X) \xrightarrow{\sim} \check{H}^1_{\mathrm{fl}}(X, G) \xrightarrow{\sim} H^1_{\mathrm{fl}}(X, G)$ (cf. Raynaud [2]). Therefore we can calculate torsors by the cohomology theory.

Our aim of this article is to give concrete descriptions of $\mathbf{Z}/p^2\mathbf{Z}$-torsors over an A-scheme X, that is to say, unramified cyclic coverings of degree p^2 over an A-scheme X. In order to give them, we use arguments similar to those using in the Kummer theory and the Artin-Schreier-Witt theory (cf. Section 1). Our main result is as follows:

ASSERTION 1 (cf. Section 3, 3.3). *Let X be an A-scheme, $\mathcal{U} = \{U_j\}$ an affine open covering on X. Let $\boldsymbol{f}_{ij} \in Z^1(\mathcal{U}, \mathcal{W}_2)$ be a 1-cocycle such that $\Psi^2([\boldsymbol{f}_{ij}]) = 0$. Then, if*

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necessary by taking a refinement of $\mathcal{U}$, *there exists* $\boldsymbol{b}_j \in \Gamma(U_j, \mathcal{V}_2)$ *for each* j, *such that* $\Psi^2(\boldsymbol{f}_{ij}) = (\Lambda_0^G(\boldsymbol{b}_j, I^G(\boldsymbol{b}_i)), \Lambda_1^G(\boldsymbol{b}_j, I^G(\boldsymbol{b}_i)))$ *on* $U_j \cap U_i$. *Let* $\boldsymbol{h} \in \Gamma(X, \mathcal{V}_2)$. *Then a* $\mathbf{Z}/p^2\mathbf{Z}$*-torsor over* X *is described by* $\pi : X' \to X$ *locally given by the covering*

$$\Psi^2(\boldsymbol{z}_j) = (\Lambda_0^G(\boldsymbol{b}_j, \boldsymbol{h}), \Lambda_1^G(\boldsymbol{b}_j, \boldsymbol{h})) \quad \text{on} \quad U_j \times \mathbf{A}^2 = \operatorname{Spec} \Gamma(U_j, \mathcal{V}_2) \otimes_A A[\boldsymbol{z}_j]\,,$$

the gluing being given by

$$(\Lambda_0^F(z_j, I_0^F(z_i)), \Lambda_1^F(z_j, I_1^F(z_i))) = \boldsymbol{f}_{ij} \quad \text{on} \quad (U_j \times \mathbf{A}^2) \cap (U_i \times \mathbf{A}^2)\,,$$

and an action of $\mathbf{Z}/p^2\mathbf{Z}$ *on* X' *by*

$$(z_j, s) \longmapsto (\Lambda_0^F(z_j, i_2(s)), \Lambda_1^F(z_j, i_2(s))) \quad \text{for} \quad s \in \mathbf{Z}/p^2\mathbf{Z}\,.$$

Here Λ_0^F *and* Λ_1^F (*resp.* Λ_0^G *and* Λ_1^G) *are the polynomials which define the multiplication on* $\mathcal{W}_2$ (*resp.* $\mathcal{V}_2$), *and* I_0^F *and* I_1^F (*resp.* I_0^G *and* I_1^G) *are the polynomials which define the inverse on* $\mathcal{W}_2$ (*resp.* $\mathcal{V}_2$).

We consider the special two cases, one is the case $H^1(X, \mathbf{Z}/p^2\mathbf{Z}) \xrightarrow{\sim} \operatorname{Coker}[\Psi^2 : \Gamma(X, \mathcal{W}_2) \to \Gamma(X, \mathcal{V}_2)]$, and the other is the case $H^1(X, \mathbf{Z}/p^2\mathbf{Z}) \xrightarrow{\sim} \operatorname{Ker}[\Psi^2 : H^1(X, \mathcal{W}_2) \to H^1(X, \mathcal{V}_2)]$.

ASSERTION 2 (cf. Section 3, 3.4). *Let* B *be an* A*-algebra. We assume that* B *is a local ring or* p *is a nilpoint in* B. *Let* $X = \operatorname{Spec} B$. *Then for any unramified* p^2*-cyclic extension* C *of* B, *there exists a morphism* $f : \operatorname{Spec} B \to \mathcal{V}_2$ *such that*

$$\begin{array}{ccc} \operatorname{Spec} C & \longrightarrow & \mathcal{W}_2 \\ \downarrow & & \downarrow{\scriptstyle \Psi^2} \\ \operatorname{Spec} B & \xrightarrow{f} & \mathcal{V}_2 \end{array}$$

is cartesian.

ASSERTION 3 (cf. Theorem 3.6). *Let* B *be a strictly Henselian noetherian local ring and faithfully flat over* A. *Let* X *be a connected flat proper scheme over* B. *Put* $X_0 = X \otimes_B B/(\zeta_1 - 1)$. *Let* $\iota_X : X_0 \to X$ *be the inclusion induced by* $\iota_B : \operatorname{Spec} B/(\zeta_1 - 1) \to \operatorname{Spec} B$. *Then we obtain an isomorphism*

$$H^1_{\mathrm{fl}}(X, \mathbf{Z}/p^2\mathbf{Z}) \xrightarrow{\sim} \{([\mathcal{L}_0], [\mathcal{L}_1]) \in \operatorname{Pic}^0(X)^2 \mid (**)\ \}$$

where $(**)$ *means the following conditions*:

$$[\iota_X{}^*\mathcal{L}_0] = [\mathcal{O}_{X_0}]\,, \quad ex_F([\iota_X{}^*\mathcal{L}_0]) = [\iota_X{}^*\mathcal{L}_1]$$
$$[\mathcal{L}_0{}^{\otimes p}] = [\mathcal{O}_X]\,, \quad [\mathcal{L}_1{}^{\otimes p}] = [\mathcal{L}_0]\,.$$

For definition of the homomorphism ex_F, *see Section* 3, 3.5.

In Assertion 3, we see that $\mathbf{Z}/p^2\mathbf{Z}$-torsors over X are described by line bundles over X satisfying suitable conditions. This fact is very interesting geometrically. In general, we can give a correspondence of a $\mathbf{Z}/p^2\mathbf{Z}$-torsor over X to a μ_{p^2}-torsor over X. Moreover we can give a $\mathbf{Z}/p^2\mathbf{Z}$-torsor over X as successive Néron blow-ups starting from a μ_{p^2}-torsor over X.

ASSERTION 4 (cf. Theorem 4.10). *Let X'' be a $\mathbf{Z}/p^2\mathbf{Z}$-torsor over X and X' a μ_{p^2}-torsor over X corresponding to X''. Then we can give the morphism $X'' \to X'$ as a composite of Néron blow-ups.*

In Section 1, we recall the Kummer theory and the Artin-Schreier-Witt theory. In Section 2, we define the Kummer-Artin-Schreier-Witt group schemes and the Kummer-Artin-Schreier-Witt exact sequence. Using these, in Section 3, we argue the Kummer-Artin-Schreier-Witt theory of degree p^2, that is to say, we concreately describe a $\mathbf{Z}/p^2\mathbf{Z}$-torsor over X. In Section 4, we give a $\mathbf{Z}/p^2\mathbf{Z}$-torsor over X as successive Néron blow-ups starting from a μ_{p^2}-torsor over X.

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NOTATIONS.

- We denote by p a prime number and ζ_2 a primitive p^2-th root of unity. We put $\zeta = \zeta_2^p$.
- Let A be a discrete valuation ring. Let $\mathfrak{m}$ denote the maximal ideal of A. For $\lambda \in \mathfrak{m} - \{0\}$, we put $A_0 = A/\lambda$ and $X_0 = X \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, A_0$. Let $\iota : \mathrm{Spec}\, A_0 \longrightarrow \mathrm{Spec}\, A$ be the canonical inclusion.
- Let R (resp. F) be a commutative ring (resp. a field). We denote by $\mathbf{G}_{a,R}$ (resp. $\mathbf{G}_{a,F}$) the additive group scheme over a ring R (resp. a field F) and by $\mathbf{G}_{m,R}$ (resp. $\mathbf{G}_{m,F}$) the multiplicative group scheme over a ring R (resp. a field F). We denote by $W_{n,F}$ the group scheme of Witt vectors of length n over a field F.
- We denote by $\mathcal{G}^{(\lambda)} = \mathrm{Spec}\, A[T, (\lambda T + 1)^{-1}]$ the Kummer-Artin-Schreier group scheme (See Sekiguchi-Oort-Suwa [3], Sekiguchi-Suwa [6]). The group structure of $\mathcal{G}^{(\lambda)}$ is as follows:
  (multiplication) $T \longmapsto \lambda T \otimes T + T \otimes 1 + 1 \otimes T$,
  (unit) $T \longmapsto 0$,
  (inverse) $T \longmapsto (-T)/(\lambda T + 1)$.
- We denote by X_{zar} (resp. $X_{\text{ét}}$, resp. X_{fl}) the small Zariski site (resp. small étale site, resp. small flat site).

1. The Kummer theory and the Artin-Schreier-Witt theory

In order to understand the Kummer-Artin-Schreier-Witt theory, we recall the Kummer theory and the Artin-Schreier-Witt theory.

1.1. We recall first the Kummer theory. Let n be an integer with $n > 1$ and μ_n the set of n-th roots of unity. Put $A = \mathbf{Z}[1/n][\mu_n]$ and $\mu_{n,A} = \operatorname{Ker}[n : \mathbf{G}_{m,A} \longrightarrow \mathbf{G}_{m,A}]$. Then we obtain the sequence of group schemes over A

$$0 \longrightarrow \mu_{n,A} \longrightarrow \mathbf{G}_{m,A} \xrightarrow{n} \mathbf{G}_{m,A} \longrightarrow 0\,. \tag{1}$$

The sequence (1) is an exact sequence of sheaves on $(\operatorname{Spec} A)_{\text{ét}}$, and hence it is an exact sequence of sheaves on $(\operatorname{Spec} A)_{\text{fl}}$. It is called the Kummer sequence. Since $\mu_n \subset A$, the group scheme $\mu_{n,A}$ is (non canonically) isomorphic to the constant group scheme $\mathbf{Z}/n\mathbf{Z}$. For an A-scheme X, the exact sequence (1) induces the cohomology long exact sequence

$$\begin{aligned} 0 &\longrightarrow \Gamma(X, \mathbf{Z}/n\mathbf{Z}) \longrightarrow \Gamma(X, \mathcal{O}_X^*) \xrightarrow{\ n\ } \Gamma(X, \mathcal{O}_X^*) \\ &\longrightarrow H^1(X, \mathbf{Z}/n\mathbf{Z}) \longrightarrow H^1(X, \mathbf{G}_{m,A}) \xrightarrow{\ n\ } H^1(X, \mathbf{G}_{m,A})\,. \end{aligned}$$

Hence we obtain the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X^*)/\Gamma(X, \mathcal{O}_X^*)^n \longrightarrow H^1(X, \mathbf{Z}/n\mathbf{Z}) \longrightarrow {}_n\mathrm{Pic}(X) \longrightarrow 0\,. \tag{2}$$

We describe the exact sequence (2) more concretely. Now, let $\mathcal{U} = \{U_j\}$ be an affine open covering on X and $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$ a 1-cocycle representing an element $\eta \in H^1(X, \mathcal{O}_X^*)$ such that $n\eta = 0$. This means that (f_{ij}^n) is a 1-coboundary, and if necessary replacing $\mathcal{U}$ a refinement, we can write

$$f_{ij}^n = b_j/b_i \quad \text{on} \quad U_j \cap U_i\,,$$

where $b_j \in \Gamma(U_j, \mathcal{O}_X^*)$. Let $h \in \Gamma(X, \mathcal{O}_X^*)$. We define $\pi : X' \to X$ locally by the Kummer covering

$$z_j^n = b_j h \quad \text{on} \quad U_j \times \mathbf{A}^1 = \operatorname{Spec} \Gamma(U_j, \mathcal{O}_X^*) \otimes_A A[z_j]\,,$$

the gluing being given by

$$z_j/z_i = f_{ij} \quad \text{on} \quad (U_j \times \mathbf{A}^1) \cap (U_i \times \mathbf{A}^1)\,,$$

and an action of μ_n on X', that of $\mathbf{Z}/n\mathbf{Z}$ on X' by

$$(\zeta, z_j) \longmapsto \zeta z_j\,.$$

Then X' is a $\mathbf{Z}/n\mathbf{Z}$-torsor over X, and $[X'] \in H^1(X, \mathbf{Z}/n\mathbf{Z})$ is mapped to $\eta \in H^1(X, \mathcal{O}_X^*)$.

(A) Let B be a local A-algebra and $X = \operatorname{Spec} B$. Since

$$H^1(X, \mathbf{G}_{m,K}) = \mathrm{Pic}(X) = 0$$

by the Hilbert theorem 90, we obtain an isomorphism

$$B^*/(B^*)^n \xrightarrow{\sim} H^1(X, \mathbf{Z}/n\mathbf{Z}).$$

Hence, for any unramified n-cyclic extension C of B, there exists a morphism $f : \operatorname{Spec} B \to \mathbf{G}_{m,A}$ such that

$$\begin{array}{ccc} \operatorname{Spec} C & \longrightarrow & \mathbf{G}_{m,A} \\ \downarrow & & \downarrow{\scriptstyle n} \\ \operatorname{Spec} B & \xrightarrow{f} & \mathbf{G}_{m,A} \end{array}$$

is cartesian.

(B) Let K be an algebraically closed field such that n is an invertible element and X a connected proper K-scheme. Then, since $\Gamma(X, \mathcal{O}_X^*) = K^*$ and the morphism $n : K^* \longrightarrow K^*$ is surjective, we obtain an isomorphism

$$H^1(X, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} {}_n\operatorname{Pic}(X).$$

1.2. We recall the Artin-Schreier-Witt theory. Let X be an $\mathbf{F}_p$-scheme and F the Frobenius map over $W_{n,\mathbf{F}_p}$. Then we obtain the sequence of group schemes

$$0 \longrightarrow \mathbf{Z}/p^n\mathbf{Z} \longrightarrow W_{n,\mathbf{F}_p} \xrightarrow{F-1} W_{n,\mathbf{F}_p} \longrightarrow 0. \tag{3}$$

The sequence (3) is an exact sequence of sheaves on $(\operatorname{Spec} A)_{\text{ét}}$, and hence it is an exact sequence of sheaves on $(\operatorname{Spec} A)_{\text{fl}}$. It is called the Artin-Schreier-Witt sequence. The exact sequence (3) induces the cohomology long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(X, \mathbf{Z}/p^n\mathbf{Z}) & \longrightarrow & \Gamma(X, W_{n,\mathbf{F}_p}) & \xrightarrow{F-1} & \Gamma(X, W_{n,\mathbf{F}_p}) \\ & \longrightarrow & H^1(X, \mathbf{Z}/p^n\mathbf{Z}) & \longrightarrow & H^1(X, W_{n,\mathbf{F}_p}) & \xrightarrow{F-1} & H^1(X, W_{n,\mathbf{F}_p}). \end{array}$$

Now, let $\mathcal{U} = \{U_j\}$ be an affine open covering on X and $(\boldsymbol{f}_{ij}) \in Z^1(\mathcal{U}, W_{n,\mathbf{F}_p})$ a 1-cocycle representing an element $\boldsymbol{\eta} \in H^1(X, W_{n,\mathbf{F}_p})$ such that $F\boldsymbol{\eta} = \boldsymbol{\eta}$. This means that $(\boldsymbol{f}_{ij}^p - \boldsymbol{f}_{ij})$ is a 1-coboundary, and we can write

$$\boldsymbol{f}_{ij}^p - \boldsymbol{f}_{ij} = \boldsymbol{b}_j - \boldsymbol{b}_i \quad \text{on} \quad U_{ij} := U_j \cap U_i$$

where $\boldsymbol{b}_j \in \Gamma(U_j, W_{n,\mathbf{F}_p})$. Let $\boldsymbol{h} \in \Gamma(X, W_{n,\mathbf{F}_p})$. We define $\pi : X' \to X$ locally by the Artin-Schreier-Witt covering

$$\boldsymbol{z}_j^p - \boldsymbol{z}_j = \boldsymbol{b}_j + \boldsymbol{h} \quad \text{on} \quad U_j \times \mathbf{A}^n = \operatorname{Spec} \Gamma(U_j, W_{n,\mathbf{F}_p}) \otimes_A A[\boldsymbol{z}_j],$$

the gluing being given by

$$\boldsymbol{z}_j - \boldsymbol{z}_i = \boldsymbol{f}_{ij} \quad \text{on} \quad (U_j \times \mathbf{A}^n) \cap (U_i \times \mathbf{A}^n),$$

and an action of $\mathbf{Z}/p^n\mathbf{Z}$ on X' by

$$(z_j, s) \longmapsto z_j + s\,, \quad \text{for} \quad s \in \mathbf{Z}/p^n\mathbf{Z}\,.$$

Then X' is a $\mathbf{Z}/p^n\mathbf{Z}$-torsor over X, and $[X'] \in H^1(X, \mathbf{Z}/p^n\mathbf{Z})$ is mapped to $\boldsymbol{\eta} \in H^1(X, W_{n,\mathbf{F}_p})$.

(A) Let B be an $\mathbf{F}_p$-algebra and $X = \operatorname{Spec} B$. Since

$$H^1(X, W_{n,\mathbf{F}_p}) = 0\,,$$

we obtain an isomorphism

$$\operatorname{Coker}[F - 1 : W_n(B) \longrightarrow W_n(B)] \xrightarrow{\sim} H^1(X, \mathbf{Z}/p\mathbf{Z})\,.$$

Hence, for any unramified p^n-cyclic extension C of B, there exists a morphism $f : \operatorname{Spec} B \to W_{n,\mathbf{F}_p}$ such that

$$\begin{array}{ccc} \operatorname{Spec} C & \longrightarrow & W_{n,\mathbf{F}_p} \\ \downarrow & & \downarrow{\scriptstyle F-1} \\ \operatorname{Spec} B & \xrightarrow{f} & W_{n,\mathbf{F}_p} \end{array}$$

is cartesian.

(B) Let k be an algebraically closed field with characteristic $p > 0$ and X a connected proper k-scheme. Then, since $\Gamma(X, W_n) = W_n(k)$ and the morphism $F - 1$ is surjective over $W_n(k)$, we obtain an isomorphism

$$H^1(X, \mathbf{Z}/p^n\mathbf{Z}) \xrightarrow{\sim} \operatorname{Ker}[F - 1 : H^1(X, W_n) \longrightarrow H^1(X, W_n)]\,.$$

REMARK 1.3. If X is smooth over k, $H^1(X, W_n)$ is isomorphic to the Dieudonné module of ${}_{F^n}\underline{\operatorname{Pic}}_{X/k}$. We see the case $n = 1$.

Let $k[\varepsilon]$ be the ring of dual numbers ($k[\varepsilon] \xrightarrow{\sim} k[T]/(T^2)$). The exact sequence

$$0 \longrightarrow \mathbf{G}_{a,k} \longrightarrow \prod_{k[\varepsilon]/k} \mathbf{G}_{m,k[\varepsilon]} \longrightarrow \mathbf{G}_{m,k} \longrightarrow 0$$

induces the exact sequence

$$0 \longrightarrow H^1(X, \mathbf{G}_{a,k}) \longrightarrow \underline{\operatorname{Pic}}_{X/k}(k[\varepsilon]) \longrightarrow \underline{\operatorname{Pic}}_{X/k}(k)\,,$$

where $\prod_{k[\varepsilon]/k}$ is the Weil restriction functor. Then we get an isomorphism

$$H^1(X, \mathbf{G}_a) \xrightarrow{\sim} \operatorname{Lie}(\underline{\operatorname{Pic}}_{X/k}) \xrightarrow{\sim} \operatorname{Lie}({}_F\underline{\operatorname{Pic}}_{X/k})\,.$$

2. The Kummer-Artin-Schreier-Witt group schemes

In this section, we define the Kummer-Artin-Schreier-Witt group schemes and the Kummer-Artin-Schreier-Witt exact sequence to unify the Kummer theory and the Artin-Schreier-Witt theory. For details, see [5], [8], [7].

Hereafter, let $\lambda = \zeta - 1$, $\lambda_2 = \zeta_2 - 1$ and $A = \mathbf{Z}_{(p)}[\zeta_2]$. Then A is a discrete valuation ring and λ_2 is a uniformizing parameter of A. K (resp. k) denotes the fraction field (resp. the residue field) of A. Then $K = \mathbf{Q}(\zeta_2)$ and $k = \mathbf{F}_p$.

2.1. Put

$$\eta = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\lambda_2^k \quad \text{and} \quad \tilde{\eta} = \frac{\lambda^{p-1}}{p}(p\eta - \lambda)\,.$$

Put

$$\begin{aligned}
&F(T) = \sum_{k=0}^{p-1} \frac{(\eta T)^k}{k!}\,, \quad G(T) = \sum_{k=0}^{p-1} \frac{(\tilde{\eta} T)^k}{k!}\,,\\
&\Lambda_0^F(X_0, Y_0) = \lambda X_0 Y_0 + X_0 + Y_0\,,\ \Lambda_0^G(X_0, Y_0) = \lambda^p X_0 Y_0 + X_0 + Y_0\,,\\
&\Lambda_1^F(X_0, X_1, Y_0, Y_1) = \lambda X_1 Y_1 + X_1 F(Y_0) + F(X_0) Y_1\\
&\qquad\qquad + \frac{1}{\lambda}\{F(X_0)F(Y_0) - F(\lambda X_0 Y_0 + X_0 + Y_0)\}\,,\\
&\Lambda_1^G(X_0, X_1, Y_0, Y_1) = \lambda^p X_1 Y_1 + X_1 G(Y_0) + G(X_0) Y_1\\
&\qquad\qquad + \frac{1}{\lambda^p}\{G(X_0)G(Y_0) - G(\lambda^p X_0 Y_0 + X_0 + Y_0)\}\,,\\
&\Psi_0(T_0) = \frac{1}{\lambda^p}\{(\lambda T_0 + 1)^p - 1\}\,,\\
&\Psi_1(T_0, T_1) = \frac{1}{\lambda^p}\left\{\frac{(\lambda T_1 + F(T_0))^p}{\lambda T_0 + 1} - G\left(\frac{1}{\lambda^p}\{(\lambda T_0 + 1)^p - 1\}\right)\right\}\,,\\
&\mathcal{W}_2 = \operatorname{Spec} A\left[T_0, T_1, \frac{1}{\lambda T_0 + 1}, \frac{1}{\lambda T_1 + F(T_0)}\right],\\
&\mathcal{V}_2 = \operatorname{Spec} A\left[T_0, T_1, \frac{1}{\lambda^p T_0 + 1}, \frac{1}{\lambda^p T_1 + G(T_0)}\right].
\end{aligned}$$

Let v denote the p-adic valuation normalized by $v(p) = 1$. Then

$$v(\lambda) = \frac{1}{p-1}\,, \quad v(\lambda_2) = v(\eta) = \frac{1}{p(p-1)}\,.$$

In fact, $\lambda^{p-1} \sim p$ and $\lambda_2^p \sim \lambda$ in A. Moreover, $\lambda_2 | \eta$ and $\lambda | \tilde{\eta}$. $\mathcal{W}_2$ and $\mathcal{V}_2$ are open subschemes of the affine space $\mathbf{A}^2$. Sekiguchi-Suwa showed the following:

THEOREM 2.2 (Sekiguchi-Suwa [8], Theorem 5.2).

(1) *The polynomials* $\Lambda_1^F(X_0, X_1, Y_0, Y_1)$, $\Lambda_1^G(X_0, X_1, Y_0, Y_1)$ *have their coefficients in A. Moreover,*

$$(T_0, T_1) \longmapsto (\Lambda_0^F(T_0 \otimes 1, 1 \otimes T_0), \Lambda_1^F(T_0 \otimes 1, T_1 \otimes 1, 1 \otimes T_0, 1 \otimes T_1))$$

defines a structure of group on $\mathcal{W}_2$, *and*

$$(T_0, T_1) \longmapsto (\Lambda_0^G(T_0 \otimes 1, 1 \otimes T_0), \Lambda_1^G(T_0 \otimes 1, T_1 \otimes 1, 1 \otimes T_0, 1 \otimes T_1))$$

defines a structure of group on $\mathcal{V}_2$.

(2) *The fraction* $\Psi_1(T_0, T_1)$ *belongs to* $A[T_0, T_1, (\lambda T_0+1)^{-1}, (\lambda T_1+F(T_0))^{-1}]$. *Moreover,*

$$(T_0, T_1) \longmapsto (\Psi_0(T_0), \Psi_1(T_0, T_1))$$

defines an A-homomorphism $\Psi^2 : \mathcal{W}_2 \to \mathcal{V}_2$, *and* $\mathrm{Ker}[\Psi^2 : \mathcal{W}_2 \to \mathcal{V}_2]$ *is isomorphic to the constant group scheme* $\mathbf{Z}/p^2\mathbf{Z}$.

(3) $(U_0, U_1) \mapsto (\lambda T_0 + 1, \lambda T_1 + F(T_0))$ *defines a homomorphism* $\alpha^{(F)} : \mathcal{W}_2 \to \mathbf{G}_m^2$ *of group schemes over A, and* $(U_0, U_1) \mapsto (\lambda^p T_0 + 1, \lambda^p T_1 + G(T_0))$ *defines a homomorphism* $\alpha^{(G)} : \mathcal{V}_2 \to \mathbf{G}_m^2$ *of group schemes over A. Moreover,* $\alpha_K^{(F)} : \mathcal{W}_{2,K} \to \mathbf{G}_{m,K}^2$ *and* $\alpha_K^{(G)} : \mathcal{V}_{2,K} \to \mathbf{G}_{m,K}^2$ *are isomorphisms.*

(4) *The diagram of group schemes over A*

$$\begin{array}{ccc} \mathcal{W}_2 & \xrightarrow{\alpha^{(F)}} & \mathbf{G}_m^2 \\ {\scriptstyle \Psi^2}\downarrow & & {\scriptstyle \Theta^2}\downarrow \\ \mathcal{V}_2 & \xrightarrow{\alpha^{(G)}} & \mathbf{G}_m^2 \end{array}$$

is commutative. Here Θ^2 *is defined by*

$$(U_0, U_1) \longmapsto (U_0^p, U_0^{-1}U_1^p).$$

(5) *The special fiber of the exact sequence of group schemes over A*

$$0 \longrightarrow (\mathbf{Z}/p^2\mathbf{Z})_A \xrightarrow{i_2} \mathcal{W}_2 \xrightarrow{\Psi^2} \mathcal{V}_2 \longrightarrow 0$$

is isomorphic to the Artin-Schreier-Witt sequence (3).

Sekiguchi-Suwa have verified this theorem in [8]. We see an outline of the proof of (2). For the proof, it is enough to show the following congruence relations:

(I) $\quad F(T)^p \equiv p\eta T + \dfrac{(p\eta)^2}{2!}T^2 + \dfrac{(p\eta)^3}{3!}T^3 + \sum_{k=0}^{p-1}\dfrac{\eta^{kp}}{k!}T^{kp} \bmod \lambda^p\,;$

(II) $$(\lambda T+1)G\Bigl(\frac{(\lambda T+1)^p-1}{\lambda^p}\Bigr) \equiv p\eta T + \frac{(p\eta)^2}{2!}T^2 + \frac{(p\eta)^3}{3!}T^3 + \sum_{k=0}^{p-1}\frac{\tilde{\eta}^k}{k!}T^{kp} \bmod \lambda^p\,;$$

(III) $\quad \eta^p \equiv \tilde{\eta} \bmod \lambda^p\,.$

Our proof which is independent of a general case is different from the one in Sekiguchi-Suwa [8]. It was given by Suwa. It is as follows:

LEMMA 2.3. *Let $f(T), g(T) \in A[[T]]$. If $f(T) \equiv g(T) \bmod \lambda$, $f(T)^p \equiv g(T)^p \bmod \lambda^p$.*

PROOF. Put

$$f(T) = g(T) + \lambda h(T)\,, \quad h(T) \in A[[T]]\,.$$

Then

$$f(T)^p = g(T)^p + \sum_{k=1}^{p}\binom{p}{k}\lambda^k g(T)^{p-k}h(T)^k\,.$$

Note that $\lambda^p | \binom{p}{k}\lambda^k$ if $k \geq 1$.

LEMMA 2.4. *$E_p(T)^p = \exp(pT)E_p(T^p)$, where $E_p(T)$ is the Artin-Hasse exponential series*:

$$E_p(T) = \exp\Biggl(\sum_{k=0}^{\infty}\frac{T^{p^k}}{p^k}\Biggr)\,.$$

PROOF.

$$E_p(T)^p = \exp\Biggl(\sum_{k=0}^{\infty}\frac{T^{p^k}}{p^{k-1}}\Biggr) = \exp(pT)\exp\Biggl(\sum_{k=0}^{\infty}\frac{(T^p)^{p^k}}{p^k}\Biggr) = \exp(pT)E_p(T^p)\,.$$

LEMMA 2.5. *Let $a \in A$. Then*

$$E_p(aT) \equiv \sum_{k=0}^{p-1}\frac{(aT)^k}{k!} \bmod a^p\,.$$

PROOF. Note that

$$E_p(T) \in \mathbf{Z}_{(p)}[[T]]\,, \quad E_p(T) \equiv \exp(T) \bmod T^p\,.$$

PROOF OF (I). By Lemma 2.5, we have

$$F(T) \equiv E_p(\eta T) \bmod \lambda$$

since $\lambda | \eta^p$. Hence

$$F(T)^p \equiv E_p(\eta T)^p \bmod \lambda^p$$

by Lemma 2.3. Furthermore, by Lemma 2.4,

$$E_p(\eta T)^p \equiv \exp(p\eta T) E_p(\eta^p T^p) \bmod \lambda^p .$$

Now

$$\exp(p\eta T) \equiv 1 + p\eta T + \frac{(p\eta)^2}{2!} T^2 + \frac{(p\eta)^3}{3!} T^3 \bmod \lambda^p .$$

In fact,

$$\begin{aligned} v\left(\frac{(p\eta)^k}{k!}\right) = kv(p\eta) - v(k!) &= k\left\{1 + \frac{1}{p(p-1)}\right\} - \sum_{i=1}^{\infty}\left[\frac{k}{p^i}\right] \\ &\geq k\left\{1 + \frac{1}{p(p-1)}\right\} - k\frac{1}{p-1} \\ &= k\frac{p-1}{p} . \end{aligned}$$

Hence, if $k \geq 4$,

$$v\left(\frac{(p\eta)^k}{k!}\right) \geq k\frac{p-1}{p} \geq v(\lambda^p) = \frac{p}{p-1} .$$

By Lemma 2.5,

$$E_p(\eta^p T^p) \equiv \sum_{k=0}^{p-1} \frac{\eta^{pk}}{k!} T^{pk} \bmod \lambda^p .$$

Therefore

$$\begin{aligned} E_p(\eta T)^p &\equiv \exp(p\eta T) E_p(\eta^p T^p) \bmod \lambda^p \\ &\equiv \left(1 + p\eta T + \frac{(p\eta)^2}{2!} T^2 + \frac{(p\eta)^3}{3!} T^3\right) \sum_{k=0}^{p-1} \frac{\eta^{pk}}{k!} T^{pk} \bmod \lambda^p . \end{aligned}$$

Now

$$\left(1 + p\eta T + \frac{(p\eta)^2}{2!} T^2 + \frac{(p\eta)^3}{3!} T^3\right) \sum_{k=0}^{p-1} \frac{\eta^{pk}}{k!} T^{pk}$$

$$\equiv\ p\eta T + \frac{(p\eta)^2}{2!}T^2 + \frac{(p\eta)^3}{3!}T^3 + \sum_{k=0}^{p-1}\frac{\eta^{pk}}{k!}T^{pk} \bmod \lambda^p\,.$$

These imply (I).

LEMMA 2.6.

$$\frac{1}{p}\binom{p}{k} \equiv \frac{(-1)^{k-1}}{k} \bmod p \quad (1 \le k \le p-1)\,.$$

PROOF.

$$\frac{1}{p}\binom{p}{k} = \frac{1}{p}\frac{p(p-1)\cdots(p-k+1)}{k!} \equiv (-1)^{k-1}\frac{(k-1)!}{k!} \bmod p\,.$$

LEMMA 2.7. *Let $a \in A$. Then*

$$\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}(aT)^k \equiv \frac{(1+aT)^p - 1 - (aT)^p}{p} \bmod pa^2\,.$$

PROOF. Apply Lemma 2.6, developing the right hand side.

LEMMA 2.8. *Let $a \in A$. Then*

$$\sum_{k=0}^{p-1}\frac{1}{k!}\left\{\sum_{j=1}^{p-1}\frac{(-1)^{j-1}}{j}(aT)^j\right\}^k \equiv 1 + aT \bmod a^p\,.$$

PROOF.

$$\log(1+T) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}T^k\,, \quad \exp(T) = \sum_{k=0}^{\infty}\frac{1}{k!}T^k\,, \quad \exp(\log(1+T)) = 1+T\,.$$

Hence

$$\sum_{k=0}^{p-1}\frac{1}{k!}\left\{\sum_{j=1}^{p-1}\frac{(-1)^{j-1}}{j}(T)^j\right\}^k \equiv 1 + T \bmod T^p\,,$$

and we get the assertion by substituting aT for T.

LEMMA 2.9. *Let $a \in A$. Suppose that $a^{p-2}|p$. Then*

(1) $\quad 1 + aT \equiv \displaystyle\sum_{k=0}^{p-1}\frac{1}{k!}\left\{\frac{(1+aT)^p - 1 - (aT)^p}{p}\right\}^k \bmod a^p\,;$

(2) $\quad (1+aT)\displaystyle\sum_{k=0}^{p-1}\frac{1}{k!}\left\{\frac{1 + (aT)^p - (1+aT)^p}{p}\right\}^k \equiv 1 \bmod a^p\,.$

PROOF. By Lemma 2.7,

$$\sum_{k=0}^{p-1}\frac{1}{k!}\left\{\sum_{j=1}^{p-1}\frac{(-1)^{j-1}}{j}(aT)^j\right\}^k \equiv \sum_{k=0}^{p-1}\frac{1}{k!}\left\{\frac{(1+aT)^p-1-(aT)^p}{p}\right\}^k \bmod pa^2 .$$

Hence, by Lemma 2.8, we get

$$\begin{aligned}1+aT &\equiv \sum_{k=0}^{p-1}\frac{1}{k!}\left\{\sum_{j=1}^{p-1}\frac{(-1)^{j-1}}{j}(aT)^j\right\}^k \bmod a^p\\ &\equiv \sum_{k=0}^{p-1}\frac{1}{k!}\left\{\frac{(1+aT)^p-1-(aT)^p}{p}\right\}^k \bmod a^p\end{aligned}$$

since $a^{p-2}|p$.

PROOF OF (II).

$$G\left(\frac{(\lambda T+1)^p-1}{\lambda^p}\right) \equiv E_p\left(\tilde{\eta}\frac{(\lambda T+1)^p-1}{\lambda^p}\right) \bmod \lambda^p$$

since $\lambda|\tilde{\eta}$. Now

$$\tilde{\eta}\frac{(\lambda T+1)^p-1}{\lambda^p} = \tilde{\eta}T^p + \tilde{\eta}\frac{(\lambda T+1)^p-1-(\lambda T)^p}{\lambda^p}$$

and

$$\begin{aligned}\tilde{\eta}\frac{(\lambda T+1)^p-1-(\lambda T)^p}{\lambda^p} &= \frac{\lambda^{p-1}}{p}(p\eta-\lambda)\frac{(\lambda T+1)^p-1-(\lambda T)^p}{\lambda^p}\\ &= \frac{\eta}{\lambda}\{(\lambda T+1)^p-1-(\lambda T)^p\} - \frac{(\lambda T+1)^p-1-(\lambda T)^p}{p} .\end{aligned}$$

Since $\lambda^p|p\lambda$,

$$\frac{\eta}{\lambda}\{(\lambda T+1)^p-1-(\lambda T)^p\} = \sum_{k=1}^{p-1}\binom{p}{k}\eta\lambda^{k-1}T^k \equiv p\eta T \bmod \lambda^p .$$

If $i+j+k \geq p$, $\lambda^p|(p\eta)^i\tilde{\eta}^j\lambda^k$. Hence

$$E_p\left(\tilde{\eta}\frac{(\lambda T+1)^p-1}{\lambda^p}\right) \equiv E_p(p\eta T)E_p(\tilde{\eta}T^p)E_p\left(\frac{1+(\lambda T)^p-(1+\lambda T)^p}{p}\right) \bmod \lambda^p .$$

Applying Lemma 2.9 to $a=\lambda$, we obtain

$$(1+\lambda T)\sum_{k=0}^{p-1}\frac{1}{k!}\left\{\frac{1+(\lambda T)^p-(1+\lambda T)^p}{p}\right\}^k \equiv 1 \bmod \lambda^p.$$

By Lemma 2.5,

$$E_p(p\eta T) \equiv 1 + p\eta T + \frac{(p\eta)^2}{2!}T^2 + \frac{(p\eta)^3}{3!}T^3 \bmod \lambda^p\,,$$
$$E_p(\tilde{\eta}T) \equiv \sum_{k=0}^{p-1} \frac{\tilde{\eta}^k}{k!}T^k \bmod \lambda^p\,.$$

Hence

$$(1+\lambda T)G\left(\frac{(\lambda T+1)^p-1}{\lambda^p}\right) \equiv \left(1 + p\eta T + \frac{(p\eta)^2}{2!}T^2 + \frac{(p\eta)^3}{3!}T^3\right)\sum_{k=0}^{p-1} \frac{\tilde{\eta}^k}{k!}T^{pk} \bmod \lambda^p\,.$$

Now

$$\left(1 + p\eta T + \frac{(p\eta)^2}{2!}T^2 + \frac{(p\eta)^3}{3!}T^3\right)\sum_{k=0}^{p-1} \frac{\tilde{\eta}^k}{k!}T^{pk}$$
$$\equiv\ p\eta T + \frac{(p\eta)^2}{2!}T^2 + \frac{(p\eta)^3}{3!}T^3 + \sum_{k=0}^{p-1} \frac{\tilde{\eta}^k}{k!}T^{pk} \bmod \lambda^p$$

since $\lambda^p | p\tilde{\eta}$. These imply (II).

LEMMA 2.10.

(1) $\eta \equiv \dfrac{\lambda - \lambda_2^p}{p} \bmod p\lambda_2^p$.

(2) $\lambda \equiv \lambda_2^p + p\eta \bmod p\lambda_2^p$. *Hence* $\lambda \equiv \lambda_2^p + p\eta \bmod \lambda^p$.

(3) $\lambda^k \equiv \lambda_2^{pk} + kp\eta\lambda_2^{(k-1)p} \bmod p\lambda_2^p$. *Hence* $\lambda^k \equiv \lambda_2^{pk} + kp\eta\lambda_2^{(k-1)p} \bmod \lambda^p (k \geq 2)$.

PROOF. By Lemma 2.6,

$$\eta = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\lambda_2^k \equiv \frac{(\lambda_2+1)^p - \lambda_2^p - 1}{p} \bmod \lambda^p\,.$$

Now

$$(\lambda_2+1)^p - 1 = \lambda\,.$$

These imply (1), (2) and (3).

LEMMA 2.11.

$$\eta^p \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k}\lambda_2^{pk} \equiv \sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}\lambda_2^{pk} \bmod \lambda^p\,.$$

PROOF. By the definition,

$$\eta = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \lambda_2^k .$$

Then we obtain

$$\eta^p \equiv \left\{ \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \lambda_2^k \right\}^p \bmod \lambda^p ,$$

noting that $\lambda | \lambda_2^p$. Now

$$\left\{ \frac{(-1)^{k-1}}{k} \right\}^p \equiv \frac{(-1)^{k-1}}{k} \bmod p .$$

Hence

$$\left\{ \frac{(-1)^{k-1}}{k} \lambda_2^k \right\}^p \equiv \frac{(-1)^{k-1}}{k} \lambda_2^{pk} \bmod \lambda^p .$$

LEMMA 2.12.

$$\frac{\lambda^{p-1}}{p} = -\sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \lambda^{k-1} .$$

PROOF. Develop and divide by $p\lambda$ the right hand side of $\lambda^p = \lambda^p + 1 - (\lambda + 1)^p$.

PROOF OF (III). By Lemma 2.12,

$$\begin{aligned} \tilde{\eta} = \frac{\lambda^{p-1}}{p}(p\eta - \lambda) &= -\left\{ \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \lambda^{k-1} \right\} (p\eta - \lambda) \\ &= -\sum_{k=1}^{p-1} \binom{p}{k} \lambda^{k-1} \eta + \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \lambda^k . \end{aligned}$$

Now

$$-\sum_{k=1}^{p-1} \binom{p}{k} \lambda^{k-1} \eta + \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \lambda^k \equiv -p\eta + \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \lambda^k \bmod \lambda^p ,$$

since $\lambda^p | p\lambda$. Hence

$$\tilde{\eta} \equiv -p\eta + \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \lambda^k \bmod \lambda^p .$$

On the other hand, by Lemma 2.11,

$$\eta^p \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \lambda_2^{pk} \equiv \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \lambda_2^{pk} \bmod \lambda^p .$$

These, together with Lemma 2.10, imply (III).

EXAMPLE 2.13. $p = 2$

$$\zeta = -1, \quad \zeta_2 = i, \quad \lambda = -2, \quad \lambda_2 = i - 1, \quad \eta = \lambda_2 = i - 1, \quad \tilde{\eta} = -2i$$

$$\begin{aligned} F(T) &= 1 + (i-1)T , \\ G(T) &= 1 - 2iT , \\ \Psi_0(T_0) &= T_0^2 - T_0 , \\ \Psi_1(T_0, T_1) &= \frac{T_1^2 - T_1 + iT_0^2 - iT_0^3 - (i-1)T_0T_1}{-2T_0 + 1} , \\ \Lambda_0^F(X_0, Y_0) &= -2X_0Y_0 + X_0 + Y_0 , \\ \Lambda_1^F(X_0, X_1, Y_0, Y_1) &= -2X_1Y_1 + X_1\{1 + (i-1)Y_0\} + \{1 + (i-1)X_0\}Y_1 + X_0Y_0 . \end{aligned}$$

2.14. We supplement the previous subsection. $\mathcal{W}_2$ has a structure of group scheme as follows:
(multiplication) $(T_0, T_1) \longmapsto (\Lambda_0^F(T_0 \otimes 1, 1 \otimes T_0), \Lambda_1^F(T_0 \otimes 1, T_1 \otimes 1, 1 \otimes T_0, 1 \otimes T_1))$,
(unit) $(T_0, T_1) \longmapsto (0, 0)$,
(inverse) $(T_0, T_1) \longmapsto I^F(T_0, T_1) = (I_0^F(T_0), I_1^F(T_0, T_1))$,
where

$$I_0^F(T_0) = \frac{-T_0}{\lambda T_0 + 1} , \quad I_1^F(T_0, T_1) = \frac{1}{\lambda} \left\{ \frac{1}{\lambda T_1 + F(T_0)} - F\left(\frac{-T_0}{\lambda T_0 + 1} \right) \right\} .$$

The group scheme $\mathcal{W}_2$ is called by the Kummer-Artin-Schreier-Witt group scheme.

$\mathcal{V}_2$ has a structure of group scheme as follows:
(multiplication) $(T_0, T_1) \longmapsto (\Lambda_0^G(T_0 \otimes 1, 1 \otimes T_0), \Lambda_1^G(T_0 \otimes 1, T_1 \otimes 1, 1 \otimes T_0, 1 \otimes T_1))$,
(unit) $(T_0, T_1) \longmapsto (0, 0)$,
(inverse) $(T_0, T_1) \longmapsto I^G(T_0, T_1) = (I_0^G(T_0), I_1^G(T_0, T_1))$,
where

$$I_0^G(T_0) = \frac{-T_0}{\lambda^p T_0 + 1} , \quad I_1^G(T_0, T_1) = \frac{1}{\lambda^p} \left\{ \frac{1}{\lambda^p T_1 + G(T_0)} - G\left(\frac{-T_0}{\lambda^p T_0 + 1} \right) \right\} .$$

The sequence of group schemes

$$0 \longrightarrow (\mathbf{Z}/p^2\mathbf{Z})_A \xrightarrow{i_2} \mathcal{W}_2 \xrightarrow{\Psi^2} \mathcal{V}_2 \longrightarrow 0 \tag{4}$$

is an exact sequence of sheaves on $(\operatorname{Spec} A)_{\text{ét}}$, hence it is an exact sequence of sheaves on $(\operatorname{Spec} A)_{\text{fl}}$. We call this sequence the Kummer-Artin-Schreier-Witt sequence. The exact sequence (4) has the Artin-Schreier-Witt sequence

$$0 \longrightarrow \mathbf{Z}/p^2\mathbf{Z} \longrightarrow W_{2,k} \stackrel{F-1}{\longrightarrow} W_{2,k} \longrightarrow 0$$

as the special fiber, and the exact sequence of Kummer type

$$0 \longrightarrow \mu_{p^2} \longrightarrow \mathbf{G}_{m,K}^2 \stackrel{\Theta^2}{\longrightarrow} \mathbf{G}_{m,K}^2 \longrightarrow 0$$

as the generic fiber.

3. The Kummer-Artin-Schreier-Witt theory

For an A-scheme X, we concretely describe a $\mathbf{Z}/p^2\mathbf{Z}$-torsor X' over X.

3.1. The exact sequence (4)

$$0 \longrightarrow \mathbf{Z}/p^2\mathbf{Z} \stackrel{i_2}{\longrightarrow} \mathcal{W}_2 \stackrel{\Psi^2}{\longrightarrow} \mathcal{V}_2 \longrightarrow 0$$

induces the cohomology long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(X, \mathbf{Z}/p^2\mathbf{Z}) & \stackrel{i_2}{\longrightarrow} & \Gamma(X, \mathcal{W}_2) & \stackrel{\Psi^2}{\longrightarrow} & \Gamma(X, \mathcal{V}_2) \\ & \longrightarrow & H^1_{\text{ét}}(X, \mathbf{Z}/p^2\mathbf{Z}) & \stackrel{i_2}{\longrightarrow} & H^1_{\text{ét}}(X, \mathcal{W}_2) & \stackrel{\Psi^2}{\longrightarrow} & H^1_{\text{ét}}(X, \mathcal{V}_2)\,. \end{array} \tag{5}$$

Since the group scheme $\mathcal{W}_2$ is smooth, $H^1_{\text{ét}}(X, \mathcal{W}_2) \simeq H^1_{\text{fl}}(X, \mathcal{W}_2)$.

PROPOSITION 3.2. *Let X be an A-scheme. Then*

$$H^1_{\text{fl}}(X, \mathcal{W}_2) \simeq H^1_{\text{zar}}(X, \mathcal{W}_2)\,.$$

PROOF. The exact sequence

$$0 \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow \mathcal{W}_2 \longrightarrow \mathcal{G}^{(\lambda)} \longrightarrow 0$$

induces the cohomology long exact sequence

$$H^1_{\text{fl}}(X, \mathcal{G}^{(\lambda)}) \longrightarrow H^1_{\text{fl}}(X, \mathcal{W}_2) \longrightarrow H^1_{\text{fl}}(X, \mathcal{G}^{(\lambda)})\,.$$

Let B be an A-algebra. We assume that B is a local ring or p is a nilpoint in B. Since $H^1_{\text{fl}}(\operatorname{Spec} B, \mathcal{G}^{(\lambda)}) = 0$ (Sekiguchi-Oort-Suwa [3]),

$$H^1_{\text{fl}}(\operatorname{Spec} B, \mathcal{W}_2) = 0\,. \tag{6}$$

Let $\varphi : X_{\text{fl}} \to X_{\text{zar}}$ be a natural morphism of sites. Since $R^1\varphi_*\mathcal{W}_2 = 0$ by (6), we have

$$H^1_{\text{fl}}(X, \mathcal{W}_2) \simeq H^1_{\text{zar}}(X, \mathcal{W}_2)$$

by the Leray spectral sequence

$$H^p_{\mathrm{zar}}(X, R^q\varphi_*\mathcal{W}_2) \Longrightarrow H^{p+q}_{\mathrm{fl}}(X, \mathcal{W}_2)\,. \tag{7}$$

3.3. We describe the exact sequence (5) more concretely. Let X be an A-scheme. Now, let $\mathcal{U} = \{U_j\}$ be an affine open covering on X and $\boldsymbol{f}_{ij} = (f_{ij}, g_{ij}) \in Z^1(\mathcal{U}, \mathcal{W}_2)$ a 1-cocycle representing an element $\boldsymbol{\eta} = (\eta_0, \eta_1) \in H^1(X, \mathcal{W}_2)$ such that $\Psi^2(\boldsymbol{\eta}) = 0$. This means that $(\Psi^2(\boldsymbol{f}_{ij}))$ is a 1-coboundary, and if necessary replacing $\mathcal{U}$ a refinement, we can write

$$\Psi^2(\boldsymbol{f}_{ij}) = (\Lambda_0^G(\boldsymbol{b}_j, I^G(\boldsymbol{b}_i)), \Lambda_1^G(\boldsymbol{b}_j, I^G(\boldsymbol{b}_i))) \quad \text{on} \quad U_{ij} := U_j \cap U_i\,,$$

where $\boldsymbol{b}_j = (b_{0j}, b_{1j}) \in \Gamma(U_j, \mathcal{V}_2)$. Let $\boldsymbol{h} = (h_0, h_1) \in \Gamma(X, \mathcal{V}_2)$. We define $\pi : X' \to X$ locally by the covering

$$\Psi^2(\boldsymbol{z}_j) = \Psi^2(z_{0j}, z_{1j}) = (\Lambda_0^G(\boldsymbol{b}_j, \boldsymbol{h}), \Lambda_1^G(\boldsymbol{b}_j, \boldsymbol{h}))$$

$$\text{on} \quad U_j \times \mathbf{A}^2 = \operatorname{Spec} \Gamma(U_j, \mathcal{V}_2) \otimes_A A[\boldsymbol{z}_j]\,,$$

the gluing being given by

$$(\Lambda_0^F(\boldsymbol{z}_j, I^F(\boldsymbol{z}_i)), \Lambda_1^F(\boldsymbol{z}_j, I^F(\boldsymbol{z}_i))) = \boldsymbol{f}_{ij} \quad \text{on} \quad (U_j \times \mathbf{A}^2) \cap (U_i \times \mathbf{A}^2)\,,$$

and an action of $\mathbf{Z}/p^2\mathbf{Z}$ on X' by

$$(\boldsymbol{z}_j, s) \longmapsto (\Lambda_0^F(\boldsymbol{z}_j, i_2(s)), \Lambda_1^F(\boldsymbol{z}_j, i_2(s))) \quad \text{for} \quad s \in \mathbf{Z}/p^2\mathbf{Z}\,.$$

Then X' is a $\mathbf{Z}/p^2\mathbf{Z}$-torsor over X, and $[X'] \in H^1(X, \mathbf{Z}/p^2\mathbf{Z})$ is mapped to $\boldsymbol{\eta} = (\eta_0, \eta_1) \in H^1(X, \mathcal{W}_2)$.

3.4. Let B be an A-algebra. We assume that B is a local ring or p is a nilpoint in B. Let $X = \operatorname{Spec} B$. Then

$$H^1_{\mathrm{fl}}(X, \mathcal{W}_2) = 0\,.$$

by (6). Hence

$$\operatorname{Coker}[\Psi^2 : \mathcal{W}_2(B) \longrightarrow \mathcal{V}_2(B)] \xrightarrow{\sim} H^1(X, \mathbf{Z}/p^2\mathbf{Z})$$

is an isomorphism. Hence, for any unramified p^2-cyclic extension C of B, there exists a morphism $f : \operatorname{Spec} B \to \mathcal{V}_2$ such that

$$\begin{array}{ccc} \operatorname{Spec} C & \longrightarrow & \mathcal{W}_2 \\ \big\downarrow & & \big\downarrow{\scriptstyle \Psi^2} \\ \operatorname{Spec} B & \xrightarrow{\ f\ } & \mathcal{V}_2 \end{array}$$

is cartesian. That is to say, for any unramified p^2-cyclic extension C of B, there exists an element $(b_0, b_1) \in \Gamma(X, \mathcal{V}_2)$ such that $X' = \operatorname{Spec} B[\alpha, \beta]$, where α and β satisfy

$$\begin{aligned}\Psi^2(\alpha, \beta) &= \left(\frac{1}{\lambda^p}\{(\lambda\alpha + 1)^p - 1\}, \frac{1}{\lambda^p}\left\{\frac{(\lambda\beta + F(\alpha))^p}{\lambda\alpha + 1} - G\left(\frac{1}{\lambda^p}\{(\lambda\alpha + 1)^p - 1\}\right)\right\}\right) \\ &= (b_0, b_1)\,.\end{aligned}$$

Now, let X' and X'' be $\mathbf{Z}/p^2\mathbf{Z}$-torsors. Then, there exist elements (b_0, b_1) and $(b'_0, b'_1) \in \Gamma(X, \mathcal{V}_2)$ such that $X' = \operatorname{Spec} B[\alpha, \beta]$ and $X'' = \operatorname{Spec} B[\alpha', \beta']$, where $\Psi^2(\alpha, \beta) = (b_0, b_1)$ and $\Psi^2(\alpha', \beta') = (b'_0, b'_1)$. Then by the exact sequence (5), the following are equivalent:

(i) X' is isomorphic to X'' as $\mathbf{Z}/p^2\mathbf{Z}$-torsors over X.

(ii) $B[\alpha, \beta]$ and $B[\alpha', \beta']$ are $\mathbf{Z}/p^2\mathbf{Z}$-equivariant over B.

(iii) There is an element $(c_0, c_1) \in \Gamma(X, \mathcal{W}_2)$ such that $(\Lambda_0^G((b_0, b_1), I^G(b'_0, b'_1)), \Lambda_1^G((b_0, b_1), I^G(b'_0, b'_1)) = \Psi^2(c_0, c_1)$.

3.5. Let B be an A-algebra. We assume that B is a noetherian local ring and faithfully flat over A. And we assume that X is a connected flat proper scheme over B. We define homomorphisms $f_2 : \mathcal{W}_2 \to \mathcal{G}^{(\lambda)} \times_{\operatorname{Spec} A} \mathbf{G}_{m,A}$ and $g_2 : \mathcal{G}^{(\lambda)} \times_{\operatorname{Spec} A} \mathbf{G}_{m,A} \to \iota_* \mathbf{G}_{m,A_0}$ by

$$f_2(x_0, x_1) = (x_0, F(x_0) + \lambda x_1) \quad \text{and} \quad g_2(y, t) = \frac{t}{F(y)} \bmod \lambda\,,$$

for local sections $(x_0, x_1) \in \mathcal{W}_2,\ y \in \mathcal{G}^{(\lambda)}$ and $t \in \mathbf{G}_{m,A}$, respectively. Since X is flat over A, we can see that the sequence of sheaves on $(\operatorname{Spec} A)_{\mathrm{zar}}$, $(\operatorname{Spec} A)_{\text{ét}}$ and $(\operatorname{Spec} A)_{\mathrm{fl}}$

$$0 \longrightarrow \mathcal{W}_2 \xrightarrow{f_2} \mathcal{G}^{(\lambda)} \times_{\operatorname{Spec} A} \mathbf{G}_{m,A} \xrightarrow{g_2} \iota_* \boldsymbol{g}_{m,A_0} \longrightarrow 0 \tag{8}$$

is exact. The exact sequence (8) induces the cohomology long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(X, \mathcal{W}_2) & \xrightarrow{f_2} & \Gamma(X, \mathcal{G}^{(\lambda)} \times_X \mathbf{G}_{m,X}) & \xrightarrow{g_2} & \Gamma(X_0, \mathbf{G}_{m,X_0}) \\ & \longrightarrow & H^1_{\text{ét}}(X, \mathcal{W}_2) & \xrightarrow{f_2} & H^1_{\text{ét}}(X, \mathcal{G}^{(\lambda)} \times_X \mathbf{G}_{m,X}) & \xrightarrow{g_2} & H^1_{\text{ét}}(X_0, \mathbf{G}_{m,X_0})\,. \end{array}$$

Now, put $C = \Gamma(X, \mathcal{O}_X)$. Then C is finite over B and a semi local ring (cf. [1]). Then by assumption on X,

$$\Gamma(X, \mathcal{G}^{(\lambda)} \times_X \mathbf{G}_{m,X}) = \Gamma(X, \mathcal{G}^{(\lambda)}) \times C^*\,, \quad \Gamma(X_0, \mathbf{G}_{m,X_0}) = (C/\lambda)^*\,.$$

Since λ belongs to the Jacobson radical of C, the morphism $C^* \longrightarrow (C/\lambda)^*$ is surjective. Hence we obtain an isomorphism

$$H^1_{\text{ét}}(X, \mathcal{W}_2) \xrightarrow{\sim} \operatorname{Ker}[g_2 : H^1_{\text{ét}}(X, \mathcal{G}^{(\lambda)}) \times \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(X_0)]\,.$$

Now, the homomorphism

$$F : \mathcal{G}^{(\lambda)} \longrightarrow \iota_* \mathbf{G}_{m,A_0}$$

induces the homomorphism

$$F_* : H^1_{\text{ét}}(X, \mathcal{G}^{(\lambda)}) \longrightarrow H^1_{\text{ét}}(X, \iota_* \mathbf{G}_{m,X_0}) .$$

Then there is a homomorphism

$$ex_F : H^1_{\text{ét}}(X_0, \mathcal{G}^{(\lambda)}) \longrightarrow H^1_{\text{ét}}(X, \iota_* \mathbf{G}_{m,X_0})$$

such that the diagram

$$\begin{array}{ccc} H^1(X, \mathcal{G}^{(\lambda)}) & \xrightarrow{F_*} & H^1(X, \iota_* \mathbf{G}_{m,X_0}) \\ \downarrow & \nearrow_{ex_F} & \\ H^1(X_0, \mathcal{G}^{(\lambda)}) & & \end{array}$$

is commutative. Hence we obtain an isomorphism

$$H^1_{\text{ét}}(X, \mathcal{W}_2) \xrightarrow{\sim} \{(c, d) \in H^1(X, \mathcal{G}^{(\lambda)}) \times \mathrm{Pic}(X) | \; d \bmod \lambda = ex_F(c \bmod \lambda) \} .$$

Let $\iota_X : X_0 \to X$ be the inclusion induced by $\iota_B : \mathrm{Spec}\, B_0 \to \mathrm{Spec}\, B$. Then we obtain an isomorphism

$$\alpha^{(F)} : H^1_{\text{ét}}(X, \mathcal{W}_2) \xrightarrow{\sim} \{([\mathcal{L}_0], [\mathcal{L}_1]) \in \mathrm{Pic}(X)^2 | \; (*) \} , \tag{9}$$

where $(*)$ means the following conditions:

$$[\iota_X{}^* \mathcal{L}_0] = [\mathcal{O}_{X_0}] , \quad ex_F([\iota_X{}^* \mathcal{L}_0]) = [\iota_X{}^* \mathcal{L}_1] .$$

Using the isomorphism (9), we describe a $\mathbf{Z}/p^2\mathbf{Z}$-torsor X' over X geometrically. We assume that B is a strictly Henselian local ring. For any $\mathbf{Z}/p^2\mathbf{Z}$-torsor X', let $i_2(X') = [(f_{ij}, g_{ij})] \in H^1_{\text{ét}}(X, \mathcal{W}_2)$. We put $(\eta_0, \eta_1) = [(f_{ij}, g_{ij})]$. By the isomorphism (9), we have a one-to-one correspondence between (η_0, η_1) and $([\mathcal{L}_0], [\mathcal{L}_1]) \in \mathrm{Pic}(X)^2$ with the conditions $(*)$. Since (η_0, η_1) is the image of X', by the exact sequence (5) we have $\Psi^2(\eta_0, \eta_1) = 0$. Hence $\Theta^2(([\mathcal{L}_0], [\mathcal{L}_1])) = ([\mathcal{O}_X], [\mathcal{O}_X])$, that is $[\mathcal{L}_0{}^{\otimes p}] = [\mathcal{O}_X]$ and $[\mathcal{L}_1{}^{\otimes p}] = [\mathcal{L}_0]$. Then $[\mathcal{L}_0], [\mathcal{L}_1] \in \mathrm{Pic}^0(X)$.

Inversely, we take $([\mathcal{L}_0], [\mathcal{L}_1]) \in \mathrm{Pic}^0(X)^2$ with the conditions $(*)$, $[\mathcal{L}_0{}^{\otimes p}] = [\mathcal{O}_X]$ and $[\mathcal{L}_1{}^{\otimes p}] = [\mathcal{L}_0]$. Then by the isomorphism (9), we obtain $[(f_{ij}, g_{ij})] \in H^1_{\text{ét}}(X, \mathcal{W}_2)$ with $\alpha^{(F)}([(f_{ij}, g_{ij})]) = ([\mathcal{L}_0], [\mathcal{L}_1])$ uniquely. Now, since $\Theta^2([\mathcal{L}_0], [\mathcal{L}_1]) = ([\mathcal{O}_X], [\mathcal{O}_X])$, $(\Psi^2(f_{ij}, g_{ij}))$ is a 1-coboundary. Then we can construct a $\mathbf{Z}/p^2\mathbf{Z}$-torsor X' over X (cf. Subsection 3.3).

On the other hand, since C is a strictly Henselian local ring, $\Psi^2 : \Gamma(X, \mathcal{W}_2) \to \Gamma(X, \mathcal{V}_2)$ is surjective. Hence we obtain the following:

THEOREM 3.6. *We obtain an isomorphism*

$$\alpha^{(F)} : H^1_{\text{ét}}(X, \mathbf{Z}/p^2\mathbf{Z}) \xrightarrow{\sim} \{([\mathcal{L}_0], [\mathcal{L}_1]) \in \text{Pic}^0(X)^2 |\ (**)\ \} \tag{10}$$

where $(**)$ *means the following conditions*:

$$[\iota_X{}^*\mathcal{L}_0] = [\mathcal{O}_{X_0}]\,, \quad ex_F([\iota_X{}^*\mathcal{L}_0]) = [\iota_X{}^*\mathcal{L}_1]\,, \quad [\mathcal{L}_0{}^{\otimes p}] = [\mathcal{O}_X]\,, \quad [\mathcal{L}_1{}^{\otimes p}] = [\mathcal{L}_0]\,.$$

We assume that X is an abelian scheme over B. Let G be a smooth affine commutative group scheme over B. Then

$$\text{Ext}^1_B(X, G) \longrightarrow H^1(X, G_X)$$

is injective. Moreover, the image is the set of primitive elements of $H^1(X, G_X)$ (Serre [9]). Here, $a \in H^1(X, G_X)$ is primitive if $m^*(a) = p_1^*(a) + p_2^*(a)$, where $m : X \times_B X \to X$ is the multiplication and $p_i : X \times_B X \to X$ is the projection to the i-th factor ($i = 1, 2$). In particular,

$$\text{Ext}^1_B(X, \mathbf{G}_{m,B}) = \text{Pic}^0(X) \subset \text{Pic}(X) = H^1(X, \mathbf{G}_{m,B})\,.$$

Moreover, we have

$$\text{Ext}^1_B(X, \mathbf{Z}/p^2\mathbf{Z}) \xrightarrow{\sim} H^1(X, \mathbf{Z}/p^2\mathbf{Z})$$

by the Künneth formula. Hence we obtain the following corollary.

COROLLARY 3.7. *We obtain isomorphisms*

$$\text{Ext}^1_B(X, \mathbf{Z}/p^2\mathbf{Z}) \xrightarrow{\sim} H^1_{\text{ét}}(X, \mathbf{Z}/p^2\mathbf{Z}) \tag{11}$$

$$\xrightarrow{\sim} \{([\mathcal{L}_0], [\mathcal{L}_1]) \in \text{Pic}^0(X)^2 |\ (**)\ \} \tag{12}$$

where $(**)$ *is the conditions given in Theorem* 3.6.

REMARK 3.8. The arguments that we gave in Subsections 3.4 and 3.5 have already been given by Sekiguchi-Suwa [7].

4. Néron blow-ups

In Theorem 3.6, we saw that $\mathbf{Z}/p^2\mathbf{Z}$-torsors over X are described by line bundles over X. In general, we get the homomorphism $H^1(X, \mathbf{Z}/p^2\mathbf{Z}) \to H^1(X, \mu_{p^2})$ induced by the homomorphism $\alpha^{(F)} : \mathcal{W}_2 \to \mathbf{G}_m^2$. In this section, we shall give a $\mathbf{Z}/p^2\mathbf{Z}$-torsor X'' as successive " Néron blow-ups" starting from a μ_{p^2}-torsor X'. Note that $\alpha^{(F)} : \mathcal{W}_2 \to \mathbf{G}_m^2$ is given by a composite of Néron blow-ups (cf. Sekiguchi-Suwa [4]). Using this fact, we shall locally describe $X'' \to X'$ as a composite of Néron blow-ups.

A Néron blow-up defined over an affine group scheme was used by Waterhouse-Weisfeiler [11] to give a classification of one-dimensional affine group schemes. We extend this argument to schemes (not necessarily affine schemes).

Let A be a discrete valuation ring and K (resp. k) the fraction field (resp. the residue field) of A. We denote by π a uniformizing parameter of A.

4.1. We recall the Néron blow-up for a group scheme. For details, see [4], [11]. Let G be a flat affine group A-scheme of finite type. We denote by G_K (resp. G_k) the generic (resp. the special) fiber of G over A. We denote by $A[G]$ (resp. $K[G]$, resp. $k[G]$) the coordinate ring of G (resp. G_K, resp. G_k).

Let H be a closed k-subgroup of G_k. Let $I(H)$ be the inverse image in $A[G]$ of the defining ideal of H in $k[G]$. Then the structure of Hopf algebra on $K[G]$ induces a structure of Hopf A-algebra on the A-subalgebra $A[\pi^{-1}I(H)]$ of $K[G]$. Then

$$G^H := \operatorname{Spec} A[\pi^{-1}I(H)]$$

is a flat affine group A-scheme of finite type. The injection

$$A[G] \subset A[G^H] = A[\pi^{-1}I(H)]$$

induces an A-homomorphism $G^H \to G$. By the definition, the generic fiber $G^H_K \to G_K$ is an isomorphism. We call the group A-scheme G^H the Néron blow-up of H in G.

EXAMPLE 4.2.

(1) The Néron blow-up of $\{0\}$ in $\mathbf{G}_{a,A} := \operatorname{Spec} A[T]$:

$$\begin{aligned} \mathbf{G}_{a,A} \longleftarrow \mathbf{G}_{a,A}^{\{0\}} &= \operatorname{Spec} A[Y] \simeq \mathbf{G}_{a,A} \\ T \longmapsto \pi Y\,. \end{aligned}$$

(2) The Néron blow-up of $\{1\}$ in $\mathbf{G}_{m,A} := \operatorname{Spec} A[T, T^{-1}]$:

$$\begin{aligned} \mathbf{G}_{m,A} \longleftarrow \mathbf{G}_{m,A}^{\{1\}} &= \operatorname{Spec} A[Y, (\pi Y + 1)^{-1}] \simeq \mathcal{G}^{(\pi)} \\ T \longmapsto \pi Y + 1\,. \end{aligned}$$

Waterhouse-Weisfeiler [11] showed the following theorem.

THEOREM (Waterhouse-Weisfeiler [11], Theorem 1.4.). *Let G and G' be flat affine group A-schemes of finite type. Let $f : G' \to G$ be an A-homomorphism. If a K-homomorphism $f_K : G'_K \to G_K$ is an isomorphism, then the A-homomorphism $f : G' \to G$ is isomorphic to a composite of Néron blow-ups.*

The homomorphism $\alpha^{(F)} : \mathcal{W}_2 \to \mathbf{G}_m^2$ is defined by $(U_0, U_1) \mapsto (\lambda T_0 + 1, \lambda T_1 + F(T_0))$, and the generic fiber $\alpha_K^{(F)} : \mathcal{W}_{2,K} \to \mathbf{G}_{m,K}^2$ is an isomorphism (cf. Theorem 2.2 (3)). The homomorphism $\alpha^{(F)}$ is described using Néron blow-ups by Sekiguchi-Suwa [4].

4.3. Let X be a flat A-scheme. We denote by X_K (resp. X_k) the generic (resp. the special) fiber of X over A. For a closed subscheme Z of X, let $\mathcal{I}$ be the ideal $\mathcal{O}_X$-sheaf defining the scheme Z. Then

$$\operatorname{Spec} A[\mathcal{O}_X, \pi^{-1}\mathcal{I}]$$

is a flat A-scheme. The injection

$$\mathcal{O}_X \subset A[\mathcal{O}_X, \pi^{-1}\mathcal{I}]$$

induces an A-morphism $\operatorname{Spec} A[\mathcal{O}_X, \pi^{-1}\mathcal{I}] \to X$. By the definition, the generic fiber is an isomorphism. We denote by X^Z or $X^{\mathcal{I}}$ a flat A-scheme $\operatorname{Spec} A[\mathcal{O}_X, \pi^{-1}\mathcal{I}]$ and call it the Néron blow-up of Z in X or the Néron blow-up of $\mathcal{I}$ in X.

PROPOSITION 4.4. *Let X be a flat A-scheme. Then*

$$X^X = X \quad \textit{and} \quad X^{\emptyset} = X_K\,.$$

PROOF. Let $\mathcal{I}_0$ be the ideal $\mathcal{O}_X$-sheaf defined by the scheme X. Since $\Gamma(X, \mathcal{I}_0) = (0)$,

$$A[\mathcal{O}_X, \pi^{-1}\mathcal{I}_0] = A[\mathcal{O}_X]\,.$$

Hence

$$\begin{aligned} X^X &= \operatorname{Spec} A[\mathcal{O}_X] \\ &= X\,. \end{aligned}$$

Let $\mathcal{I}_1$ be the ideal $\mathcal{O}_X$-sheaf defined by $\emptyset$. Since $\Gamma(X, \mathcal{I}_1) = \Gamma(X, \mathcal{O}_X)$,

$$\begin{aligned} A[\mathcal{O}_X, \pi^{-1}\mathcal{I}_1] &= A[\mathcal{O}_X\,, \pi^{-1}\mathcal{O}_X] \\ &= A[\pi^{-1}\mathcal{O}_X] \\ &= K[\mathcal{O}_X]\,. \end{aligned}$$

Hence

$$\begin{aligned} X^{\emptyset} &= \operatorname{Spec} K[\mathcal{O}_X] \\ &= X_K\,. \end{aligned}$$

EXAMPLE 4.5. We consider the affine line $\mathbf{A}^1_A = \operatorname{Spec} A[T]$.

(1) We calculate the Néron blow-up of $\{0\}$ in $\mathbf{A}^1_A$. Let $\mathcal{I}_0$ be the ideal $\mathcal{O}_{\mathbf{A}^1_A}$-sheaf defined by $\{0\}$. Since $\Gamma(\mathbf{A}^1_A, \mathcal{I}_0) = (T) \subset A[T]$,

$$\begin{aligned} A[\mathcal{O}_{\mathbf{A}^1_A}, \pi^{-1}\mathcal{I}_0] &= A[A[T] + \pi^{-1}T A[T]] \\ &= A[\pi^{-1}T] \\ &\xrightarrow{\sim} A[Y]\,, \end{aligned}$$

where the morphism $A[\pi^{-1}T] \xrightarrow{\sim} A[Y]$ is defined by $T \mapsto \pi Y$. Hence

$$(\mathbf{A}^1_A)^{\{0\}} = \operatorname{Spec} A[\mathcal{O}_{\mathbf{A}^1_A}, \pi^{-1}\mathcal{I}_0]$$

$$\overset{\sim}{\leftarrow} \operatorname{Spec} A[Y]$$
$$= \mathbf{A}^1_A .$$

(2) We calculate the Néron blow-up of $V((T^N))$ in $\mathbf{A}^1_A$, where $N \in \mathbf{N}$. Let $\mathcal{I}_N$ be the ideal $\mathcal{O}_{\mathbf{A}^1_A}$-sheaf defined by $V((T^N))$. Since $\Gamma(\mathbf{A}^1_A, \mathcal{I}_N) = (T^N) \subset A[T]$,

$$\begin{aligned} A[\mathcal{O}_{\mathbf{A}^1_A}, \pi^{-1}\mathcal{I}_N] &= A[A[T] + \pi^{-1}T^N A[T]] \\ &= A[T, Y]/(T^N - \pi Y) . \end{aligned}$$

Hence

$$\begin{aligned} (\mathbf{A}^1_A)^{V((T^N))} &= \operatorname{Spec} A[\mathcal{O}_{\mathbf{A}^1_A}, \pi^{-1}\mathcal{I}_N] \\ &= \operatorname{Spec} A[T, Y]/(T^N - \pi Y) . \end{aligned}$$

EXAMPLE 4.6. We calculate the Néron blow-up of $V((T-1))$ in $X := \operatorname{Spec} A[T, T^{-1}]$. Let $\mathcal{I}$ be the ideal $\mathcal{O}_X$-sheaf defined by $V((T-1))$. Since $\Gamma(X, \mathcal{I}) = (T-1) \subset A[T, T^{-1}]$,

$$\begin{aligned} A[\mathcal{O}_X, \pi^{-1}\mathcal{I}] &= A[A[T, T^{-1}] + \pi^{-1}(T-1)A[T, T^{-1}]] \\ &= A[\pi^{-1}(T-1), T^{-1}] \\ &\overset{\sim}{\rightarrow} A[Y, (\pi Y + 1)^{-1}] , \end{aligned}$$

where the morphism $A[\pi^{-1}(T-1), T^{-1}] \overset{\sim}{\rightarrow} A[Y, (\pi Y + 1)^{-1}]$ is defined by $T \mapsto \pi Y + 1$. Hence

$$\begin{aligned} X^{V((T-1))} &= \operatorname{Spec} A[\mathcal{O}_X, \pi^{-1}\mathcal{I}] \\ &\overset{\sim}{\leftarrow} \operatorname{Spec} A[Y, (\pi Y + 1)^{-1}] . \end{aligned}$$

EXAMPLE 4.7. We consider the projective line $\mathbf{P}^1_A = \operatorname{Proj} A[T_0, T_1]$. Put

$$U_0 = \operatorname{Spec} A[T_1/T_0] = \operatorname{Spec} A[t_0] \quad \text{and} \quad U_1 = \operatorname{Spec} A[T_0/T_1] = \operatorname{Spec} A[t_1] .$$

Then $\mathbf{P}^1_A$ is given by gluing U_0 and U_1 with isomorphisms

$$\begin{aligned} U_{10} &\overset{\sim}{\longleftarrow} U_{01} \\ t_1 &\longmapsto t_0^{-1} , \end{aligned}$$

where

$$U_0 \supset U_{01} = \operatorname{Spec} A[t_0, t_0^{-1}] \quad \text{and} \quad U_1 \supset U_{10} = \operatorname{Spec} A[t_1, t_1^{-1}] .$$

(1) We calculate the Néron blow-up of $V_+((T_0, T_1))$ in $\mathbf{P}^1_A$. Let $\mathcal{I}_0$ be the ideal $\mathcal{O}_{\mathbf{P}^1_A}$-sheaf defined by $V_+((T_0, T_1))$. Then

$$\Gamma(U_0, \mathcal{I}_0) = (t_0) \subset A[t_0] \quad \text{and} \quad \Gamma(U_1, \mathcal{I}_0) = (t_1) \subset A[t_1] .$$

Now, put

$$V_0 = (\mathbf{P}^1_A)^{V_+((T_0,T_1))} \times_{\mathbf{P}^1_A} U_0 \quad \text{and} \quad V_1 = (\mathbf{P}^1_A)^{V_+((T_0,T_1))} \times_{\mathbf{P}^1_A} U_1 \,.$$

Then

$$\begin{aligned} V_0 = U_0^{V((t_0))} &= \operatorname{Spec} A[t_0/\pi] \\ &\overset{\sim}{\leftarrow} \operatorname{Spec} A[s_0] \\ &= \mathbf{A}^1_A \end{aligned}$$

and

$$\begin{aligned} V_1 = U_1^{V((t_1))} &= \operatorname{Spec} A[t_1/\pi] \\ &\overset{\sim}{\leftarrow} \operatorname{Spec} A[s_1] \\ &= \mathbf{A}^1_A \,, \end{aligned}$$

where the morphism $\operatorname{Spec} A[t_0/\pi] \overset{\sim}{\leftarrow} \operatorname{Spec} A[s_0]$ is defined by $t_0 \mapsto \pi s_0$ and the morphism $\operatorname{Spec} A[t_1/\pi] \overset{\sim}{\leftarrow} \operatorname{Spec} A[s_1]$ is defined by $t_1 \mapsto \pi s_1$. Now, put

$$V_{01} = V_0 \times_{\mathbf{P}^1_A} (U_0 \times_{\mathbf{P}^1_A} U_1) \quad \text{and} \quad V_{10} = V_1 \times_{\mathbf{P}^1_A} (U_0 \times_{\mathbf{P}^1_A} U_1) \,.$$

Then

$$\begin{aligned} V_{01} = U_{01}^{V((t_0))} &= \operatorname{Spec} A[A[t_0, t_0^{-1}] + (t_0/\pi) A[t_0, t_0^{-1}]] \\ &= \operatorname{Spec} A[t_0/\pi, t_0^{-1}] \\ &\overset{\sim}{\leftarrow} \operatorname{Spec} A[s_0, (\pi s_0)^{-1}] \\ &= \operatorname{Spec} K[s_0, s_0^{-1}] \,, \end{aligned}$$

where the morphism $\operatorname{Spec} A[t_0/\pi, t_0^{-1}] \overset{\sim}{\leftarrow} \operatorname{Spec} K[s_0, s_0^{-1}]$ is defined by $t_0 \mapsto \pi s_0$. Similarly

$$\begin{aligned} V_{10} = U_{10}^{V((t_1))} &= \operatorname{Spec} A[t_1/\pi, t_1^{-1}] \\ &\overset{\sim}{\leftarrow} \operatorname{Spec} K[s_1, s_1^{-1}] \,, \end{aligned}$$

where the morphism $\operatorname{Spec} A[t_1/\pi, t_1^{-1}] \overset{\sim}{\leftarrow} \operatorname{Spec} K[s_1, s_1^{-1}]$ is defined by $t_1 \mapsto \pi s_1$. Hence $(\mathbf{P}^1_A)^{V_+((T_0,T_1))}$ is obtained by gluing $V_0 \simeq \mathbf{A}^1_A$ and $V_1 \simeq \mathbf{A}^1_A$ with isomorphisms

$$\begin{aligned} V_{10} &\overset{\sim}{\longleftarrow} V_{01} \\ s_1 &\longmapsto (\pi^2 s_0)^{-1} \,. \end{aligned}$$

Now, we give the special fiber of $(\mathbf{P}^1_A)^{V_+((T_0,T_1))}$. We have $V_0 \otimes_A k \simeq \mathbf{A}^1_k$ and $V_1 \otimes_A k \simeq \mathbf{A}^1_k$. Moreover, $V_{01} \otimes_A k = \operatorname{Spec} A[t_0/\pi, t_0^{-1}] \otimes k = \emptyset$. Similarly, $V_{10} \otimes_A k = \emptyset$. Hence we have $(\mathbf{P}^1_A)^{V_+((T_0,T_1))} \otimes_A k = \mathbf{A}^1_k \coprod \mathbf{A}^1_k$.

(2) We calculate the Néron blow-up of $V_+((T_0))$ in $\mathbf{P}^1_A$. Let $\mathcal{I}_1$ be the ideal $\mathcal{O}_{\mathbf{P}^1_A}$-sheaf defined by $V_+((T_0))$. Then

$$\Gamma(U_0, \mathcal{I}_1) = (1) = A[t_0] \quad \text{and} \quad \Gamma(U_1, \mathcal{I}_1) = (t_1) \subset A[t_1]\,.$$

Now, put

$$V_0 = (\mathbf{P}^1_A)^{V_+((T_0))} \times_{\mathbf{P}^1_A} U_0 \quad \text{and} \quad V_1 = (\mathbf{P}^1_A)^{V_+((T_0))} \times_{\mathbf{P}^1_A} U_1\,.$$

Then

$$\begin{aligned} V_0 = U_0^{V((1))} &= U_0^{\emptyset} \\ &= \operatorname{Spec} K[t_0] \\ &= \mathbf{A}^1_K \end{aligned}$$

and

$$\begin{aligned} V_1 = U_1^{V((t_1))} &= \operatorname{Spec} A[t_1/\pi] \\ &\overset{\sim}{\leftarrow} \operatorname{Spec} A[s_1] \\ &= \mathbf{A}^1_A\,, \end{aligned}$$

where the morphism $\operatorname{Spec} A[t_1/\pi] \overset{\sim}{\leftarrow} \operatorname{Spec} A[s_1]$ is defined by $t_1 \mapsto \pi s_1$. Now, put

$$V_{01} = V_0 \times_{\mathbf{P}^1_A} (U_0 \times_{\mathbf{P}^1_A} U_1) \quad \text{and} \quad V_{10} = V_1 \times_{\mathbf{P}^1_A} (U_0 \times_{\mathbf{P}^1_A} U_1)\,.$$

Then

$$V_{01} = U_{01}^{\emptyset} = \operatorname{Spec} K[t_0, t_0^{-1}]$$

and

$$\begin{aligned} V_{10} = U_{10}^{V((t_1))} &= \operatorname{Spec} A[t_1/\pi, t_1^{-1}] \\ &\overset{\sim}{\leftarrow} \operatorname{Spec} K[s_1, s_1^{-1}]\,, \end{aligned}$$

where the morphism $\operatorname{Spec} A[t_1/\pi, t_1^{-1}] \overset{\sim}{\leftarrow} \operatorname{Spec} K[s_1, s_1^{-1}]$ is defined by $t_1 \mapsto \pi s_1$. Hence $(\mathbf{P}^1_A)^{V_+((T_0))}$ is obtained by gluing $V_0 \simeq \mathbf{A}^1_K$ and $V_1 \simeq \mathbf{A}^1_A$ with isomorphisms

$$\begin{aligned} V_{10} &\overset{\sim}{\longleftarrow} V_{01} \\ s_1 &\longmapsto (\pi t_0)^{-1}\,. \end{aligned}$$

Now, we give the special fiber of $(\mathbf{P}^1_A)^{V_+((T_0))}$. We have $V_0 \otimes_A k = \emptyset$ and $V_1 \otimes_A k \simeq \mathbf{A}^1_k$. Hence we have $(\mathbf{P}^1_A)^{V_+((T_0,T_1))} \otimes_A k = \mathbf{A}^1_k$.

4.8. We consider the Kummer-Artin-Schreier Theory (cf. [3], [6]). In this subsection, let $\lambda = \zeta - 1$ and $A = \mathbf{Z}_{(p)}[\zeta]$. Then A is a discrete valuation ring and λ is a uniformizing parameter of A. K (resp. k) denotes the fraction field (resp. the residue field) of A. Then $K = \mathbf{Q}(\zeta)$ and $k = \mathbf{F}_p$. Put

$$\Lambda^F(X, Y) = \lambda XY + X + Y\,,$$

$$\Lambda^G(X, Y) = \lambda^p XY + X + Y\,,$$

$$\Psi(T) = \frac{1}{\lambda^p}\{(\lambda T + 1)^p - 1\}\,.$$

Let X be a flat A-scheme and $\mathcal{U} = \{U_j\}$ an affine open covering on X. Put $U_j = \operatorname{Spec} A_j$. Let X'' be a $\mathbf{Z}/p\mathbf{Z}$-torsor over X. Then X'' is locally written by

$$V_j := X'' \times_X U_j = \operatorname{Spec} A_j[Y_j, (\lambda Y_j + 1)^{-1}]/(\Psi(Y_j) - c_j)\,,$$

where $c_j \in \mathcal{G}^{(\lambda^p)}(A_j)$. X'' is given by gluing V_j with isomorphisms

$$\begin{aligned} V_j \times_X U_{ij} &\xleftarrow{\sim} V_i \times_X U_{ij} \\ Y_j &\longmapsto \Lambda^F(g_{ij}, Y_i)\,, \end{aligned}$$

where $U_{ij} = U_j \times_X U_i$ and $g_{ij} \in \Gamma(U_{ij}, \mathcal{G}^{(\lambda)})$. For any j, put $b_j = \lambda^p c_j + 1$. Then $b_j \in A_j^\times$. We define the scheme X' locally by

$$U'_j := X' \times_X U_j = \operatorname{Spec} A_j[T_j, T_j^{-1}]/(T_j^p - b_j)\,,$$

the gluing being given by isomorphisms

$$\begin{aligned} U'_j \times_X U_{ij} &\xleftarrow{\sim} U'_i \times_X U_{ij} \\ T_j &\longmapsto f_{ij} T_i\,, \end{aligned}$$

where $f_{ij} \in \Gamma(U_{ij}, \mathbf{G}_m)$. Then X' is a μ_p-torsor over X.

Now, we describe the morphism $X'' \to X'$ using a Néron blow-up. We define the subscheme Z_j of U'_j by $V((T_j - 1))$. Then

$$\begin{aligned} V_j &= (U'_j)^{Z_j} \\ &= \operatorname{Spec} A_j[Y_j, (\lambda Y_j + 1)^{-1}]/(\lambda^p \Psi(Y_j) - (b_j - 1))\,. \end{aligned}$$

Here the morphism $\tilde{f}_j : V_j \to U'_j$ is defined by $T_j \mapsto \lambda Y_j + 1$. The scheme X'' is given by gluing V_j with isomorphisms

$$\begin{aligned} V_j \times_X U_{ij} &\xleftarrow{\sim} V_i \times_X U_{ij} \\ Y_j &\longmapsto \Lambda^F(g_{ij}, Y_i)\,, \end{aligned}$$

where

$$g_{ij} = (f_{ij} - 1)/\lambda\,.$$

Therefore we obtain the morphism

$$\tilde{f} : X'' \longrightarrow X' .$$

4.9. Hereafter we use the notations in section 2. We can write $\lambda = u_\lambda \lambda_2^p$, where $u_\lambda \in A^\times$. Put

$$\widehat{F}_k(T) = \sum_{j=0}^{k} \frac{(\eta u_\lambda^{-1} T)^j}{j!} \quad \text{and} \quad \widehat{G}_k(T) = \sum_{j=0}^{k} \frac{(\tilde{\eta} u_\lambda^{-p} T)^j}{j!}$$

for $k = 0, 1, \ldots, p-1$. Put

$$\widehat{F}'_k(T) = \widehat{F}_k(T) - \widehat{F}_{k-1}(T) \quad \text{and} \quad \widehat{G}'_k(T) = \widehat{G}_k(T) - \widehat{G}_{k-1}(T) .$$

Put

$$\Lambda_0^{(k)}(X_0, Y_0) = \lambda_2^k X_0 Y_0 + X_0 + Y_0 ,$$

$$\Lambda_1^{(k)}(X_0, X_1, Y_0, Y_1) = \lambda_2^k X_1 Y_1 + X_1 \widehat{F}_{k-1}(Y_0) + \widehat{F}_{k-1}(X_0) Y_1 + \frac{1}{\lambda_2^k}\{\widehat{F}_{k-1}(X_0)\widehat{F}_{k-1}(Y_0) - \widehat{F}_{k-1}(\Lambda_0^{(p)}(X_0, Y_0))\} ,$$

$$\Psi_0^{(k)}(X) = \frac{1}{\lambda_2^{kp}}\{(\lambda_2^k X + 1)^p - 1\} ,$$

$$\Psi_1^{(k)}(X_0, X_1) = \frac{1}{\lambda_2^{kp}}\left\{\frac{(\lambda_2^k X_1 + \widehat{F}_{k-1}(X_0))^p}{\lambda_2^p X_0 + 1} - \widehat{G}_{k-1}(\Psi_0^{(p)}(X_0))\right\} ,$$

$$\Phi_1^{(k)}(X_0, X_1) = \frac{X_1^p}{\lambda_2^k X_0 + 1} ,$$

for $k = 1, 2, \ldots, p$.

Let X be a flat A-scheme and $\mathcal{U} = \{U_j\}$ an affine open covering on X. Put $U_j = \operatorname{Spec} A_j$. Let X'' be a $\mathbf{Z}/p^2\mathbf{Z}$-torsor over X. Then X'' is locally written by

$$\begin{aligned} V_j &:= X'' \times_X U_j \\ &= \operatorname{Spec} A_j[Y_{0j}, Y_{1j}, (\lambda Y_{0j} + 1)^{-1}, (\lambda Y_{1j} + F(Y_{0j}))^{-1}] \\ &\quad /(\Psi_0(Y_{0j}) - c_{0j} , \Psi_1(Y_{0j}, Y_{1j}) - c_{1j}) , \end{aligned}$$

where $(c_{0j}, c_{1j}) \in \mathcal{V}_2(A_j)$. X'' is given by gluing V_j with isomorphisms

$$V_j \times_X U_{ij} \overset{\sim}{\longleftarrow} V_i \times_X U_{ij}$$

$$(Y_{0j}, Y_{1j}) \longmapsto (\Lambda_0^F(f'_{ij}, Y_{0i}) , \Lambda_1^F(f'_{ij}, g'_{ij}, Y_{0i}, Y_{1i}))$$

where $U_{ij} = U_j \times_X U_i$ and $(f'_{ij}, g'_{ij}) \in \Gamma(U_{ij}, \mathcal{W}_2)$. For any j, put $(b_{0j}, b_{1j}) = \alpha^{(G)}(c_{0j}, c_{1j}) = (\lambda^p c_{0j} + 1, \lambda^p c_{1j} + G(c_{0j}))$. Then $(b_{0j}, b_{1j}) \in (A_j^\times)^2$. We define the scheme X' locally by

$$U'_j := X' \times_X U_j = \operatorname{Spec} A_j[T_{0j}, T_{1j}, T_{0j}^{-1}, T_{1j}^{-1}]/(T_{0j}^p - b_{0j}, T_{0j}^{-1}T_{1j}^p - b_{1j}),$$

the gluing being given by isomorphisms

$$\begin{aligned} U'_j \times_X U_{ij} &\xleftarrow{\sim} U'_i \times_X U_{ij} \\ (T_{0j}, T_{1j}) &\longmapsto (f_{ij}T_{0i}, g_{ij}T_{1i}), \end{aligned}$$

where $(f_{ij}, g_{ij}) \in \Gamma(U_{ij}, \mathbf{G}_m^2)$. Then X' is a μ_{p^2}-torsor over X.

Now, we describe the morphism $X'' \to X'$ using Néron blow-ups.

(1) **Step 1.** We define the subscheme $Z_j^{(1,0)}$ of U'_j by $V((T_{0j}-1))$. Put $V_j^{(1,0)} = (U'_j)^{Z_j^{(1,0)}}$. Then

$$V_j^{(1,0)} = \operatorname{Spec} A_j[Y_{0j}, Y_{1j}, (\lambda_2 Y_{0j}+1)^{-1}, Y_{1j}^{-1}]/(\psi_0^{(1,0)}(Y_{0j}), \psi_1^{(1,0)}(Y_{0j}, Y_{1j})),$$

where

$$\begin{aligned} \psi_0^{(1,0)}(Y_{0j}) &= \lambda_2^p \Psi_0^{(1)}(Y_{0j}) - (b_{0j}-1) \quad \text{and} \\ \psi_1^{(1,0)}(Y_{0j}, Y_{1j}) &= \Phi_1^{(1)}(Y_{0j}, Y_{1j}) - b_{1j}. \end{aligned}$$

Here the morphism $\tilde{f}_j^{(1,0)} : V_j^{(1,0)} \to U'_j$ is defined by $(T_{0j}, T_{1j}) \mapsto (\lambda_2 Y_{0j}+1, Y_{1j})$. The scheme $X'^{(1,0)}$ is given by gluing $V_j^{(1,0)}$ with isomorphisms

$$\begin{aligned} V_j^{(1,0)} \times_X U_{ij} &\xleftarrow{\sim} V_i^{(1,0)} \times_X U_{ij} \\ (Y_{0j}, Y_{1j}) &\longmapsto (\Lambda_0^{(1)}(f_{ij}^{(1,0)}, Y_{0i}), g_{ij}^{(1,0)}Y_{1i}), \end{aligned}$$

where

$$(f_{ij}^{(1,0)}, g_{ij}^{(1,0)}) = ((f_{ij}-1)/\lambda_2, g_{ij}).$$

Therefore we obtain the morphism

$$\tilde{f}^{(1,0)} : X'^{(1,0)} \longrightarrow X'.$$

(2) **Step 2.** For any $2 \le k \le p$, we define the subscheme $Z_j^{(k,0)}$ of $V_j^{(k-1,0)}$ by $V((T_{0j}))$. Put $V_j^{(k,0)} = (V_j^{(k-1,0)})^{Z_j^{(k,0)}}$. Then

$$V_j^{(k,0)} = \operatorname{Spec} A_j[Y_{0j}, Y_{1j}, (\lambda_2^k Y_{0j}+1)^{-1}, Y_{1j}^{-1}]/(\psi_0^{(k,0)}(Y_{0j}), \psi_1^{(k,0)}(Y_{0j}, Y_{1j})),$$

where

$$\psi_0^{(k,0)}(Y_{0j}) = \lambda_2^{kp} \Psi_0^{(k)}(Y_{0j}) - (b_{0j}-1) \quad \text{and}$$

$$\psi_1^{(k,0)}(Y_{0j}, Y_{1j}) = \Phi_1^{(k)}(Y_{0j}, Y_{1j}) - b_{1j}\,.$$

Here the morphism $\tilde{f}_j^{(k,0)} : V_j^{(k,0)} \to V_j^{(k-1,0)}$ is defined by $(T_{0j}, T_{1j}) \mapsto (\lambda_2 Y_{0j}, Y_{1j})$. The scheme $X'^{(k,0)}$ is given by gluing $V_j^{(k,0)}$ with isomorphisms

$$\begin{aligned} V_j^{(k,0)} \times_X U_{ij} &\overset{\sim}{\longleftarrow} V_i^{(k,0)} \times_X U_{ij} \\ (Y_{0j}, Y_{1j}) &\longmapsto (\Lambda_0^{(k)}(f_{ij}^{(k,0)}, Y_{0i}), g_{ij}^{(k,0)} Y_{1i})\,, \end{aligned}$$

where

$$(f_{ij}^{(k,0)}, g_{ij}^{(k,0)}) = (f_{ij}^{(k-1,0)}/\lambda_2, g_{ij}^{(k-1,0)})\,.$$

Therefore we obtain the morphism

$$\tilde{f}^{(k,0)} : X'^{(k,0)} \longrightarrow X'^{(k-1,0)}\,.$$

(3) **Step 3.** We define the subscheme $Z_j^{(p,1)}$ of $V_j^{(p,0)}$ by $V((T_{1j} - 1))$. Put $V_j^{(p,1)} = (V_j^{(p,0)})^{Z_j^{(p,1)}}$. Then

$$V_j^{(p,1)} = \operatorname{Spec} A_j[Y_{0j}, Y_{1j}, (\lambda_2^p Y_{0j} + 1)^{-1}, (\lambda_2 Y_{1j} + 1)^{-1}]/(\psi_0^{(p,1)}(Y_{0j}), \psi_1^{(p,1)}(Y_{0j}, Y_{1j}))\,,$$

where

$$\begin{aligned} \psi_0^{(p,1)}(Y_{0j}) &= \lambda_2^{p^2} \Psi_0^{(p)}(Y_{0j}) - (b_{0j} - 1) \quad \text{and} \\ \psi_1^{(p,1)}(Y_{0j}, Y_{1j}) &= \lambda_2^p \Psi_1^{(1)}(Y_{0j}, Y_{1j}) - (b_{1j} - 1)\,. \end{aligned}$$

Here the morphism $\tilde{f}_j^{(p,1)} : V_j^{(p,1)} \to V_j^{(p,0)}$ is defined by $(T_{0j}, T_{1j}) \mapsto (Y_{0j}, \lambda_2 Y_{1j} + 1)$. The scheme $X'^{(p,1)}$ is given by gluing $V_j^{(p,1)}$ with isomorphisms

$$\begin{aligned} V_j^{(p,1)} \times_X U_{ij} &\overset{\sim}{\longleftarrow} V_i^{(p,1)} \times_X U_{ij} \\ (Y_{0j}, Y_{1j}) &\longmapsto (\Lambda_0^{(p)}(f_{ij}^{(p,1)}, Y_{0i}), \Lambda_0^{(1)}(g_{ij}^{(p,1)}, Y_{1i}))\,, \end{aligned}$$

where

$$(f_{ij}^{(p,1)}, g_{ij}^{(p,1)}) = (f_{ij}^{(p,0)}, (g_{ij}^{(p,0)} - 1)/\lambda_2)\,.$$

Therefore we obtain the morphism

$$\tilde{f}^{(p,1)} : X'^{(p,1)} \longrightarrow X'^{(p,0)}\,.$$

(4) **Step 4.** For any $2 \le k \le p$, we define the subscheme $Z_j^{(p,k)}$ of $V_j^{(p,k-1)}$ by $V((T_{1j} - \widehat{F}'_{k-1}(T_{0j})/\lambda_2^{k-1}))$. Put $V_j^{(p,k)} = (V_j^{(p,k-1)})^{Z_j^{(p,k)}}$. Then

$$V_j^{(p,k)} = \mathrm{Spec}\, A_j[Y_{0j}, Y_{1j}, (\lambda_2^p Y_{0j}+1)^{-1}, (\lambda_2^k Y_{1j} + \widehat{F}_{k-1}(Y_{0j}))^{-1}] \\ /(\psi_0^{(p,k)}(Y_{0j}), \psi_1^{(p,k)}(Y_{0j}, Y_{1j}))\,,$$

where

$$\psi_0^{(p,k)}(Y_{0j}) = \lambda_2^{p^2}\Psi_0^{(p)}(Y_{0j}) - (b_{0j}-1) \quad \text{and}$$
$$\psi_1^{(p,k)}(Y_{0j}, Y_{1j}) = \lambda_2^{kp}\Psi_1^{(k)}(Y_{0j}, Y_{1j}) - (b_{1j} - \widehat{G}_{k-1}(\Psi_0^{(p)}(Y_{0j})))\,.$$

Here the morphism $\tilde{f}_j^{(p,k)} : V_j^{(p,k)} \to V_j^{(p,k-1)}$ is defined by $(T_{0j}, T_{1j}) \mapsto (Y_{0j}, \lambda_2 Y_{1j} + \widehat{F}'_{k-1}(Y_{0j})/\lambda_2^{k-1})$. The scheme $X'^{(p,k)}$ is given by gluing $V_j^{(p,k)}$ with isomorphisms

$$V_j^{(p,k)} \times_X U_{ij} \overset{\sim}{\longleftarrow} V_i^{(p,k)} \times_X U_{ij}$$
$$(Y_{0j}, Y_{1j}) \longmapsto (\Lambda_0^{(p)}(f_{ij}^{(p,k)}, Y_{0i}), \Lambda_1^{(k)}(f_{ij}^{(p,k)}, g_{ij}^{(p,k)}, Y_{0i}, Y_{1i}))\,,$$

where

$$(f_{ij}^{(p,k)}, g_{ij}^{(p,k)}) = (f_{ij}^{(p,k-1)}, g_{ij}^{(p,k-1)}/\lambda_2 - \widehat{F}'_{k-1}(f_{ij}^{(p,k-1)})/\lambda_2^k)\,.$$

Therefore we obtain the morphism

$$\tilde{f}^{(p,k)} : X'^{(p,k)} \longrightarrow X'^{(p,k-1)}\,.$$

We define the morphism $\tilde{f}' : X'' \to X'^{(p,p)}$ locally by

$$\mathrm{Spec}\, A_j[T_{0j}, T_{1j}, (\lambda_2^p T_{0j}+1)^{-1}, (\lambda_2^p T_{1j} + \widehat{F}_{p-1}(T_{0j}))^{-1}]/(\psi_0^{(p,p)}(T_{0j}), \psi_1^{(p,p)}(T_{0j}, T_{1j}))$$
$$\longleftarrow \mathrm{Spec}\, A_j[Y_{0j}, Y_{1j}, (\lambda Y_{0j}+1)^{-1}, (\lambda Y_{1j} + F(Y_{0j}))^{-1}]$$
$$/(\Psi_0(Y_{0j}) - c_{0j}, \Psi_1(Y_{0j}, Y_{1j}) - c_{1j})$$
$$(T_{0j}, T_{1j}) \longmapsto (u_\lambda Y_{0j}, u_\lambda Y_{1j})\,.$$

Summing up the above argument, we obtain the following theorem.

THEOREM 4.10. *Under the above notations, we obtain the morphism $X'' \to X'$ as the following*:

$$X'' \xrightarrow{\tilde{f}'} X'^{(p,p)} \xrightarrow{\tilde{f}^{(p,p)}} \cdots \xrightarrow{\tilde{f}^{(p,2)}} X'^{(p,1)} \xrightarrow{\tilde{f}^{(p,1)}} X'^{(p,0)} \xrightarrow{\tilde{f}^{(p,0)}} \cdots \xrightarrow{\tilde{f}^{(2,0)}} X'^{(1,0)} \xrightarrow{\tilde{f}^{(1,0)}} X'\,.$$

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