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Local Orbit Types of S-representations of Symmetric R-spaces

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Dedicated to Professor Katuhiro SHIOHAMA on his sixtieth birthday

E. Heintze and C. Olmos [HeiOlm] have investigated the local orbit types of s-representation of semisimple symmetric spaces in terms of restricted root systems. Their results have been generalized by H. Tamaru [Tama2]. But, as far as the author knows, there are no complete lists of all orbit types of s-representations of Riemannian symmetric spaces. The main purpose of this paper is to obtain a complete list of all local orbit types of s-representations of the following "symmetric R-spaces"; (i) the classical types of the rank 2: $T \cdot AI_2$, $T \cdot AII_2$, $AIII_2$, BDI_2, CI_2, CII_2, $CII_2 = Gr_2(\mathbf{H}^4)$, $DIII_2$, (ii) the classical types of the rank 3: $T \cdot AI_3$, $T \cdot AII_3$, $AIII_3$, BDI_3, CI_3, CII_3, $DIII_3$, (iii) the exceptional types: $EIII$, EIV, $EVII$, $FII = P^2(\mathbf{O})$; $G = G_2/SO(4)$ (as a normal space), (iv) the classical groups of the rank 2: $SO(4)$, $SO(5)$, $U(3)$, $Sp(2)$, (v) the classical groups of the rank 3: $SO(6)$, $SO(7)$, $U(4)$, $Sp(3)$, (vi) the real quadrics: $S^p \cdot S^q$ $(p \le q)$, which is our main results (see Section 3). For a compact semisimple symmetric space, we get the result stated in Section 2 as follows;

THEOREM 0.1 (Criterion theorem 2.6 in Section 2). *Any two orbits of a compact semisimple symmetric space are locally diffeomorphic if and only if their closed subsystems in the restricted root system are conjugate.*

COROLLARY 0.2 (Corollary 2.8 in Section 2). *The number of the local orbit types of s-representations of a compact semisimple symmetric space is less than or equal to 2^r, where r is the rank of the symmetric space.*

Let $M = G/K$ be a compact semisimple symmetric space, where G is the identity component of the isometry group. Let H, H' be two points in the tangent space T_oM to M at the origin $o \in M$, and let K_H and $K_{H'}$ be the isotropy subgroups of K (identified with the linear isotropy group) at H and H', respectively. We denote by $\mathfrak{k}_H$ and $\mathfrak{k}_{H'}$ the Lie algebras of K_H and $K_{H'}$, respectively. We say that two orbits $K(H) = K/K_H$ and $K(H') = K/K_{H'}$ are of the *same orbit type* if K_H is conjugate to $K_{H'}$ in K under the automorphism group of K. Thus, we say that two orbits $K(H) = K/K_H$ and $K(H') = K/K_{H'}$ are of the *same local orbit type* if $\mathfrak{k}_H$ is conjugate to $\mathfrak{k}_{H'}$ in $\mathfrak{k}$ under the automorphism group of $\mathfrak{k}$. We say that

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the conjugate class $[\mathfrak{k}_H]$ is a *local orbit type* of the K-orbit $K(H)$ in T_oM. For a compact semisimple symmetric space $M = G/K$, Tamaru [Tama2] found a method to determine the Lie algebras of the isotropy subgroups of K for each K-orbit in T_oM. We apply Tamaru's method to each symmetric R-space in Section 3 to get the main results.

The organization of this paper is as follows. In Section 1, we give preliminaries for the restricted root systems Δ of a compact semisimple symmetric space $M = G/K$ (cf. [Hel]). And we describe the Tamaru's method. In Section 2, we give the proof of Theorem 0.1 (Theorem 2.6), and mention that representatives of any K-orbit can be choosen to be a sum of elements of the positive Weyl chamber (Theorem 2.7 due to Tamaru [Tama1]). Finally, we give the proof of Corollary 0.2 (Corollary 2.8). In Subsection 2.3, we give an "algorithm" to get the isotropy subalgebras of any K-orbit in T_oM, which is used throughout Section 3. This paper is a part of the author's master thesis, Sophia University.

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1. Preliminaries

Let (G, K) be a compact semisimple symmetric pair. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. There exists the involutive automorphism σ of $\mathfrak{g}$ such that $\mathfrak{k}$ is the $(+1)$-eigenspace of σ. Let $\mathfrak{m}$ be the (-1)-eigenspace of σ. Then we have the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{m}$, which is uniquely determined up to conjugacy. We call $\dim \mathfrak{a}$ the *rank* of the symmetric space $M := G/K$. Let α be a linear form on $\mathfrak{a}$, and $\mathfrak{g}(\alpha) := \{X \in \mathfrak{g} \mid [H, [H, X]] = -\alpha(H)^2 X \text{ for any } H \in \mathfrak{a}\}$. A non-zero linear form α is said to be a *root*, if $\mathfrak{g}(\alpha) \neq 0$. Remark that $\mathfrak{g}(\alpha) = \mathfrak{g}(-\alpha)$. Let $\mathfrak{k}(\alpha) := \mathfrak{k} \cap \mathfrak{g}(\alpha)$ and $\mathfrak{m}(\alpha) := \mathfrak{m} \cap \mathfrak{g}(\alpha)$. Observe that $\mathfrak{m}(0) = \mathfrak{a}$. The subalgebra $\mathfrak{k}(0)$ is called the *pricipal isotropy subalgebra* of $\mathfrak{k}$. In fact, $\mathfrak{k}(0)$ coincides with the Lie algebra of the principal isotropy subgroup of s-representation. We call $\dim \mathfrak{k}(\alpha)$ the *multiplicity* of a root α. The set of all roots with the multiplicities is called the *restricted root system* of the symmetric space. The *restricted Dynkin diagram* is given by the inner product of elements of the restricted root system. Let Δ denote the restricted root system of the compact irreducible symmetric pair $(\mathfrak{g}, \mathfrak{k})$ with respect to $\mathfrak{a}$. Then we have the following decompositions, which are called the *root space decompositions*: $\mathfrak{k} = \mathfrak{k}(0) \oplus \sum_{\alpha \in \Delta} \mathfrak{k}(\alpha)$, and $\mathfrak{m} = \mathfrak{m}(0) \oplus \sum_{\alpha \in \Delta} \mathfrak{m}(\alpha)$. For each pair $(\mathfrak{g}, \mathfrak{k})$, Tamaru gave the table of corresponding Dinkin diagram (with multiplicities) and the principal isotropy subalgebra $\mathfrak{k}(0)$ of $\mathfrak{k}$. See Table 1 of [Tama2]. A subset Δ' in a root system Δ is called a *closed subsystem*, if the following two properties hold: (i) if $\alpha, \beta \in \Delta'$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta'$, (ii) $\Delta' = -\Delta'$. We can easily see that every closed subsystem is a root system. Furthermore, it is the restricted root system of certain symmetric space. We denote $\mathfrak{k}(0)$ the principal isotropy subalgebra.

THEOREM 1.1 (Tamaru [Tama2]). *Let Δ' be a closed subsystem. Then*, (i) *there exists a compact semisimple symmetric pair* $(\mathfrak{g}', \mathfrak{k}')$ *whose restricted root system is* Δ', *and* (ii) *the principal isotropy subalgebra of the pair* $(\mathfrak{g}', \mathfrak{k}')$ *is an ideal of the principal isotropy subalgebra* $\mathfrak{k}(0)$ *of the pair* $(\mathfrak{g}, \mathfrak{k})$.

Tamaru [Tama2] considered the following subalgebra of the closed subsystem in the restricred root system of the pair $(\mathfrak{g}, \mathfrak{k})$; For a closed subsystem Δ' in Δ, we call $\mathfrak{k}(\Delta') := \mathfrak{k}(0) \oplus \sum_{\alpha \in \Delta'} \mathfrak{k}(\alpha)$ the Δ'-*subalgebra*. By the above theorem, there exists the compact semisimple symmetric pair $(\mathfrak{g}', \mathfrak{k}')$ whose restricted root system is Δ'. Let $\mathfrak{k}'(0)$ denote the principal isotropy subalgebra of that pair $(\mathfrak{g}', \mathfrak{k}')$.

THEOREM 1.2 (Tamaru [Tama2]). $\mathfrak{k}(\Delta') = (\mathfrak{k}(0)/\mathfrak{k}'(0)) \oplus \mathfrak{k}'$, *where* $(\mathfrak{k}(0)/\mathfrak{k}'(0))$ *means the orthogonal complement of* $\mathfrak{k}'(0)$ *in* $\mathfrak{k}(0)$.

2. Representatives of K-orbit

2.1. The isotropy subalgebra of S-representation. Let $(\mathfrak{g}, \mathfrak{k})$ be a compact semisimple symmetric pair, and let Δ be the restricted root system of the pair $(\mathfrak{g}, \mathfrak{k})$ with respect to $\mathfrak{a}$.

PROPOSITION 2.1. *For given $H \in \mathfrak{a}$, let $\Delta_H := \{\alpha \in \Delta \mid \alpha(H) = 0\}$, and $\mathfrak{k}_H = \{X \in \mathfrak{k} \mid [X, H] = 0\}$. Then*, (i) *$\Delta_H$ is the closed subsystem*, *and* (ii) *$\mathfrak{k}_H = \mathfrak{k}(\Delta_H)$ is the isotropy subalgebra of the s-representation at $H \in \mathfrak{a}$, i.e.*, *the Δ_H-subalgebra.*

PROOF. By the definition, $\Delta_H := \{\alpha \in \Delta \mid \alpha(H) = 0\}$ is the closed subsystem of Δ. On the other hand, by the definition of the root, we have the following relation such that $\mathfrak{k}_H = \mathfrak{k}(0) \oplus \sum_{\alpha(H) \in \Delta_H} \mathfrak{k}(\alpha)$. □

We define $\mathfrak{g}^H$ and $\mathfrak{k}^H$ corresponding to the compact semisimple symmetric pair $(\mathfrak{g}', \mathfrak{k}')$ in Theorem 1.1 as follows. Let $\mathfrak{k}^H(0) := \sum_{\alpha \in \Delta_H} [\mathfrak{k}(\alpha), \mathfrak{k}(\alpha)]_{\mathfrak{k}(0)}$, where the subscript $\mathfrak{k}(0)$ denote the $\mathfrak{k}(0)$-component, and let $\mathfrak{a}^H$ be the subspace in $\mathfrak{a}$ spaned by Δ_H, $\mathfrak{k}^H := \mathfrak{k}^H(0) \oplus \sum_{\alpha \in \Delta_H} \mathfrak{k}(\alpha)$, $\mathfrak{m}^H := \mathfrak{a}^H \oplus \sum_{\alpha \in \Delta_H} \mathfrak{m}(\alpha)$, and $\mathfrak{g}^H := \mathfrak{k}^H \oplus \mathfrak{m}^H$. Then, we have the corollary to Theorem 1.1 and Theorem 1.2;

COROLARY 2.2. *Let $\Delta_H = \{\alpha \in \Delta \mid \alpha(H) = 0\}$, i.e., the closed subsystem. Then,* (i) *$(\mathfrak{g}^H, \mathfrak{k}^H)$ is a compact semisimple symmetric pair whose restricted root system is Δ_H,* (ii) *the principal isotropy subalgebra $\mathfrak{k}^H(0)$ of the pair $(\mathfrak{g}^H, \mathfrak{k}^H)$ is an ideal of $\mathfrak{k}(0)$, and* (iii) *$\mathfrak{k}_H = (\mathfrak{k}(0)/\mathfrak{k}^H(0)) \oplus \mathfrak{k}^H$, where $(\mathfrak{k}(0)/\mathfrak{k}^H(0))$ means the orthogonal complement of $\mathfrak{k}^H(0)$ in $\mathfrak{k}(0)$.*

2.2. Representatives of K-orbit. We give a criterion for the same local orbit types (Theorem 2.6). We will find good representatives of the linear isotropy orbit (Theorem 2.7). Finally, we give an evaluation of the number of the local orbit types (Corollary 2.8).

LEMMA 2.3. *Let* $\{\alpha_1, \cdots, \alpha_r\}$ *be a set of all simple roots of* Δ, *where* r *is the rank. Let* $H := \sum_{i=1}^{k} \varpi_i$, *where* $\Omega := \{\varpi_1, \cdots, \varpi_r\}$ *is the set of all fundamental weights. Then,* $\{\alpha_{k+1}, \cdots, \alpha_r\}$ *is a set of all simple roots of* Δ_H.

PROOF. Let $\alpha = \sum_{j=1}^{r} c_j \alpha_j \in \Delta_H$, where all c_j are non-positive or non-negative integers. Since $0 = \alpha(H) = \frac{1}{2}\sum_{i=1}^{k} c_i(\alpha_i, \alpha_i)$, we have $c_1 = \cdots = c_k = 0$. Then, α can be expressed as the linear combination of $\{\alpha_{k+1}, \cdots, \alpha_r\}$ with non-positive or non-negative coefficients. □

COROLLARY 2.4. *Let* Π *be a simple root system of* Δ, *and let* Π' *be a subset of* Π. *Let* $\Omega = \{\varpi_1, \cdots, \varpi_r\}$ *be the set of all fundamental weights, where* r *is the rank. Then, there exists a unique element* $H = \sum_{l=1}^{k} \varpi_{i_l}$ *such that* $\Pi' = \Pi_H$ *satisfying the following conditions*: (i) *if* $\alpha_i \in \Pi'$, $\alpha_i(H) = 0$ (ii) *if* $\alpha_i \notin \Pi'$, $\alpha_i(H) = \frac{1}{2}(\alpha_i, \alpha_i)$, *so that,* $H = \sum_{\alpha_i \notin \Pi'} \varpi_i$.

LEMMA 2.5. *Let* $N_K(\mathfrak{a}) := \{g \in K \mid \mathrm{Ad}(g)\mathfrak{a} = \mathfrak{a}\}$ *be a normalizer of* $\mathfrak{a}$ *in* K, *and* $\Omega = \{\varpi_1, \cdots, \varpi_r\}$ *be the set of all fundamental weights, where* r *is the rank, and* Π_H *and* $\Pi_{H'}$ *be simple root systems of* Δ_H *and* $\Delta_{H'}$ *respectively. Let* $H := \sum_{j=1}^{k} \varpi_j$, $H' := \sum_{s=1}^{t} \varpi_s$. *For given* $g \in N_K(\mathfrak{a})$, $g\Delta_H = \Delta_{H'}$ *if and only if* $g'\Pi_H = \Pi_{H'}$ *for a certain* $g' \in N_K(\mathfrak{a})$.

PROOF. If $g\Delta_H = \Delta_{H'}$ is satisfied, we have $g\Pi_H \subset \Delta_{H'}$. Then, there exists an element w of Weyl groups $W(\Delta_{H'})$ of $\Delta_{H'}$ such that $w(g\Pi_H) = \Pi_{H'}$. Conversely, if $g'\Pi_H = \Pi_{H'}$ for a certain $g' \in N_K(\mathfrak{a})$, one can easily see the assertion, by identifying $\alpha_j \in \Pi_H$ with ϖ_j and doing $\alpha_s \in \Pi_{H'}$ with ϖ_s. □

THEOREM 2.6. *Let* $\mathcal{C} := \{\sum_{i=1}^{r} c_i\varpi_i \mid c_i \geq 0 \text{ for } i = 1, \cdots, r\}$, *where each* ϖ_i *is an element of the set of all fundamental weights* $\Omega = \{\varpi_1, \cdots, \varpi_r\}$. *For any* $H, H' \in \mathcal{C}$, Δ_H *is equivalent to* $\Delta_{H'}$ *if and only if* $\mathfrak{k}_H$ *is conjugate to* $\mathfrak{k}_{H'}$, *i.e., for given* $g \in N_K(\mathfrak{a})$, $g\Delta_H = \Delta_{H'}$ *if and only if* $\mathrm{Ad}(g)\mathfrak{k}_H = \mathfrak{k}_{H'}$.

PROOF. For given $g \in N_K(\mathfrak{a})$, put $g : \mathfrak{a}^* \to \mathfrak{a}^*$ such that $\alpha \mapsto g \cdot \alpha := \alpha \circ \mathrm{Ad}(g^{-1})$, where $\mathfrak{a}^*$ is a dual space of $\mathfrak{a}$. Then, we claim first that $\mathrm{Ad}(g)\mathfrak{k}(\alpha) = \mathfrak{k}(g \cdot \alpha)$. Indeed, since $\mathrm{Ad}(g)\mathfrak{k}(\alpha) = \mathfrak{k} \cap \mathrm{Ad}(g)\mathfrak{g}(\alpha)$, it suffices to observe that $\mathrm{Ad}(g)\mathfrak{g}(\alpha) = \mathfrak{g}(g \cdot \alpha)$. For any $X_\alpha \in \mathfrak{g}(\alpha)$ and any $H \in \mathfrak{a}$, we have $[H, [H, \mathrm{Ad}(g)X_\alpha]] = -g \cdot \alpha(H)^2 \mathrm{Ad}(g)X_\alpha$, which implies that $\mathrm{Ad}(g)X_\alpha \in \mathfrak{g}(g \cdot \alpha)$. This proves the first assertion. Let $\mathfrak{k}_H := \mathfrak{k}(0) \oplus \sum_{\alpha \in \Delta_H} \mathfrak{k}(\alpha)$, and $\mathfrak{k}_{H'} := \mathfrak{k}(0) \oplus \sum_{\beta \in \Delta'_H} \mathfrak{k}(\beta)$. If $g\Delta_H = \Delta_{H'}$ is satisfied, we have $\mathrm{Ad}(g)\mathfrak{k}_H = \mathfrak{k}_{H'}$. Conversely, if $\mathrm{Ad}(g)\mathfrak{k}_H = \mathfrak{k}_{H'}$ is satisfied, we have $g\Delta_H = \Delta_{H'}$. Indeed, we have $\mathrm{Ad}(g)\mathfrak{k}_H = \mathfrak{k}(0) \oplus \sum_{g \cdot \alpha \in \Delta_H} \mathfrak{k}(g \cdot \alpha)$ and $\mathfrak{k}_{H'} = \mathfrak{k}(0) \oplus \sum_{\beta \in \Delta_{H'}} \mathfrak{k}(\beta)$. By hypothesis, we have $\sum_{g \cdot \alpha \in \Delta_H} \mathfrak{k}(g \cdot \alpha) = \sum_{\beta \in \Delta_{H'}} \mathfrak{k}(\beta)$. Suppose that $g \cdot \alpha \notin \Delta_H$ for all $\alpha \in \Delta_H$. Then, we have $\mathfrak{k}(g \cdot \alpha) \not\subset \sum_{\beta \in \Delta_{H'}} \mathfrak{k}(\beta)$. This contradicts with $\sum_{g \cdot \alpha \in \Delta_H} \mathfrak{k}(g \cdot \alpha) = \sum_{\beta \in \Delta_{H'}} \mathfrak{k}(\beta)$. This proves that

$g\Delta_H \subset \Delta_{H'}$. One can easily see that $g\Delta_H \supset \Delta_{H'}$ in a similar way. It follows that $g\Delta_H = \Delta_{H'}$. $\square$

In [Tama1], Tamaru showed the following;

THEOREM 2.7 (Tamaru [Tama1]). *Let $\Omega = \{\varpi_1, \cdots, \varpi_r\}$ be the set of all fundamental weights, where r is the rank. For any vector $H \in \mathfrak{m}$, $\mathfrak{k}_H$ is conjugate to the isotropy subalgebra at $\sum_{j=1}^{k} \varpi_{i_j}$, where $0 \leq k \leq r$.*

COROLLARY 2.8. *The number of the local orbit types of s-representations of a compact semisimple symmetric space is less than or equal to 2^r, where r is the rank of the symmetric space.*

PROOF OF COROLLARY 2.8. By Theorem 2.7, how to choose $\{\varpi_{i_j}\}_{1\leq j\leq k}$ out of $\Omega = \{\varpi_1, \cdots, \varpi_r\}$ is ${}_rC_0 + {}_rC_1 + \cdots + {}_rC_r = (1+1)^r = 2^r$. But, by Theorem 2.6, the number of the local orbit types is less than 2^r. $\square$

2.3. The algorithm to get the isotropy subalgebras. The method to get the isotropy subalgebras $\mathfrak{k}_H$ of $\mathfrak{k}$ at H is as follows : Let $(\mathfrak{g}, \mathfrak{k})$ be a compact semisimple symmetric pair. Then,

(Algorithm 1) By inspecting Tamaru's Table 1, we find the pair $(\mathfrak{g}, \mathfrak{k})$ to get the corresponding type, rank, and multiplicities. Then, we know the restricted Dynkin diagram, and the principal isotropy subalgebra $\mathfrak{k}(0)$ also. For example, in the case of $(\mathfrak{g}, \mathfrak{k}) = AI_3 = (su(4), so(4))$, the Dynkin diagram of the pair $(su(4), so(4))$ is $\circ^1 - \circ^1 - \circ^1$, where the number on each vertex is the multiplicity, and the principal isotropy subalgebra $\mathfrak{k}(0)$ of the pair $(su(4), so(4))$ is 0, by $A(1)_3$-type of Tamaru's Table 1.

(Algorithm 2) By Corollary 2.4, we can list all subset $\Pi' = \Pi_H \subset \Pi$ corresponding to all possible subdiagrams of the Dynkin diagram obtained in Algorithm 1. And, by Corollary 2.4, we can easily determine each H from the Dynkin diagrams of each Π_H. In the case of the above example, the Dynkin diagrams of each Π_H are (1) $\circ^1 - \circ^1 - \circ^1$, (2) $\circ^1 - \circ^1$, (3) $\circ^1 \quad \circ^1$, (4) $\circ^1$, and (5) $\{\emptyset\}$, by the Dynkin diagram $\circ^1 - \circ^1 - \circ^1$ of the pair $(su(4), so(4))$. Thus, each H are $H = \{0\}$ for (1), $H = \varpi_1$, or ϖ_3 for (2), $H = \varpi_2$ for (3), $H = \varpi_1 + \varpi_2$, $\varpi_2 + \varpi_3$ or $\varpi_1 + \varpi_3$ for (4), and $H = \sum_{i=1}^{3} \varpi_i$ for (5).

(Algorithm 3) From the Dynkin subdiagram corresponding to each subset $\Pi' = \Pi_H$ of Π, we find the corresponding symmetric pair $(\mathfrak{g}^H, \mathfrak{k}^H)$, and the principal isotropy subalgebra $\mathfrak{k}^H(0)$, by using Tamaru's Table 1. For the case of (2) of the above example, $(\mathfrak{g}^H, \mathfrak{k}^H)$ and $\mathfrak{k}^H(0)$ are $(su(3), so(3))$ and 0, respectively, by $A(1)_2$-type of Tamaru's Table 1. For the other cases, one can get explicitly each $(\mathfrak{g}^H, \mathfrak{k}^H)$ in the same way.

(Algorithm 4) By Corollary 2.2, we can determine explicitly isotropy subalgebras of each Π_H, i.e., $\mathfrak{k}_H = (\mathfrak{k}(0)/\mathfrak{k}^H(0)) \oplus \mathfrak{k}^H$. For the case of (2) of the above example, $\mathfrak{k}_{\varpi_1} = \mathfrak{k}_{\varpi_3} = (0/0) \oplus so(3) = so(3)$. Here, $\mathfrak{k}_{\varpi_1} = \mathfrak{k}_{\varpi_3}$ is true by Theorem 2.6. In addition, the number of the isotropy subalgebras of s-representations of AI_3 is 5, i.e., less than 2^3. This fact is suitable to Corollary 2.8.

3. The classification of local orbit types of S-representations

Let M be a compact symmetric space, and let G be the identity component of the isometry group of M. We say that M is a *symmetric R-space* iff there exists a subgroup $L \subset \mathrm{Diff}(M)$ such that L is the maximal compact semisimple subgroup of G. Now, let K be the isotropy subgroup of G at the origin $o \in M$. Thus, we have the symmetric R-space $M = G/K$. Let K_H be the isotropy subgroup of K (identified with the linear isotropy group) at H in the tangent space T_oM. We denote by $\mathfrak{k}_H$ the Lie algebra of K_H. Let Δ be the restricted root system of symmetric R-space $M = G/K$, and Π be the simple root system of Δ. We denote by $\Delta_H \subset \Delta$ the closed subsystem. Thus, we have the simple root system $\Pi_H := \Pi \cap \Delta_H$ of Δ_H. Throughout this section, we obeyed the Algorithm in Subsection 2.3. One can easily give the proofs of the following results in a similar way, so we omitted the proof.

3.1. The classification of local orbit types $[\mathfrak{k}_H]$ of S-representations of symmetric R-spaces of classical types of the rank 2

$(T \cdot AI_2 = U(3)/O(3))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^1 - \circ^1$	$\{o\}$	$so(3)$	$\{o\}$	0	
(2)	$\circ^1$	ϖ_1 (ϖ_2)	$so(2)$	S^2	2	Symmetric space
(3)	$\emptyset$	$\varpi_1 + \varpi_2$	0	$SO(3)$	3	Symmetric R-space

$(T \cdot AII_2 = U(6)/Sp(3))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^4 - \circ^4$	$\{o\}$	$sp(3)$	$\{o\}$	0	
(2)	$\circ^4$	ϖ_1 (ϖ_2)	$sp(1) \oplus sp(2)$	$P^2(\mathbf{H})$	8	Symmetric R-space
(3)	$\emptyset$	$\varpi_1 + \varpi_2$	$sp(1)^3$	$\frac{Sp(3)}{Sp(1)^3}$	12	$P^1(\mathbf{H})$-bundle over $P^2(\mathbf{H})$

$(AIII_2 = Gr_2(\mathbf{C}^{2+q}) = SU(2+q)/S(U(2) \times U(q)),\ q \geq 2)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^2 \Longleftrightarrow \circledcirc^{2(q-2),1}$	$\{o\}$	$su(2) \oplus su(q) \oplus \mathbf{R}$	$\{o\}$	0	
(2)	$\circ^2$	ϖ_2	$u(q-2) \oplus so(3)$	$\frac{U(q)}{U(q-2)}$	$4q-4$	Stiefel manifold
(3)	$\circledcirc^{2(q-2),1}$	ϖ_1	$\mathbf{R} \oplus u(q-1)$	$S^2 \times S^{2q-1}$	$2q+1$	Symmetric space
(4)	$\emptyset$	$\varpi_1 + \varpi_2$	$\mathbf{R} \oplus u(q-2)$	$S^2 \times \frac{U(q)}{U(q-2)}$	$4q-2$	S^2-bundle over (2)

$(BDI_2 = SO(q+2)/SO(q) \times SO(2),\ q \geq 2)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^1 \Longrightarrow \circ^{q-2}$	$\{o\}$	$so(2) \oplus so(q)$	$\{o\}$	0	
(2)	$\circ^{q-2}$	ϖ_1	$so(q-1)$	$S^1 \times S^{q-1}$	q	Symmetric space
(3)	$\circ^1$	ϖ_2	$so(2) \oplus so(q-2)$	$\frac{SO(q)}{SO(q-2)}$	$2q-3$	Stiefel manifold
(4)	$\emptyset$	$\varpi_1 + \varpi_2$	$so(q-2)$	$S^1 \times \frac{SO(q)}{SO(q-2)}$	$2q-2$	S^1-bundle over (3)

$(CI_2 = Sp(2)/U(2))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^1 \Longleftarrow \circ^1$	$\{o\}$	$u(2)$	$\{o\}$	0	
(2)	$\circ^1$	$\varpi_1 (\varpi_2)$	$u(1)$	S^3	3	Symmetric R-space
(3)	$\emptyset$	$\varpi_1 + \varpi_2$	0	$U(2)$	4	Symmetric R-space

$(CII_2 = Gr_2(\mathbf{H}^{2+q}) = Sp(2+q)/Sp(2) \times Sp(q),\ q \geq 2)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^4 \Longleftrightarrow \circledcirc^{4(q-2),3}$	$\{o\}$	$sp(2) \oplus sp(q)$	$\{o\}$	0	
(2)	$\circ^4$	ϖ_2	$sp(q-2) \oplus so(5)$	$\frac{Sp(q)}{Sp(q-2)}$	$8q-6$	Stiefel manifold
(3)	$\circledcirc^{4(q-2),3}$	ϖ_1	$sp(1)^2 \oplus sp(q-1)$	$S^4 \times S^{4q-1}$	$4q+3$	Symmetric space
(4)	$\emptyset$	$\varpi_1 + \varpi_2$	$sp(1)^2 \oplus sp(q-2)$	$S^4 \times \frac{Sp(q)}{Sp(q-2)}$	$8q-2$	S^4-bundle over (2)

$(CII_2 = Gr_2(\mathbf{H}^4) = Sp(4)/Sp(2) \times Sp(2))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^4 \Longleftarrow \circ^3$	$\{o\}$	$sp(2) \oplus sp(2)$	$\{o\}$	0	
(2)	$\circ^4$	ϖ_2	$so(5)$	$SO(5)$	10	Symmetric R-space
(3)	$\circ^3$	ϖ_1	$sp(1) \oplus so(4)$	$S^7 \times S^4$	11	Symmetric space
(4)	$\emptyset$	$\varpi_1 + \varpi_2$	$sp(1)^2$	$S^7 \times S^7$	14	Symmetric space

$(D\mathit{III}_2 = SO(8)/U(4))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^4 \Longleftarrow \circ^1$	$\{o\}$	$u(4)$	$\{o\}$	0	
(2)	$\circ^4$	ϖ_2	$so(5)$	$S^1 \times S^5$	6	Symmetric space
(3)	$\circ^1$	ϖ_1	$su(2)^2 \oplus so(2)$	$\frac{SO(6)}{SO(3)\times SO(3)}$	9	Symmetric R-space
(4)	$\emptyset$	$\varpi_1 + \varpi_2$	$sp(1)^2$	$\frac{U(4)}{Sp(1)^2}$	10	S^1-bundle over (3)

$(D\mathit{III}_2 = SO(10)/U(5))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^4 \Longleftrightarrow \circledcirc^{4,1}$	$\{o\}$	$u(5)$	$\{o\}$	0	
(2)	$\circledcirc^{4,1}$	ϖ_1	$sp(1) \oplus u(3)$	$\frac{SU(5)}{SU(2)\times SU(3)}$	13	Symmetric R-space
(3)	$\circ^4$	ϖ_2	$\mathbf{R} \oplus so(5)$	$\frac{SU(5)}{SO(5)}$	14	Symmetric R-space
(4)	$\emptyset$	$\varpi_1 + \varpi_2$	$su(2)^2 \oplus \mathbf{R}$	$\frac{SU(5)}{SU(2)\times SU(2)}$	18	Flag manifold (S^4-bundle over (2))

3.2. The classification of local orbit types $[\mathfrak{k}_H]$ of S-representations of symmetric R-spaces of classical types of the rank 3

$(T \cdot AI_3 = U(4)/O(4))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^1 - \circ^1 - \circ^1$	$\{o\}$	$so(4)$	$\{o\}$	0	
(2)	$\circ^1 - \circ^1$	ϖ_1 (ϖ_3)	$so(3)$	S^3	3	Symmetric R-space
(3)	$\circ^1 \quad \circ^1$	ϖ_2	$so(2)^2$	$\frac{SO(4)}{SO(2)\times SO(2)}$	4	Symmetric R-space
(4)	$\circ^1$	$\varpi_i + \varpi_j$ $(i \neq j)$	$so(2)$	$\frac{SO(4)}{SO(2)}$	5	Stiefel manifold
(5)	$\emptyset$	$\varpi_1 + \varpi_2 + \varpi_3$	0	$SO(4)$	6	Symmetric R-space

$(T \cdot AII_3 = SU(8)/Sp(4)\mathbf{Z}_2)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^4 - \circ^4 - \circ^4$	$\{o\}$	$sp(4)$	$\{o\}$	0	
(2)	$\circ^4 - \circ^4$	ϖ_1 (ϖ_3)	$sp(1) \oplus sp(3)$	$P^3(\mathbf{H}^4)$	12	Symmetric R-space
(3)	$\circ^4 \quad \circ^4$	ϖ_2	$so(5) \oplus so(5)$	$Gr_2(\mathbf{H}^4)$	16	Symmetric R-space
(4)	$\circ^4$	$\varpi_i + \varpi_j$ $(i \neq j)$	$sp(1)^2 \oplus so(5)$	$\frac{Sp(4)}{Sp(1)^2 \times Sp(2)}$	20	2nd kind GLA
(5)	$\emptyset$	$\varpi_1 + \varpi_2 + \varpi_3$	$sp(1)^4$	$\frac{Sp(4)}{Sp(1)^4}$	24	1st kind GLA

$(AIII_3 = Gr_3(\mathbf{C}^{3+q}),\ q \geq 3)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^2 - \circ^2 \Longleftrightarrow \circledcirc^{2(q-3),1}$	$\{o\}$	$su(3) \oplus su(q) \oplus \mathbf{R}$	$\{o\}$	0	
(2)	$\circ^2 - \circ^2$	ϖ_3	$u(q-3) \oplus su(3)$	$\frac{SU(q)}{SU(q-3)}$	$6q-9$	Stiefel manifold
(3)	$\circ^2 \Longleftrightarrow \circledcirc^{2(q-3),1}$	ϖ_1	$\mathbf{R} \oplus su(2) \oplus$ $su(q-1) \oplus \mathbf{R}$	$P^2(\mathbf{C}) \times S^{2q-1}$	$2q+3$	Symmetric space
(4)	$\circ^2 \quad \circledcirc^{2(q-3),1}$	ϖ_2	$u(q-3) \oplus so(3)$ $\oplus su(q-2) \oplus \mathbf{R}$	$\frac{(SU(3) \times U(q))}{(U(q-3) \times SU(2) \times U(q-2))}$	$-q^2+10q-8$	
(5)	$\circ^2$	$\varpi_1 + \varpi_3$ $(\varpi_2 + \varpi_3)$	$\mathbf{R} \oplus u(q-3)$ $\oplus so(3)$	$S^5 \times SU(q)/U(q-3)$	$6q-5$	
(6)	$\circledcirc^{2(q-3),1}$	$\varpi_1 + \varpi_2$	$\mathbf{R} \oplus u(q-3)$ $\oplus su(q-2) \oplus \mathbf{R}$	$\frac{(SU(3) \times U(q))}{(U(1) \times U(q-3) \times U(q-2))}$	$-q^2+10q-6$	
(7)	$\emptyset$	$\sum_{i=1}^{3} \varpi_i$	$\mathbf{R}^2 \oplus u(q-3)$	$\frac{(SU(3) \times SU(q))}{(U(1) \times U(q-3))}$	$6q-3$	

$(BDI_3 = Gr_3(\mathbf{R}^{3+q}),\ q \geq 3)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^1 - \circ^1 \Longrightarrow \circ^{(q-3)}$	$\{o\}$	$so(q) \oplus so(3)$	$\{o\}$	0	
(2)	$\circ^1 - \circ^1$	ϖ_3	$so(q-3) \oplus so(3)$	$\frac{SO(q)}{SO(q-3)}$	$3q-6$	Stiefel manifold
(3)	$\circ^1 \Longrightarrow \circ^{(q-3)}$	ϖ_1	$so(2) \oplus so(q-1)$	$S^{q-1} \times S^2$	$q+1$	Symmetric space
(4)	$\circ^1 \quad \circledcirc^{(q-3)}$	ϖ_2	$so(2) \oplus so(q-2)$	$\frac{SO(q)}{SO(q-2)} \times S^2$	$2q-1$	$(S^{q-3} \times S^2)$-bundle over (2)
(5)	$\circ^1$	$\varpi_1 + \varpi_3$ $(\varpi_2 + \varpi_3)$	$so(q-3) \oplus so(2)$	$\frac{SO(q)}{SO(q-3)} \times S^2$	$3q-4$	S^{q-3}-bundle over (4)
(6)	$\circ^{(q-3)}$	$\varpi_1 + \varpi_2$	$so(q-2)$	$\frac{SO(q)}{SO(q-2)} \times S^3$	$2q$	S^1-bundle over (4)
(7)	$\emptyset$	$\sum_{i=1}^{3} \varpi_i$	$so(q-3)$	$\frac{SO(q)}{SO(q-3)} \times S^3$	$3q-3$	S^3-bundle over (2)

$(CI_3 = Sp(3)/U(3))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^1 - \circ^1 \Longleftarrow \circ^1$	$\{o\}$	$u(3)$	$\{o\}$	0	
(2)	$\circ^1 - \circ^1$	ϖ_3	$so(3)$	$S^5 \times S^1$	6	Symmetric space
(3)	$\circ^1 \Longleftarrow \circ^1$	ϖ_1	$u(2)$	S^5	5	Symmetric space
(4)	$\circ^1 \quad \circ^1$	ϖ_2	$so(2)^2$	$\frac{U(3)}{(U(1)\times U(1))}$	7	Flag manifold
(5)	$\circ^1$	$\varpi_1 + \varpi_3$ $(\varpi_2 + \varpi_3)$	$so(2)$	$SU(3)$	8	S^1-bundle over (4)
(6)	$\emptyset$	$\sum_{i=1}^3 \varpi_i$	$\{0\}$	$U(3)$	9	Symmetric R-space

$(CII_3 = Gr_3(\mathbf{H}^{3+q}),\ q \geq 3)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^4 - \circ^4 \Longleftrightarrow \circledcirc^{4(q-3),3}$	$\{o\}$	$sp(3) \oplus sp(q)$	$\{o\}$	0	
(2)	$\circ^4 - \circ^4$	ϖ_3	$sp(q-3) \oplus sp(3)$	$\frac{Sp(q)}{Sp(q-3)}$	$12q - 15$	Stiefel manifold
(3)	$\circ^4 \Longleftrightarrow \circledcirc^{4(q-3),3}$	ϖ_1	$sp(1) \oplus sp(2)$ $\oplus sp(q-1)$	$P^2(\mathbf{H}) \times S^{4q-1}$	$4q + 7$	Symmetric space
(4)	$\circ^4 \quad \circledcirc^{4(q-3),3}$	ϖ_2	$sp(2) \oplus sp(1)$ $\oplus sp(q-2)$	$P^2(\mathbf{H}) \times \frac{Sp(q)}{Sp(q-2)}$	$8q + 2$	S^{4q-5}-bundle over (3)
(5)	$\circ^4$	$\varpi_1 + \varpi_3$ $(\varpi_2 + \varpi_3)$	$sp(1) \oplus sp(q-3)$ $\oplus sp(2)$	$P^2(\mathbf{H}) \times \frac{Sp(q)}{Sp(q-3)}$	$12q - 7$	S^{4q-9}-bundle over (4)
(6)	$\circledcirc^{4(q-3),3}$	$\varpi_1 + \varpi_2$	$sp(1)^3 \oplus sp(q-2)$	$\frac{Sp(3)}{Sp(1)^3} \times \frac{Sp(q)}{Sp(q-2)}$	$8q + 6$	$P^1(\mathbf{H})$-bundle over (4)
(7)	$\emptyset$	$\sum_{i=1}^3 \varpi_i$	$sp(1)^3 \oplus sp(q-3)$	$\frac{Sp(3)}{Sp(1)^3} \times \frac{Sp(q)}{Sp(q-3)}$	$12q - 3$	S^{4q-9}-bundle over (4)

$(DIII_3 = SO(12)/U(6))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^4 - \circ^4 \Longleftarrow \circ^1$	$\{o\}$	$u(6)$	$\{o\}$	0	
(2)	$\circ^4 - \circ^4$	ϖ_3	$sp(3)$	$\frac{U(6)}{Sp(3)}$	15	Symmetric R-space
(3)	$\circ^4 \Longleftarrow \circ^1$	ϖ_1	$su(2) \oplus u(4)$	$\frac{SU(6)}{SU(2)\times SU(4)}$	17	
(4)	$\circ^4 \quad \circ^1$	ϖ_2	$su(2) \oplus sp(2) \oplus so(2)$	$\frac{SU(6)}{SU(2)\times Sp(2)}$	22	S^5-bundle over (3)
(5)	$\circ^4$	$\varpi_1 + \varpi_3$ $(\varpi_2 + \varpi_3)$	$su(2) \oplus sp(2)$	$\frac{U(6)}{SU(2)\times Sp(2)}$	23	S^1-bundle over (4)
(6)	$\circ^1$	$\varpi_1 + \varpi_2$	$su(2)^3 \oplus so(2)$	$\frac{SU(6)}{SU(2)^3}$	26	
(7)	$\emptyset$	$\sum_{i=1}^3 \varpi_i$	$su(2)^3$	$\frac{U(6)}{SU(2)^3}$	27	S^1-bundle over (6)

$(D III_3 = SO(14)/U(7))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$
(1)	$\circ^4 - \circ^4 \Longleftrightarrow \circledcirc^{4,1}$	$\{o\}$	$u(7)$	$\{o\}$	0
(2)	$\circ^4 \Longleftrightarrow \circledcirc^{4,1}$	ϖ_1	$su(2) \oplus u(5)$	$\frac{SU(7)}{SU(2)\times SU(5)}$	21
(3)	$\circ^4 - \circ^4$	ϖ_3	$\mathbf{R} \oplus sp(3)$	$\frac{SU(7)}{Sp(3)}$	27
(4)	$\circ^4 \quad \circledcirc^{4,1}$	ϖ_2	$sp(2) \oplus u(3)$	$\frac{SU(7)}{Sp(2)\times SU(3)}$	30
(5)	$\circledcirc^{4,1}$	$\varpi_1 + \varpi_2$	$su(2)^2 \oplus u(3)$	$\frac{SU(7)}{SU(2)^2\times SU(3)}$	34
(6)	$\circ^4$	$\varpi_1 + \varpi_3$ $(\varpi_2 + \varpi_3)$	$su(2) \oplus \mathbf{R} \oplus sp(2)$	$\frac{SU(7)}{SU(2)\times Sp(2)}$	35
(7)	$\emptyset$	$\sum_{i=1}^{3} \varpi_i$	$su(2)^3 \oplus \mathbf{R}$	$\frac{SU(7)}{SU(2)^3}$	39

3.3. The classification of local orbit types $[\mathfrak{k}_H]$ of S-representations of symmetric R-spaces of exceptional types

$(E III = E_6/Spin(10) \cdot T)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^6 \Longleftrightarrow \circledcirc^{8,1}$	$\{o\}$	$so(2) \oplus so(10)$	$\{o\}$	0	
(2)	$\circledcirc^{8,1}$	ϖ_1	$so(2) \oplus su(5)$	$\frac{Spin(10)}{SU(5)}$	21	Homogeneous CR-manifold
(3)	$\circ^6$	ϖ_2	$so(2) \oplus so(7)$	$\frac{Spin(10)}{Spin(7)}$	24	Stiefel manifold
(4)	$\emptyset$	$\varpi_1 + \varpi_2$	$u(4)$	$\frac{Spin(10)}{Spin(6)}$	30	Stiefel manifold

$(E IV = T \cdot E_6/F_4)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^8 - \circ^8$	$\{o\}$	f_4	$\{o\}$	0	
(2)	$\circ^8$	$\varpi_i \ (i = 1, 2)$	$so(9)$	$P^2(\mathbf{O}) = \frac{F_4}{SO(9)}$	16	Symmetric R-space
(3)	$\emptyset$	$\varpi_1 + \varpi_2$	$so(8)$	$\frac{F_4}{SO(8)}$	24	S^8-bundle over $P^2(\mathbf{O})$

$(E\,VII = E_7/E_6 \cdot T)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^8 - \circ^8 \Longleftarrow \circ^1$	$\{o\}$	$so(2) \oplus e_6$	$\{o\}$	0	
(2)	$\circ^8 - \circ^8$	ϖ_3	f_4	$\frac{T \cdot E_6}{F_4}$	27	Symmetric R-space
(3)	$\circ^8 \Longleftarrow \circ^1$	ϖ_1	$so(2) \oplus so(10)$	$\frac{E_6}{Spin(10)}$	33	Homogeneous CR-manifold
(4)	$\circ^8 \quad \circ^1$	ϖ_2	$so(2) \oplus so(9)$	$\frac{E_6}{Spin(9)}$	42	S^9-bundle over (3)
(5)	$\circ^8$	$\varpi_1 + \varpi_3$ $(\varpi_2 + \varpi_3)$	$so(9)$	$\frac{T \cdot E_6}{Spin(9)}$	43	$P^2(\mathbf{O})$-bundle over (2) (or S^1-bundle over (4))
(6)	$\circ^1$	$\varpi_1 + \varpi_2$	$so(8) \oplus so(2)$	$\frac{E_6}{Spin(8)}$	50	S^8-bundle over (4)
(7)	$\emptyset$	$\sum_{i=1}^3 \varpi_i$	$so(8)$	$\frac{T \cdot E_6}{Spin(8)}$	51	S^1-bundle over (6)

$(F\,II = P^2(\mathbf{O}) = F_4/SO(9))$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circledcirc^{8,7}$	$\{o\}$	$so(9)$	$\{o\}$	0	
(2)	$\emptyset$	ϖ_1	$so(7)$	$S^{15} = \frac{Spin(9)}{Spin(7)}$	15	Symmetric space

$(G = G_2/SO(4)$ as a normal space$)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^1 \Lleftarrow \circ^1$	$\{o\}$	$so(4)$	$\{o\}$	0	
(2)	$\circ^1$	ϖ_1 (ϖ_2)	$so(2)$	$\frac{SO(4)}{SO(2)}$	5	Stiefel manifold
(3)	$\emptyset$	$\varpi_1 + \varpi_2$	0	$SO(4)$	6	Symmetric R-space

3.4. The classification of local orbit types $[\mathfrak{k}_H]$ of S-representations of symmetric R-spaces of classical groups of the rank 2

$(SO(4),\ r = 2)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^2 \quad \circ^2$	$\{o\}$	$so(4)$	$\{o\}$	0	
(2)	$\circ^2$	ϖ_1 (ϖ_2)	$\mathbf{R} \oplus so(3)$	S^2	2	Symmetric space
(3)	$\emptyset$	$\varpi_1 + \varpi_2$	$\mathbf{R}^2$	$\frac{SO(4)}{SO(2) \times SO(2)}$	4	Symmetric R-space

$(SO(5),\ r=2)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	dim K/K_H	
(1)	$\circ^2 \Longrightarrow \circ^2$	$\{o\}$	$so(5)$	$\{o\}$	0	
(2)	$\circ^2$	$\varpi_1\ (\varpi_2)$	$\mathbf{R} \oplus so(3)$	$\frac{SO(5)}{SO(2)\times SO(3)}$	6	Symmetric R-space
(3)	$\emptyset$	$\varpi_1 + \varpi_2$	$\mathbf{R}^2$	$\frac{SO(5)}{SO(2)\times SO(2)}$	8	Flag manifold

$(U(3),\ r=2)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	dim K/K_H	
(1)	$\circ^2 - \circ^2$	$\{o\}$	$su(3)$	$\{o\}$	0	
(2)	$\circ^2$	$\varpi_1\ (\varpi_2)$	$\mathbf{R} \oplus su(2)$	$P^2(\mathbf{C})$	4	Symmetric R-space
(3)	$\emptyset$	$\varpi_1 + \varpi_2$	$\mathbf{R}^2$	$\frac{SU(3)}{S(U(1)\times U(1))}$	6	Flag manifold

$(Sp(2),\ r=2)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	dim K/K_H	
(1)	$\circ^2 \Longleftarrow \circ^2$	$\{o\}$	$sp(2)$	$\{o\}$	0	
(2)	$\circ^2$	$\varpi_1\ (\varpi_2)$	$\mathbf{R} \oplus sp(1)$	$\frac{SO(5)}{SO(2)\times SO(3)}$	6	Symmetric R-space
(3)	$\emptyset$	$\varpi_1 + \varpi_2$	$\mathbf{R}^2$	$\frac{SO(5)}{SO(2)\times SO(2)}$	8	Flag manifold

3.5. The classification of local orbit types $[\mathfrak{k}_H]$ of *S*-representations of symmetric R-spaces of classical groups of the rank 3

$(SO(6),\ r=3)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	dim K/K_H	
(1)	$\circ^2 - \circ^2 - \circ^2$	$\{o\}$	$so(6)$	$\{o\}$	0	
(2)	$\circ^2 - \circ^2$	$\varpi_1\ (\varpi_3)$	$\mathbf{R} \oplus su(3)$	$\frac{SO(6)}{U(3)}$	6	Symmetric R-space
(3)	$\circ^2 \quad \circ^2$	ϖ_2	$\mathbf{R} \oplus so(3)^2$	$\frac{SU(4)}{S(U(2)\times U(2))}$	8	Symmetric R-space
(4)	$\circ^2$	$\varpi_i + \varpi_j\ (i \neq j)$	$\mathbf{R}^2 \oplus so(3)$	$\frac{SU(4)}{U(1)\times U(2)}$	10	
(5)	$\emptyset$	$\sum_{i=1}^3 \varpi_i$	$\mathbf{R}^3$	$\frac{SO(6)}{SO(2)^3}$	12	S^2-bundle over (4)

$(SO(7),\ r=3)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^2 - \circ^2 \Longrightarrow \circ^2$	$\{o\}$	$so(7)$	$\{o\}$	0	
(2)	$\circ^2 \Longrightarrow \circ^2$	ϖ_1	$\mathbf{R} \oplus so(5)$	$\frac{SO(7)}{SO(2)\times SO(5)}$	10	Symmetric R-space
(3)	$\circ^2 - \circ^2$	ϖ_3	$\mathbf{R} \oplus su(3)$	$\frac{SO(7)}{U(3)}$	12	
(4)	$\circ^2 \quad \circ^2$	ϖ_2	$\mathbf{R} \oplus so(3)^2$	$\frac{SO(7)}{SO(2)\times SO(3)^2}$	14	
(5)	$\circ^2$	$\varpi_i + \varpi_j\ (i \neq j)$	$\mathbf{R}^2 \oplus so(3)$	$\frac{SO(7)}{SO(2)^2\times SO(3)}$	16	S^2-bundle over (4)
(6)	$\emptyset$	$\sum_{i=1}^3 \varpi_i$	$\mathbf{R}^3$	$\frac{SO(7)}{SO(2)^3}$	18	S^2-bundle over (5)

$(U(4),\ r=3)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^2 - \circ^2 - \circ^2$	$\{o\}$	$su(4)$	$\{o\}$	0	
(2)	$\circ^2 - \circ^2$	$\varpi_1\ (\varpi_3)$	$\mathbf{R} \oplus su(3)$	$P^3(\mathbf{C})$	6	Symmetric R-space
(3)	$\circ^2 \quad \circ^2$	ϖ_2	$\mathbf{R} \oplus so(3)^2$	$\frac{SU(4)}{S(U(2)\times U(2))}$	8	Symmetric R-space
(4)	$\circ^2$	$\varpi_i + \varpi_j\ (i \neq j)$	$\mathbf{R}^2 \oplus so(3)$	$\frac{SO(6)}{SO(2)^2\times SO(3)}$	10	S^2-bundle over (3)
(5)	$\emptyset$	$\sum_{i=1}^3 \varpi_i$	$\mathbf{R}^3$	$\frac{SO(6)}{SO(2)^3}$	12	S^2-bundle over (4)

$(Sp(3),\ r=3)$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^2 - \circ^2 \Longleftarrow \circ^2$	$\{o\}$	$sp(3)$	$\{o\}$	0	
(2)	$\circ^2 \Longleftarrow \circ^2$	ϖ_1	$\mathbf{R} \oplus sp(2)$	$P^5(\mathbf{C})$	10	Symmetric R-space
(3)	$\circ^2 - \circ^2$	ϖ_3	$\mathbf{R} \oplus su(3)$	$\frac{Sp(3)}{U(3)}$	12	Symmetric R-space
(4)	$\circ^2 \quad \circ^2$	ϖ_2	$\mathbf{R} \oplus so(3)^2$	$\frac{Sp(3)}{S^1\times Sp(1)^2}$	14	
(5)	$\circ^2$	$\varpi_i + \varpi_j\ (i \neq j)$	$\mathbf{R}^2 \oplus so(3)$	$\frac{Sp(3)}{SO(2)^2\times SO(3)}$	16	S^2-bundle over (4)
(6)	$\emptyset$	$\sum_{i=1}^3 \varpi_i$	$\mathbf{R}^3$	$\frac{Sp(3)}{SO(2)^3}$	18	S^2-bundle over (5)

3.6. The classification of local orbit types $[\mathfrak{k}_H]$ of S-representations of the real quadrics $SO(p+1)\times SO(q+1)/S(O(p)\times O(q))$, $p\leq q$

	Π_H	H	$\mathfrak{k}_H$	K/K_H	$\dim K/K_H$	
(1)	$\circ^{p-1}\ \circ^{q-1}$	$\{o\}$	$so(p)\oplus so(q)$	$\{o\}$	0	
(2)	$\circ^{q-1}$	ϖ_1	$so(p-1)\oplus so(q)$	S^{p-1}	$p-1$	Symmetric space
(3)	$\circ^{p-1}$	ϖ_2	$so(q-1)\oplus so(p)$	S^{q-1}	$q-1$	Symmetric space
(4)	$\emptyset$	$\varpi_1+\varpi_2$	$so(p-1)\oplus so(q-1)$	$S^{p-1}\times S^{q-1}$	$p+q-2$	Symmetric space

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